An Exploration of Manipulatives in Math Education

Jade Monte

Follow this and additional works at: https://vc.bridgew.edu/honors_proj

Part of the Elementary Education Commons, Mathematics Commons, Science and Mathematics Education Commons, and the Social Psychology Commons

Recommended Citation
Copyright © 2021 Jade Monte

This item is available as part of Virtual Commons, the open-access institutional repository of Bridgewater State University, Bridgewater, Massachusetts.
An Exploration of Manipulatives in Math Education

Jade Monte

Submitted in Partial Completion of the Requirements for
Commonwealth Interdisciplinary Honors in Elementary Education and Mathematics

Bridgewater State University

May 11, 2021

Dr. Jacquelynne Boivin, Thesis Advisor
Dr. Nguyenho Ho, Thesis Advisor
Dr. Rebecca Metcalf, Committee Member
Dr. Jeanne Carey Ingle, Committee Member
ABSTRACT

Pre-existing literature has shown that the education system needs to re-evaluate mathematical teaching practices in a manner that can boost students’ confidence in mathematics. Thus, the research is to investigate the use of manipulatives in reducing students’ anxiety by increasing their learning experience and engagement in mathematics. Furthermore, the purpose of this thesis is to explain the interconnectedness of math manipulatives, student engagement, and problem-solving. An in-depth literature review is conducted, which contains definitions, important benefits and methodologies of manipulatives, as well as the teacher’s role regarding these three terms. When manipulatives, student engagement, and problem-solving are in harmony, students can achieve intellectual comprehension of abstract mathematical concepts. The thesis will conclude with a list of future research ideas and recommendations to strengthen mathematical teaching practices.

Key words: manipulative, student engagement, problem-solving, math, teacher, student
# TABLE OF CONTENTS

1. Introduction .......................................................................................................................... 5

2. Manipulatives .......................................................................................................................... 5
   2.1 Definition .......................................................................................................................... 5
   2.2 Benefits of Manipulatives ............................................................................................... 6
   2.3 The Three Principles for Effectiveness ........................................................................... 13
   2.4 Types of Manipulatives ................................................................................................... 19
   2.5 Four Commonalities of Manipulatives .......................................................................... 26
   2.6 Teacher’s Perspective in Action ................................................................................... 34

3. Student Engagement ............................................................................................................. 37
   3.1 Definition ....................................................................................................................... 37
   3.2 Why Student Engagement Matters .............................................................................. 38
   3.3 Reasons and Results of Disengagement ...................................................................... 38
   3.4 Reasons and Results of Engagement .......................................................................... 40
   3.5 The Three Types of Student Engagement .................................................................. 41
   3.6 Differentiated Learning ............................................................................................... 49
   3.7 Concrete, Representational, Abstract (CRA) Instruction ........................................... 50

4. Problem-Solving ................................................................................................................... 57
   4.1 Definition ....................................................................................................................... 57
1. INTRODUCTION

“I hear and I forget. I see and I remember. I do and I understand.” -Confucius (551-479 BC)

Many students find themselves struggling to understand abstract mathematical concepts. To fully grasp a concept, students must go beyond hearing and seeing mathematics, and actively participate in mathematics problems. One way to foster students’ understanding is to present problems in a way that students can process to comprehend easily. Manipulatives are tools that can demonstrate how effective this aid can be in the learning process. With some help and support, teachers can adjust their mathematical teaching so they can engage students with the use of manipulatives, and in return create stronger and more confident problem solvers.

2. MANIPULATIVES

2.1 Definition

Teachers need the right tools to help their students learn and succeed. This can be particularly difficult with mathematics since concepts are so abstract, but research has shown that one specific tool called a manipulative is effective. So, what exactly is a manipulative? Although scholars’ definitions differ, they all tend to have the same key words: concrete, physical, object, engage, mathematics, and hands-on. By taking all of these terms, a manipulative can be defined as a physical object that is used to engage students in the hands-on learning of mathematics (Jones & Tiller, 2017). They can be purchased, brought from home, or teacher/student made. The main purpose of a manipulative is to introduce, practice, and/or remediate a mathematical concept. Ever since the 1900s, they have been considered essential in teaching elementary mathematics (Boggan et al., 2010). They are mainly used in the primary grades and tend to fade away as a child climbs up the ladder to each grade level. However, the National Council of
Teachers of Mathematics (NCTM) has recommended the use of manipulatives at all grade levels for teaching mathematical concepts (Boggan et al., 2010). This research entails a detailed exploration of manipulatives and the teacher’s role while using them in mathematics education.

2.2 Benefits of Manipulatives

Manipulatives are widely used and known worldwide because there are numerous benefits for using them to teach mathematical concepts. Besides the NCTM, teachers and scholars also suggest using them in the classroom. Their use is supported by both learning theory and educational research. Countless studies have been conducted by researchers over the past few decades which look at their benefits (Boggan et al., 2010). These studies have been administered in all different grade levels and in several countries (Boggan et al., 2010). This section will discuss four benefits of mathematical manipulatives.

One benefit is that manipulatives increase math achievement (Boggan et al., 2010). Although manipulatives are mainly used in elementary classrooms, research suggests using them at all levels of education to assist students. The mathematical skills that students gain from manipulative use are going to benefit them in future settings. This is because the concepts that they learn early on are going to follow them as they progress through each grade. Two studies that show the increase in math achievement while using manipulatives will be discussed. The first study is about a first-grade class using clocks to learn time. The second study is about secondary education students with learning disabilities using a balance to learn algebra.

The first study that will be discussed involved a first-grade class that used clocks to learn time, which took place in 2016. The study was directed by education professor David S. Benders at Union College, but primarily done by first grade teacher Tracy Craft. Their study focused on
using flexible small groups to differentiate instruction in a first-grade classroom to learn the concept of telling time to the hour and half-hour (Benders & Craft, 2016). Differentiated instruction means tailoring instruction to meet individual needs (Alber, 2014). Many researchers have found manipulatives to be beneficial for a variety of learners, particularly low-achievers, students with disabilities, and English language learners (Boggan et al., 2010). This information is helpful for teachers for when they assemble students in small groups based on their needs.

At the start of Benders and Craft’s research, a pre-test was given to eleven students, who had been performing below grade level, with the aim to concentrate on academic development. Ms. Craft also incorporated other methods including “Work with the Teacher” sessions, practice sheets, and exit slips to look for student’s conceptual understanding of the concept of time. In the first “Work with the Teacher” session, Ms. Craft used a large demonstration clock, which is a manipulative used to demonstrate the concept of time. The students had their own clock to practice setting different times that the teacher called out (1 o’clock, 4 o’clock). In Figure 1 below, the large demonstration clock is seen in the middle with student clocks underneath. Using the clocks allowed them to correctly identify the minute and hour hands. At the end of this study, a post-test was given and the eleven students who were performing below grade level improved tremendously. The average on the pre-test was 24.5%, whereas it was 90.9% on the post-test (Benders & Craft, 2016). The clocks were also beneficial for the students by allowing them to follow along with the teacher using a similar manipulative in their hands. Professor Benders and Ms. Craft’s study shows that most math achievement is greatly enhanced when using manipulatives.

*Figure 1: One Demonstration Clock and 24 Student Clocks.*

*Learning Resources Store. (n.d.)*
Another recent study conducted by researchers Rajiv Satsangi, Emily Bouck, Teresa Taber-Doughty, Laura Bofferding, and Carly Roberts (2016) supports the claim that the use of manipulatives increases math achievement. Their study sought to see the benefits of manipulatives to assist secondary students with a learning disability in mathematics to solve single-variable linear equations (Satsangi et al., 2016). This study had a specific focus on algebraic instruction because research shows students with a learning disability in mathematics benefit from using concrete manipulatives to solve algebraic equations (Strickland & Maccini, 2013). Three African American students named Chase, Kareem, and Ricky who identified with a learning disability in mathematics were selected to participate. The concrete manipulative that was used was an Ohaus School single-beam balance that possessed two pans, as seen in Figure 2. Weights, consisting of chips and canisters, were used to model numbers and variables. To illustrate each component of an equation, constants were represented by individual chips, while a product of a constant and a single variable (e.g., 2x) was represented by a certain number of canisters (e.g., two canisters), each containing a predetermined number of chips. An equilibrium dial is in the center of the balance and points to the heavier side when objects are placed on one or both pans (Satsangi et al., 2016).

Figure 2: An Ohaus School Balance.

Satsangi et al., 2016, p. 243.

This study consisted of three phases: baseline, intervention, and best treatment. Prior to baseline, each student was given a pre-assessment to test their ability to solve linear equations. Instructional training and prompting were not provided to students when baseline assessments were administered. Intervention consisted of ten sessions with using
concrete manipulatives and ten sessions without using manipulatives. Finally, students participated in five sessions solving linear equations using the treatment condition found most effective during intervention (Satsangi et al., 2016). All three students scored above their pre-assessment with greater accuracy after intervention with manipulatives.

The results of this study showed that Chase, Kareem, and Ricky earned higher scores from the use of concrete manipulatives after the treatment. Chase had an average of 10% for the baseline phase, and at the end of intervention he increased to an average of 98.7%. Kareem had an average of 5% for the baseline phase, and at the end of intervention he increased his average to 97.5%. Lastly, Ricky had an average of 0% during the baseline phase, and at the end of intervention he increased his average to 92.5% using concrete manipulatives (Satsangi et al., 2016). Instruction with manipulatives produced significant improvement in achievement as seen by the average scores after intervention. This study shows how manipulatives can be valuable beyond the elementary level.

The second benefit is that manipulatives greatly reduce mathematics anxiety (Boggan et al., 2010). Mathematics anxiety varies from person to person, but is described as a feeling of disorientation, where panic sets in and students no longer engage with learning. Many students develop math anxiety in the beginning of second or third grade. Mathematics anxiety has a negative impact on the students who experience it because they tend to avoid and dislike math (Barrett, 2013). Academic literature suggests that manipulatives can assist learners of all ages when it comes to handling math anxiety (Finlayson, 2014; Blazer, 2011). Using a manipulative is a constructive approach to help students overcome misunderstood concepts and anxiousness.

Dr. Diane Barrett, a professor at the University of Hawai’i at Hilo conducted a study in 2013 that investigated preservice elementary teachers' attitudes towards using mathematics
manipulatives. The point of the study was to see if their math anxiety would be reduced from participating in a class that focused on manipulative use in the classroom. The study showed that preservice elementary teachers who participated in the class significantly improved their attitudes towards using mathematics manipulatives with students. Math anxiety was not significantly decreased after a short period of time but did decrease by seven percent after one hands-on mathematics course (Barrett, 2013). The main takeaway from Dr. Barrett’s study was that manipulatives should be incorporated in the mathematics curriculum because they can reduce math anxiety. As Dr. Barrett concluded, further research is important and will be carried out to investigate other matters (Barrett, 2013).

As a result of math anxiety being reduced, the classroom environment is lighter, and happier. Working with manipulatives provides excitement, which means students are less nervous about math. Educators not only want their students to learn the content, but also have fun doing so and manipulatives in the classroom ultimately brings that satisfaction. When students are excited to learn, they have a positive outlook on attaining success (Hand2mind, n.d.). If students have a positive attitude towards learning, it may lead to long term interest, which leads to increased mathematical ability (Moore, 2014). Students will typically invest their time and energy into mathematics if it is a subject that they are interested in. Any amount of time and energy that students devote into mathematics helps improve their skillset, and knowledge to conquer specific concepts. As students build their skillset, manipulatives can reduce math anxiety when they are incorporated into classrooms.

A third benefit is that manipulatives provide the “value of interaction” when used over a long period of time. Manipulatives enable students to interact with mathematics on a deeper level, which produces opportunities for hands-on learning, discussion, and collaboration.
Research shows that using manipulatives on a long-term basis is more beneficial than using them on a short-term basis. Moore (2014) states “When students are exposed to hands-on learning on a weekly rather than a monthly basis, they prove to be 72% of a grade level ahead in mathematics” (Moore, 2014, p. 2). For math manipulatives to be effective, it is essential that they are frequently used in classroom instruction and practice (Moore, 2014). Frequent use is essential because it actively engages students in the doing of mathematics on a regular basis. A manipulative is an additional resource that allows students to touch and feel mathematics at the same time. This would not possible without students interacting with them over a long period of time.

Moreover, using manipulatives long-term allows them to psychologically and physically interact with mathematics. Hand2mind is a company that supports the field of education by providing high quality, hands-on materials to support students. They believe children learn best by doing. In an article written by Hand2mind, they discuss what areas educators have found students most improve in with the long-term use of manipulatives compared to short-term use. Some of the areas include verbalizing mathematical thinking, discussing mathematical ideas and concepts, working collaboratively, and taking ownership of their learning experience (Hand2mind, n.d.). Many other aspects of the learning process, such as the areas mentioned above are fostered when manipulatives are used long-term. This is significant because it demonstrates that manipulatives have multiple purposes as opposed to only teaching content.

Furthermore, manipulatives are a great tool not only to touch, but they also promote discussion and collaboration. They encourage students to verbalize their thoughts, which lead them to practice mathematical language that everyone can identify. This is critical because it gives students the opportunity to have an open discussion that relates back to the concepts they
are learning. It also enables everyone to have a common language, so students can understand what someone else is saying when they use a specific term. The value of interaction also comes with working collaboratively. When students interact with their peers it allows them to share ideas, solve problems together, and ask one another for help. This is specifically helpful with mathematics since concepts can be easily confusing (Hand2mind, n.d.).

A fourth benefit is that manipulatives enhance learning. Learning is enhanced by increased retention, content knowledge, and critical thinking. Studies have shown that there are gains in both long-term and short-term student retention of material when using manipulatives (Boggan et al., 2010). When students are utilizing manipulatives, they tend to retain more knowledge because they are constructing their own thoughts and practicing the same concepts. The increase in retention helps improve memory and understanding as well since students see a manipulative more than once (Kamina & Iyer, 2009). Every time students recognize familiar manipulatives, it sparks students’ preexisting memory and understanding of the concept. This enhances learning as they progress to gain a thorough understanding since concepts build upon one another.

Manipulatives also enhance learning by increasing both students and teachers content knowledge (Kilgo & White, 2014). Students typically understand the content being taught better when they fully retain it. If a student has a good grasp of the material, they tend to show improved retention in content knowledge. On the other hand, students with a low grasp of the material tend to be confused and lost. Teachers content knowledge increases when they form connections between a manipulative and a math concept. Teachers can identify those connections to students, so they can reach their own conclusions.
Critical thinking increases when manipulatives are used in a classroom because the relation between a manipulative and a concept may not be obvious (Kilgo & White, 2014). It may require some out of the box thinking. If the relationship is not immediately understood, they can turn to other strategies to decode the deeper meaning between the two. Those include finding similarities and differences between the manipulative and the concept, listing the characteristics of the manipulative, making sure the concept is fully understood, asking a classmate, or requesting help from the teacher. These strategies can get ideas flowing and start critical thinking. Learning is enhanced in classrooms when retention, content knowledge, and critical thinking are all increased.

The benefits of mathematical manipulatives were explored in this section. Research has shown in many settings that they increase achievement, reduce math anxiety, provide the value of interaction, and enhance learning. These benefits indicate that manipulatives are incredible assets to students. As discussed, studies completed by researchers in the field of mathematical education indicate that manipulatives support students’ academic growth. They can truly make a difference in their mathematical education and thus, are recommended by the National Council of Teachers of Mathematics, teachers, and scholars.

2.3 The Three Principles for Effectiveness

This next section will cover the three principles for maximizing the effectiveness of manipulatives. The principles are as follows: (1) use a manipulative consistently over a long period of time, (2) begin with concrete representations before moving to abstract representations but avoid manipulatives that resemble everyday objects, and (3) explicitly explain the relation between the manipulative and the math concept. These principles stem from the Montessori approach, named after Italian educator Maria Montessori (Laski, et al., 2015). In the early 1900s,
Montessori further advanced the idea that manipulatives are important in education. She
designed several materials to help elementary students discover and learn the basic ideas of
mathematics (Boggan, et al., 2010). The research of the three principles is explored as follows.

The first principle is to use a manipulative consistently, over a long period of time. This
principle is very similar to one of the benefits discussed in the prior section. It is more beneficial
to use a manipulative over a long period of time because it creates consistency for students.
Using the same or similar manipulative leads to a deeper understanding of the relation between
the manipulative and the mathematical concept (Laski, et al., 2015).

Research claims for manipulatives to be effective, children, particularly young ones, need
time to make the relation between the concrete materials and the abstract concepts they represent
(Laski, et al., 2015). Scholar E.J. Sowell conducted one of the first analyses of studies comparing
instruction with and without manipulatives in 1989. The results from the data collected allowed
Sowell to conclude that the benefit of manipulatives depends on how long children are exposed
to them (Sowell, 1989). More recent research explains that children, and even older children are
better able to identify the relation between a concept and a manipulative when they have multiple
opportunities to compare them (Laski, et al., 2015). Laski et al. (2015) states “Using the same or
similar manipulatives to repeatedly solve problems leads to a deeper understanding of the
relation between the physical material and the abstract concept because it allows for an
understanding of the two to co-evolve” (Laski, et al., 2015, p. 2). It is a continuous cycle that
helps students establish a basic understanding of the concept, gain deeper insights into how the
material relates to the concept, and in return leads to better understanding. This cycle is only
possible with consistent prolonged use of the same or similar manipulatives (Laski, et al., 2015).
It may be tempting for a teacher to use a different manipulative every day for the same concept. However, student comprehension will be better if there is consistency between manipulatives (Willingham, 2017). Using the same or a similar manipulative may help students avoid confusion since they would be familiar with it. Students can make more observations and conclusions each time they use the same or a similar manipulative. This concept can be compared to when someone practices a sport or an instrument over a long period. When someone continually practices something, they typically become better and understand more. Progression happens over time, and students need that time to make connections between a math manipulative and a math concept.

The second principle is to begin with highly transparent concrete representations and move to more abstract representations over time but avoid manipulatives that resemble everyday objects. Children will most likely be able to understand the relation between the manipulative and the concept it represents when there is a greater similarity between the two. Research has been conducted on how board games support learning to support this claim. Overall, a board game is a great manipulative to use because it allows students to touch and move pieces around. For example, a number board game with the numbers one through ten in squares arranged in a line leads to better improvement in preschoolers’ understanding of the magnitude of numbers than a game board with the numbers arranged in a circle. This is also known as a student’s mental number line. The linear board game is believed to be better because it is a more transparent reflection of increasing numerical magnitude (Laski, et al., 2015). A second reason a linear board game may be better is because it follows a sequence. For instance, a student would be able to see the numbers one through ten increasing in order starting from one. A student could also rearrange the numbers and put them back in order to show they understand the sequence.
Conversely, a student may find a circular board confusing because it keeps going around and around. The numbers may be in order, but a student could find the one being next to the ten confusing since that would not happen on a linear board.

A principle known as concreteness fading was proposed fifty years ago. It addresses the problem of the likelihood of successful transfer of learning and the perceptual richness of the object used as a manipulative (Willingham, 2017). Although concrete representations of mathematics are important for helping students, research suggests that instruction should progress to abstract representations over time (Laski, et al., 2015). In other words, instruction begins with concrete manipulatives, and students gradually move to more abstract symbols (Willingham, 2017). Concreteness fading will be elaborated on more in depth in a future section.

Educators may think that perceptually rich manipulatives are always the way to go, meaning they are covered with lots of details. For example, educators may use realistic-looking frog counters instead of bland green counters (Willingham, 2017). However, manipulatives that resemble everyday objects or have distracting irrelevant features should be avoided. Early advocates for manipulatives initially believed that concrete objects that resembled everyday objects helped children understand concepts. However, recent research suggests that manipulatives that represent everyday objects may impede learning. Because of this, research suggests that manipulatives should be as basic as possible. Manipulatives without irrelevant features or references to real world objects seem to promote the greatest learning (Laski, et al., 2015).

Everyday objects are only effective when used under certain circumstances. They are okay to use for general education, but in some situations it might not work well. It is not a good idea for younger children to use everyday objects as a manipulative. For example, a study found
that children who solved word problems involving money using real dollar bills and coins made more mistakes than those who solved the same problems using more basic representations such as white pieces of paper with only numbers on them (Laski, et al., 2015). The reason for this may be because real dollar bills and coins have distinctive faces, colors, and textures whereas white pieces of paper all look and feel the same. The white pieces of paper acting as money would be better for children to start off with because the only difference is the amount written. To illustrate, imagine there are three different bills, and one has a 1 on it, one has a 5 on it, and the last one has a 10 on it to represent $1, $5, and $10. With only numbers written, a child could easily see that in total there is $16 by adding 1, 5, and 10 together. This example shows it is valuable to use simple manipulatives.

When a manipulative is interesting to play with or evokes ideas that are irrelevant to mathematics, it distracts the child trying to learn. Another example that shows it is best to use simple manipulatives is using pizzas to learn about fractions. Pizzas have lots of qualities that a teacher would not want to focus on: they are edible, purchasable, and often found at parties. It is not enough for a manipulative to call attention to itself by being perceptually rich. It must call attention to the key feature, and no other features (Willingham, 2017). A distraction could ultimately prevent the student from making the relation between the manipulative and the concept it is supposed to represent. Willingham (2017) states “Manipulatives fail to aid understanding when children focus on a feature that is irrelevant to the analogy” (Willingham, 2017, p. 28). The goal is for the students to make the connection between the two, so basic manipulatives with no distracting features are best. To achieve this goal, children need to direct all their attention on thinking about how the manipulative represents a certain concept (Laski, et al., 2015).
The third principle is to explicitly explain the relation between the manipulative and the math concept. It is unreasonable to expect young children to make the relation between the concrete material and the mathematics concept it represents without explicit guidance (Laski et al., 2015). Willingham (2017) states “Students are more likely to understand the concept the manipulative is meant to convey if that parallel is made explicit to them” (Willingham, 2017, p. 30). Explicit statements about how the manipulative represents the mathematical concept helps direct children’s attention to the relevant features of the manipulative. This promotes learning by allowing children to cognitively focus on the mathematics aspect rather than try to figure out what the relation is.

Different experts have different viewpoints regarding manipulatives representing mathematical concepts. One education expert named Deborah Ball argued strongly against a constructivist view of manipulatives. Ball opposed the idea that children could independently develop an understanding of mathematics concepts by interacting with concrete materials. Ball believed that these experiences could enhance the learning process but emphasized that information and understanding do not travel right through the fingertips (Ball, 1992). It is important to remember that manipulatives are aids for learning mathematics, but a student cannot automatically learn a concept from being handed one. The effects that manipulatives have on students greatly depends on the guidance from the teacher.

The extent to which teachers provide guidance when using manipulatives contributes to differences in student learning and mathematics achievement. Studies have shown that when there are high levels of instructional guidance to support the use of manipulatives, there are greater effects compared to low levels of guidance (Laski, et al., 2015). Actively guiding a student allows room for correcting errors, explaining a relation more in depth, and provides the
opportunity for feedback. It is important to explicitly explain the relation between the manipulative and the concept because children cannot make that inference when only a manipulative is given to them. Children may notice the color, texture, and anything written on the manipulative, but it is unrealistic to expect them to connect the dots without direct guidance from an educator. The guidance provided can be either verbal or non-verbal (Laski, et al., 2015). No matter what type of guidance is given, educators want to make sure they are explicitly explaining the relation between the manipulative and the concept it represents.

This section explored the three principles which make a manipulative effective. To recap, the principles were to: (1) use a manipulative consistently over a long period of time, (2) begin with concrete representations before moving to abstract representations but avoid manipulatives that resemble everyday objects, and (3) explicitly explain the relation between the manipulative and the math concept. Laski et al. (2015) concluded with “Cognitive science research suggests that instruction that follows these principles when using manipulatives is likely to lead to greater mathematics learning than instruction that does not” (Laski, et al., 2015, p. 7). Teachers can use these principles and research to guide their decisions when it comes to using math manipulatives in the classroom, and adjust their practices as necessary.

2.4 Types of Manipulatives

There are a vast variety of manipulatives that can be used during mathematics lessons. Teachers need to know what types of manipulatives are available so they can access them for their own classrooms. Manipulatives should foster children’s concepts in specific areas such as numbers and operations, patterns, geometry, measurement, probability, reasoning, and more (Boggan, et al., 2010). There are certain manipulatives used in each of these areas. Examples of
manipulatives teachers could use in the areas of numbers and operations, geometry, and probability will now be examined.

Base-ten blocks are popular manipulatives that come in a set and are used to teach numbers and operations (Boggan, et al., 2010). The complete set, seen in Figure 3 includes a “cube” (Figure 3 (a)) representing one thousand, a “flat” (Figure 3 (b)) representing one hundred, a “long” (Figure 3 (c)) (sometimes called a “rod”) representing ten, and “units” (Figure 3 (d)) representing ones (sometimes called “minis” or “ones”). Base-ten blocks can also be printed from a template and laminated if purchasing the set is not feasible. They are vastly used in early elementary classrooms and often used to teach place value as well (Loong, 2014). Students frequently mix up the value for each digit in a number, which leads to confusion. Place value mats and base 10 blocks are jointly used to help students overcome this. Figure 3(e) shows a place value mat with the four columns, each representing a different place value with a picture of the respective base 10 block. For example, the column labeled “flats” is where students would place any number of flats to represent the hundreds place. Second grade students using place value mats and base ten blocks together is shown in Figure 3(f). Flats, longs, and units can be seen on the students’ desks as they work on modeling different numbers.
It is vital that students first understand place value before they move on to number and operations. Students can easily mix up digits when performing addition, subtraction, multiplication, and/or division if they do not understand place value. For example, if a teacher asked a student to illustrate the number 341 using the base-ten blocks, the student would need to understand there would be three flats, four longs, and 1 unit. Once students understand place value, they can use the base-ten blocks and place value mat to guide them when solving problems. Base-ten blocks are a great tool to use to teach addition, especially when digits must
be carried over. They are also helpful to teach subtraction when regrouping must be carried out (Loong, 2014). To summarize, a set of base-ten blocks is a valuable resource for students in the classroom to learn numbers and operations.

A geoboard is a manipulative used to assist in the learning of geometry. A geoboard is a board with nails or pegs lined up in rows and columns (Loong, 2014). They can come in different sizes and colors. Geoboards are used by wrapping rubber bands around the nails or pegs to create shapes and learn geometry. *Figure 4(a)* displays yellow geoboards with common shapes. Geoboard templates can also be printed, but the aspect of making shapes with the rubber bands would not be possible. This takes away the point of physically handling a manipulative. Geoboards are helpful when trying to identify simple geometric shapes such as squares, rectangles, circles, and triangles (Boggan, et al., 2010). They also develop problem-solving and teach patterning, perimeter, and symmetry (Goonen & Pittman-Shetler, n.d.).

*Figure 4:* (a) Geoboards and (b) A Student Working with a Geoboard.

(a) *SI Manufacturing. (n.d.)*

(b) *Connell, 2012.*
Other concepts geoboards can illustrate are area, perimeter, and rational number concepts (Goonen & Pittman-Shetler, n.d.). Students often confuse the terms area and perimeter. Sometimes they use the two interchangeably and the units are wrongly attributed. A geoboard is an effective tool to help students overcome misconceptions about area and perimeter as described below. The student in Figure 4(b) is creating shapes with different perimeters that the teacher has written on the board (Connell, 2012). One of the most common misconceptions students have is they think shapes with the same perimeter have the same area. By counting the number of squares or the lengths of the sides, area and perimeter can be differentiated. A geoboard is also an ideal tool to explore how the area of a shape changes with the perimeter. Geoboards promote critical thinking as students investigate the relationships among shapes, area, and perimeter (Loong, 2014). To summarize, geoboards are tremendously useful for students to learn about different concepts in geometry.

Spinners are manipulatives used in probability. Spinners are used to find the probability of landing on a designated area (Boggan, et al., 2010). Seen in Figure 5, a spinner is a circle with an arrow in the middle that gets spun around a central point. They can be split up into any number of parts, and the parts can represent anything. The most common parts that spinners are split into are colors, numbers, words, and pictures. For example, the spinner below in Figure 5(a) has six different colors (blue, green, orange, red, yellow, and purple). If a student spins it once, they have a one out of six (1/6) chance of landing on blue since there is only one blue section out of the six-total number of sections. The spinner in Figure 5(b) differs from 5(a) because it has four different colors (red, blue, green, and yellow), and is divided into eight parts instead of six. Spinners can also be used to test multiple possibilities by spinning more than once.
Figure 5: (a) Six Color Spinner and (b) Four Color Spinner.

(a) Hand2mind Store. (n.d.).
(b) Hand2mind Store. (n.d.).

Dice are manipulatives also used in probability. They are used to find the probability of rolling a certain number or combination of numbers (Boggan, et al., 2010). Seen in Figure 6, dice are cubes with a certain number of dots on each side to represent a number that ranges from one to six. For example, if a student rolls a die and wants to roll a five, they have a one out of six (1/6) chance because a die has one five out of six possible outcomes. They come in a variety of sizes and colors, and are used in many games. There are dice templates that can be printed out and assembled, but may rip easily since paper and tape is not completely sturdy. Most dice are made from plastic, but some are also made from foam.

Figure 6: Dice.

Teacher Created Resources. (n.d.).

Playing cards are a third type of manipulative used in probability. They are used to find the probability of picking a certain number, suit, or combination of the two. They are also used in many different
games. Playing cards come in decks of fifty-two cards each that contain four different suits: spades, hearts, clubs, and diamonds. *Figure 7(a)* shows a ten and ace of spades, a jack of hearts, a queen of clubs, and a king of diamonds. *Figure 7(b)* shows multiple decks of cards with a storage bin that can be used in a classroom. These are inexpensive and easily accessible manipulatives for teachers to have in their classrooms.

*Figure 7*: (a) Playing Cards and (b) Playing Cards with Storage Tote.

(a) *Regal Games. (n.d.)*  
(b) *Hand2mind. (n.d.)*

This section explored different types of manipulatives that are applicable for numbers and operations, geometry, and probability. There are many more manipulatives for those three areas, as well as for different areas. Studies have shown that students using manipulatives in specific areas are more likely to achieve success than students who do not have the opportunity to work with manipulatives. They are found to provide a strong foundation for students to master concepts in specific areas (Hand2mind, n.d.). In closing, teachers can integrate these manipulatives into lessons to assist students on their way to success.
2.5 Four Commonalities of Manipulatives

There are many commonalities among the literature on the correct way to use and implement manipulatives into classrooms. These are important for teachers, so they can make the best decision when it comes to using manipulatives to enhance the learning experience for their students. The top four commonalities were: (1) the manipulative needs to fit the developmental level of the child, (2) allow students free time to explore manipulatives, (3) have clear expectations, and (4) manipulatives do not replace the concepts they are representing. The four commonalities can show and reinforce what may or may not be working and provide valuable guidance. They will be explored in this section.

The first commonality was that the manipulative must fit the developmental level of the child. A child’s developmental level refers to where they are physically, cognitively, and socially and emotionally. Boggan et al. (2010) states “A good manipulative bridges the gap between informal math and formal math” (Boggan et al., 2010, pp. 2-3). For example, informal math could be a class wide discussion prior to a formal lesson. This is where a teacher will informally introduce a concept by asking students if they have any background knowledge, thoughts, or questions. Formal math would include a formal lesson carried out by the teacher with examples. A good manipulative bridges the gap between informal and formal math when it is at an appropriate level for the child to make connections. It must fit the mathematical ability of the child or it is useless (Boggan et al., 2010). Children can become easily bored if the concept is too easy, and in contrast, if it is too hard then the child will want to give up. In a perfect world, each student would have a manipulative that is just right for them and their developmental level. However, this is a challenge because everyone's needs are different, and each student learns differently. To expand on this, cognitive theories that focus on how our mental processes change
over time will be explored (Walker & Bobola, 2017). Particularly, theories from cognitive theorists Jean Piaget and Lev Vygotsky will be discussed.

Jean Piaget (1896-1980) is one of the most influential cognitive theorists in development. Piaget’s cognitive developmental theory is based on the idea that the developing child builds cognitive structures. Cognitive structures are schemes used to understand and respond to the physical environment. He believed the cognitive structure increased with development (Brainerd, 1978). His theory outlines four major stages of cognitive development and the processes by which children progress through them. The first stage is the sensorimotor stage (birth-2 years), where children experience the world primarily through their senses and motor skills. The second stage is the preoperational stage (2-7 years old), where children begin to master the use of symbols or words. During this stage, the child can think of the world symbolically but not yet logically. The third stage is the concrete operational stage (7-11 years old) and is marked by an ability to use logic in understanding the physical world. In the final stage, the formal operational stage (11-15 years old), adolescents learn to think abstractly and use logic in both concrete and abstract ways (Babakr et al., 2019). As students’ progress from stage to stage, it helps teachers understand their developmental level and what is appropriate for that age. However, Piaget’s theory has been criticized for overemphasizing the role that physical maturation plays in cognitive development. Many scholars believed he underestimated the role that culture, and interaction plays in development (Walker & Bobola, 2017).

Lev Vygotsky (1896-1934) is another influential cognitive theorist. Vygotsky’s theory differs from Piaget’s because his sociocultural theory emphasizes the importance of culture and interaction in the development of cognitive abilities. He believed that a person not only has a set of abilities, but also a set of potential abilities that can be realized if given the proper guidance
from others. This is known as scaffolding, where a child can learn cognitive skills within a certain range known as the zone of proximal development (ZPD) through guided participation with a teacher or peer (Walker & Bobola, 2017). He outlined scaffolding as a tool for growth. Scaffolding allows learners to complete small, manageable steps in order to reach the goal (Kurt, 2020).

Vygotsky defined the zone of proximal development as the difference between the current level of cognitive development and the potential level of cognitive development (Kurt, 2020). *Figure 8* shows the three stages that a student’s current knowledge can be divided into, and where the zone of proximal development stands. The first stage is when a student can work independently. The second stage is the zone of proximal development, where scaffolding is used to get students to the independent stage. This is where collaboration takes place with relevant tools to teach students new content. The third stage represents knowledge that is beyond a student’s comprehension, even with guided help. Vygotsky’s approach has been adopted by many educators so they can understand what students can do with the proper guidance rather than assessing students on what they are doing (Walker & Bobola, 2017).

*Figure 8: The Zone of Proximal Development.*

*Williams, 2020.*

An example will now be explored to further illustrate the difference in cognitive abilities between young elementary students and older elementary students. Young elementary students should have individual counters such as two-color counters. Two-color counters are small circular pieces with one red side and one
yellow side (see Figure 9). They can be made from foam or plastic, or cut out on paper from a template. These manipulatives allow students to count, sort, group, add, and subtract (Hand2mind, n.d.). Older elementary students can use a different manipulative such as cuisenaire rods that represent different numbers. Cuisenaire rods are a collection of rectangular rods of ten colors, each color corresponding to a different length (see Figure 10). They can be made from foam, wood, or plastic. They provide students opportunities to introduce, investigate, and reinforce concepts including counting, fractions, addition, and subtraction (Hand2mind, n.d.). The foundations for children’s math development are established early on. Older students should grasp the concept of counting which means they can use a more complex manipulative. The complexity of materials will continue to increase as children’s thinking and understanding of mathematical concepts increase (Boggan et al., 2010). It is important for students to have a strong foundation of basic math skills since concepts build upon one another. If students do not have a strong foundation, they typically fall behind in mathematics.

Figure 9: Two-Color Counters.

Learning Resources Store. (n.d.).

Figure 10: Cuisenaire Rods.

Hand2mind Store. (n.d.).

One study that has shown this rippling effect of students falling behind from a lack of math skills was conducted by mathematics researchers David Geary, Mary Hoard, Lara Nugent, and Drew Bailey (2013). They found that children need to understand that written numerals represent quantities before entering first grade. Children who do not grasp this meaning will fall behind their peers in math achievement, and most of them will not catch up. Their study
indicated that children who start first grade with low number system knowledge are at greater risk for low numeracy scores in later grades. Number system knowledge was found to be more important in predicting a child's likelihood of attaining basic math skills than improvements in math ability that happen in later grades. There are many different factors that contribute to this including poverty, socioeconomic status, and ethnicity. However, students who start to fall behind in math will continue to fall behind. Spotting math deficits early and providing intervention can help improve children’s understanding of mathematics, especially for children who are at risk of poor school performance due to social disadvantages (Baker, 2013).

The second commonality was to give children free time to explore manipulatives. Before explicitly explaining the relation between the manipulative and the concept it represents, children need free time to explore. It is important that teachers recognize that children need opportunities to play around with manipulatives with no specific goal in mind (Boggan et al., 2010). This can be achieved by center rotations. Students get to collaborate with others, as well as build up their comfort with the manipulatives. These opportunities give students the chance to see and feel the manipulative. Giving students time to explore allows them to generate their own questions and answers before a formal lesson. This promotes critical thinking before jumping into a formal lesson because the relation may not be completely obvious. It also provides the teacher the opportunity to lead an engaging, class wide discussion where students can share their observations and conclusions. Students may find that others noticed the same qualities of the manipulative, or they may think “Oh! I never thought of that”. This could lead to an “aha” moment before formally learning a concept.
Another important point is that free time to explore also demonstrates that there are multiple ways to solve a problem. Generating multiple solutions to problems is an essential strategy in mathematics (Boggan et al., 2010). One specific manipulative that shows there are multiple ways to solve a problem are pattern blocks. Seen in Figure 11 (a), they consist of six shapes in six different colors: yellow hexagons, green triangles, orange squares, red trapezoids, blue parallelograms, and tan rhombuses. Pattern blocks are seen in early elementary classrooms and can be made of plastic, wood, or foam. There are also templates available to print out small or big pattern blocks. They are used to learn about geometric shapes and patterns. Figure 11(b) shows two students using pattern blocks along with a worksheet to enhance their understanding of these concepts. They also assist with exploring fractions and percents (Goonen & Pittman-Shetler, n.d.).

Figure 11: (a) Pattern Blocks and (b) Students Working with Pattern Blocks.

(a) Hand2mind Store. (n.d.).
(b) Ewing, 2020.
The third commonality was to have clear expectations for students when using manipulatives. This is key because using manipulatives can present classroom management challenges. Teachers find that manipulatives easily get lost or broken. They can also create messy and noisy classrooms. Goonen & Pittman-Shetler (n.d.), associated with the Institute for the Professional Development of Adult Educators provide two guidelines to help teachers use manipulatives more effectively to prevent classroom management challenges.

The first guideline is to set up a simple storage system to store the manipulatives. Many teachers use clear storage containers or clear plastic bags that seal and store them on shelves, in cupboards, or in the center of desks or tables. The most important thing is that students have easy access to them. It is also important that the bins or bags are labeled. This is important for both the students and the teacher since the manipulatives can easily get mixed up or lost. If they are clearly labeled, it is easy to detect what is missing and what needs to be replaced. Lastly, students must understand the storage system to help keep the classroom organized (Goonen & Pittman-Shetler, n.d.).

The second guideline is to establish clear rules on appropriate manipulative use, so everyone in the class is on the same page. This is important to do before a teacher uses manipulatives for the first time. A chart posted in the classroom outlining the names and rules of the manipulatives will help remind students what is appropriate. To establish clear rules, appropriate uses for learning, handling, storage, distribution and return should be covered, as well as student roles and responsibilities (Goonen & Pittman-Shetler, n.d.). These two guidelines provide a common ground for everyone in the classroom.

One more expectation to emphasize is that manipulatives are not toys. Students sometimes use manipulatives for a reason other than the intended purpose (Goonen & Pittman-
Shetler, n.d.). Students must be taught how to appropriately handle manipulatives (Tichenor, 2008). If teachers do not stress the expectations and classroom rules for manipulatives, students can easily see them as a toy rather than a learning tool. Therefore, it is significant that teachers clearly explain that they are not toys, even when freely exploring. They can accomplish this by telling students that they are acting as mathematicians when working with manipulatives. Teachers can explain that mathematicians do not use toys, and instead emphasize that manipulatives must be used in a safe learning environment. When students are aware that the manipulative has a purpose, it holds them accountable for their actions. Informing a class that a manipulative is not a toy can avoid future problems with students using them inappropriately.

The fourth commonality is that manipulatives do not replace the concepts they are representing. Although manipulatives are concrete objects, understanding how they represent concepts requires abstract thinking. A manipulative still requires a student to think abstractly to understand what it is really representing. Consequently, it is important to recognize that a manipulative is a physical representation of a concept, not the concept itself (Laski et al., 2015). Since a manipulative does not replace the concept it represents, it is significant to explicitly explain the relation. A manipulative is made to further enhance students’ understanding and mathematical skills. For example, a teacher cannot only rely on geoboards to teach area and perimeter. Geoboards are another resource teachers may use to reinforce concepts and practice with, but teachers cannot expect students to fully grasp geometric concepts from only working with one. However, working with a geoboard can further advance a student’s understanding of area and perimeter, and make them more comfortable performing calculations. Therefore, teachers must be aware that manipulatives are there to help, not hinder, but they cannot solely rely on manipulatives to replace their teaching of the specific concept.
This section explored four commonalities found among research that are important points when it comes to using manipulatives in the classroom. The first was to make sure the manipulative fits the developmental level of the child. The second was to allow students to freely explore manipulatives before a formal lesson, so they can make intuitions that they can carry with them throughout the lesson. The third was to have clear expectations when using manipulatives to make classroom management trouble-free as possible. The fourth one was to remember that the manipulative does not replace the concept. These four points are important to keep in mind. They can act like a “guidebook” for teachers, so they can make sure they are providing their students with the best opportunities when using manipulatives.

2.6 Teacher’s Perspective in Action

Many teachers see manipulatives as a valuable learning tool for themselves and their students. Manipulatives tend to make teachers develop self-efficacy and self-confidence, which makes them less anxious about teaching mathematics. Teachers want the best for their students and use manipulatives if and only if they believe they will have a positive effect on students’ learning (Golafshani, 2013). However, not all teachers have this perspective for various reasons. Although much of the literature on teaching and learning with manipulatives has been positive and supportive, there are some drawbacks. Teachers play an immense role in the classroom when it comes to using manipulatives since they determine who, what, when, where, why, and how students use manipulatives. This section will further explore teacher’s views and roles regarding manipulatives.

One perspective teachers have towards manipulative use is that they believe they could use more administerial and financial support. A case study conducted by researchers Puchner, Taylor, O’Donnell, and Fick (2008) analyzed the use of manipulatives in math lessons developed
and taught by four groups of elementary teachers. The study found that in three of four lessons, the manipulative turned into a distraction rather than a tool, and in the fourth lesson manipulative use hindered student learning. Puchner et al. believes this occurred because of the “deeply embedded focus in U.S. mathematics teaching on the procedure and the product” as opposed to focusing on the process to obtain a solution (Puchner et al., 2008, n.p.). In a few lessons, manipulative use became an exercise separated from the solving of the problem. In the second-grade lesson, the students copied the teacher’s example and never attached any meaning to the manipulative. The teacher’s manipulative use and misuse allowed the researchers to conclude “teachers need support making decisions regarding manipulative use, including when and how to use manipulatives to help them and their students think about mathematical ideas more closely” (Puchner et al., 2008, n.p.). This study conveys that teachers do not feel supported enough to use manipulatives in their classroom. To perform their very best teaching, teachers could use more support and guidance to avoid using manipulatives incorrectly. Providing training for teachers to learn about manipulatives and explore them before officially incorporating them into their classrooms would allow them to receive support, guidance, and feedback.

There are several possible reasons why mathematics teachers do not use manipulatives in their lessons. One reason is because of the lack of funds to purchase manipulatives or the shortage of time to develop the hands-on materials (Goonen & Pittman-Shetler, n.d.). Depending on the school district and their budget, teachers may not have the funds to pay for manipulatives. This is unfortunate, since manipulatives are made to help students learn mathematics. As the research has shown, students benefit from having the opportunity to work with manipulatives in meaningful ways. The less funding a school has, the less resources they can provide for their students. If teachers do not have access to purchase manipulatives, they could print or make their
own. Teachers can print out and cut manipulative templates as described in the “Types of Manipulatives” section, or they could make their own with other materials. For example, a teacher could glue ten beans on a popsicle stick to make a rod, and glue ten of them together to make a flat to represent base 10 blocks. However, this is time consuming, which means teachers lose a great chunk of their time due to the lack of funding when they could be doing something else to better benefit their students.

A second reason why teachers may not use manipulatives is because they fear a breakdown in classroom management (Goonen & Pittman-Shetler, n.d.). Using manipulatives works nicely in a cooperative learning setting. However, as previously discussed, lessons using manipulatives may be noisier and messier. Manipulatives also require a great deal of planning and organizing (Goonen & Pittman-Shetler, n.d.). These aspects of manipulative use may push teachers away from the chance of using them in their lessons. Teachers may cut out the idea of using manipulatives when they think about the prep work and cleanup. Although classroom management may be a potential hurdle when using manipulatives, there are tactics teachers can use to diminish the chance. Possibilities include planning and organizing during prep periods, having a teacher’s aid to help set up and monitor students, setting up when students are at lunch and recess, having a student teacher help, or having students who complete their work early help. These possibilities make it more manageable for a teacher to use manipulatives without a fear of breakdown in classroom management.

To conclude, the way teachers view the use of manipulatives impacts whether they are used in the classroom. If teachers believe manipulatives will benefit students and are confident enough to use them, then they will most likely appear in the classroom. However, if teachers do not receive the support, guidance, and funds to implement manipulatives into their lessons then
they will most likely not appear. A classroom teacher plays the most prominent role with manipulative use, and they need to be sure they are communicating concepts effectively. Hence, if a teacher wishes to use manipulatives in their classroom, they must receive support.

A mathematics manipulative is a concrete tool used by many teachers in their classroom to assist students in learning mathematics. When manipulatives are used correctly and effectively, there are numerous benefits for the learner. Teachers play the most dominant role in manipulative use. Their influence and how they maximize their effectiveness impacts the opportunities students have to work with them. Manipulatives lead to student engagement, which will be discussed next.

3. STUDENT ENGAGEMENT

3.1 Definition

It is important to discuss student engagement since manipulatives are tools used to engage students. Student engagement is the heart of the teaching and learning process, and one of the most critical components to student success (Pedler et al., 2020). It is one of the most important issues facing educators today because teachers must cater to each student’s needs. Without engagement, students tend to withdraw physically, emotionally, and cognitively from the learning process (Conner, 2016). Student engagement is a term used to describe an individual’s interest and enthusiasm for school, which impacts their academic performance and behavior. Students vary in their level of engagement as they progress through school. They can also change within specific aspects of engagement. For example, a student may demonstrate high levels of engagement for reading but demonstrate low levels of engagement for math (Olson & Peterson, 2015). This section will explore student engagement, the three types of student
engagement, differentiated learning, and concrete, representational, and abstract (CRA) instruction.

3.2 Why Student Engagement Matters

There are many reasons why student engagement is the heart of the teaching and learning process. One of the main reasons is because student engagement truly matters, and it lays the foundation for student success. Student engagement is essential to student success because it is predictive of higher grades, achievement on test scores, academic aspirations, attendance, persistence, and school completion (Fredricks et al., 2019). Engagement is an essential part of a meaningful education because it drives students to do their best work. As a result, students stay in school because they are engaged (Knight, 2019). Engagement allows educators to identify factors that influence a student’s engagement and/or disengagement. These factors are helpful to educators when constructing strategies to reduce the likelihood of school failure (Finn & Zimmer, 2012). After specific factors are recognized, educators can adjust their teaching practices to create a more engaging environment where students want to invest in their education. Part of the appeal of studying student engagement is that it can provide a richer picture of how students think, act, and feel in school. Studying engagement is invaluable to help educators understand how to effectively implement prevention and intervention efforts. From research on student engagement, educators can pinpoint students who show varying patterns of cognitive, emotional, and behavioral engagement (Fredricks et al., 2019). The reasons for engagement and disengagement, as well as the results will be explored.

3.3 Reasons and Results of Disengagement

Although engagement is essential for student success, many students fail to become fully
engaged, and others begin to disengage at some point during their schooling. Disengagement is a gradual process driven by the interaction between the student and the environment (Olson & Peterson, 2015). Disengaged students are those who do not participate in class and school activities, do not become cognitively involved in their learning, do not fully develop or maintain a sense of school belongingness, and/or display inappropriate behavior. Research has identified several factors associated with disengagement. They can be organized into two distinct categories: status risk factors and educational risk factors (Finn & Zimmer, 2012). Status risk factors are sociodemographic characteristics that are difficult or impossible to alter through school-based interventions. These include socioeconomic status (SES), race and ethnicity, whether English is spoken in the home, family structure, and early pregnancy/parenthood. Educational risk factors are educational outcomes that interfere with later academic achievement. These include low grades and test scores, in-grade retention, misbehavior, and absenteeism (Finn & Zimmer, 2012; Fredricks et al., 2019; Olson & Peterson, 2015). These factors put students at risk for school failure and dropout.

The result of a disengaged student can take many forms. A student may lack participation, act out and disrupt class, and withdraw themselves from the learning process (Fredricks et al., 2019). It ultimately leads to academic problems, misbehavior, and in some cases dropout. Dropping out is an outcome of earlier school experiences that becomes an obstacle to further schooling. It is an educational risk factor (Finn & Zimmer, 2012). When a student is no longer engaged, their motivation declines to complete and persist their schooling. Transitional periods (e.g., transition from middle school to high school) are critical periods for increased disengagement and dropout. If a student relates to any of the factors previously discussed, they are at increased risk during and beyond the transitional period. Sixty percent of
students who dropped out of high school could have been predicted with early warning signs at
the middle school level (Olson & Peterson, 2015). Thus, it is important to distinguish the
particular needs of students to decrease the risk of dropout. It is useful for educators to know
both status and educational risk factors so they can identify signs of disengagement, and guide
students to become more engaged. Their intervention is crucial to students’ success, and in time
can prevent failure and dropout.

3.4 Reasons and Results of Engagement

Educators hope that students will be engaged in their learning, rather than disengaged.
Engagement is associated with academic achievement and lower risk behavior, which lead to
higher engagement over time (Fredricks et al., 2019). Engaged students are those who participate
in class and school activities, become cognitively involved in their learning, fully develop and
maintain a sense of school belongingness, and behave accordingly. Engagement is closely aligned
with student success. Students who are engaged with school are more likely to learn, find the
experience rewarding, graduate, and pursue higher education (Abla & Fraumeni, 2019). They
also tend to be less disruptive because they stay on track and execute their tasks. Students who
are more engaged cognitively, emotionally, and/or behaviorally may have higher levels of
academic achievement. Cognitive, emotional, and behavioral engagement are types of
engagement that will be explored in the next section. However, they lead to academic
achievement because students are emotionally responsive, have positive relationships with their
classmates and teachers, are motivated to learn, and want to persevere when they are engaged
(Fung et al., 2018). Connecting with these factors enables students to be present in their learning,
and often lowers the risk for troubling behavior, especially as they shift into middle and
secondary school.
A second result of engagement is that it is a protective factor that can buffer students from risky behaviors. Students who are engaged in school are less likely to fall victim to potential adolescent troubles. Student engagement protects against behaviors that are not part of the school environment. These behaviors may include substance abuse, risky sexual behaviors, and delinquency (Olson & Peterson, 2015). Student engagement is also correlated with mental health and students’ well-being. Engagement leads to lower levels of depression, suicidal thoughts and behaviors, and higher life satisfaction (Fredricks et al., 2019). Avoidance of these behaviors and good mental health help keep students from dropping out.

One way to engage students in mathematics is through the use of manipulatives. Lessons that students find engaging are those that include physical activity and active learning situations involving concrete materials (Attard, 2012). Using manipulatives increases engagement when students are actively touching and moving tangible objects that represent mathematical concepts. Two techniques that lead to increased engagement by using manipulatives are long-term manipulative use and CRA instruction. CRA instruction will be further explored in an upcoming section. When both techniques are effectively used, students stay focused on the task at hand, and a learning environment that encourages engagement is created (Cockett & Kilgour, 2015).

3.5 The Three Types of Student Engagement

Student engagement is a multifaceted construct with three distinct types: cognitive, emotional (also known as affective), and behavioral (Olson & Peterson, 2015). It is important to note that even though each type has its own definition, they are interdependent and share overlapping concepts (Conner, 2016). The first type of student engagement is cognitive engagement. It is focused on the student's internal investment in the learning process, which incorporates the inner psychological qualities of the students (Nguyen et al., 2018). In other
words, it mostly occurs “inside” the student rather than outside (Knight, 2019). Cognitive engagement can also be described as “engagement of the mind.” It can range from simple memorization, to the use of learning strategies that promote deep understanding (Conner, 2016). Factors relating to cognitive engagement are crucial to know to help a student succeed.

One aspect of cognitive engagement is cognitive control. Cognitive control is the ability to coordinate thoughts and actions in relation with goals and is often required in everyday life. It also serves for higher purposes including planning and reasoning. As cognitive growth occurs, students are capable of engagement for longer periods (Conner, 2016). Over the course of time, students learn how to control their thoughts and actions. This progression allows them to engage for longer periods since they can control themselves. When students are more engaged, they are generally more invested. They recognize the value of learning and are more willing to put in the effort necessary to comprehend complex ideas and master difficult skills (Conner, 2016).

There is a framework for cognitive engagement consisting of quantitative and qualitative factors. Attention, effort, and time on task indicate the quantity of cognitive engagement. Strategy use, absorption, and curiosity indicate its quality. The quantitative factors are more outwardly visible indicators that show mental energy is being put toward learning. Although these factors may include behaviors, they are behaviors reflecting the presence or absence of mental energy focused on learning. Contrarily, qualitative factors indicate the quality of engagement. These factors convey what piques students’ curiosity, and how they integrate that into their learning to solidify understanding (Halverson & Graham, 2019). Connecting these quantitative and qualitative factors together provide educators insight into what traits will lead to cognitive engagement.
Cognitive engagement is a primary factor in how students go about completing instructional related activities. The enthusiasm of the teacher can encourage students’ interest and their readiness and willingness to learn (Pedler et al., 2020). Students’ minds often wander when lessons are not engaging, and in return they become bored. Students fail to see the relevance in lessons that are dull because they cannot connect them to anything relevant in their lives (Conner, 2016). Almost all educators have heard the famous question “When are we going to use this?” or “I’m never going to use this” from students during math lessons. Teachers have an influence on student’s learning and interest in the classroom. Instructional methods and resources that promote student interest are highly effective and encouraged to use. It is how topics are presented, not simply the topics themselves, that create interest for the student. Thus, instruction that provides support for student autonomy (understanding, choice, relevance) and effective use of participation are critical to provide an effective learning environment for cognitive engagement (Pedler et al., 2020).

Cognitive engagement is also concerned with students’ perseverance. Perseverance will be explored further in the next main section. Cognitively engaged students have higher levels of academic achievement because they are more interested in learning (Fung et al., 2018). When students want to learn, they want to be better and do better. They are driven to overcome any difficulties in their learning when they are cognitively invested. This leads to a higher intensity of thinking as students work out their instructional activities through their own thought processes. Overall, a student is bound to be cognitively engaged when they are genuinely interested and view their task as relevant.

The second type of student engagement is emotional engagement. Emotional engagement includes the experience, feelings, attitudes, and perceptions a student has towards school. It
specifically focuses on students’ sense of belonging, interest, willingness to learn, and general sense of liking school (Olson & Peterson, 2015). Emotional engagement can be described as “engagement of the heart” because it emphasizes students’ feelings of connection or disconnection of their school (Conner, 2016). Three ways students stay emotionally engaged throughout school are by displaying their emotions, gaining an identity, and building student teacher relationships.

One way students stay emotionally engaged is by displaying their emotions. Students exhibit emotional engagement when they employ positive or negative emotional responses to learning activities (Conner, 2016). Positive emotional engagement includes enjoyment, happiness, and confidence. These emotions lead to an increase of student engagement. They help learners to see relatedness, and to process material in a more integrated and flexible fashion (Halverson & Graham, 2019). The hope is that positive emotions will foster motivation, engagement, and learning (Conner, 2016). Negative emotional engagement includes boredom, frustration, sadness, tiredness, and anxiety. Negative emotions lead to a lack of interest and understanding (Halverson & Graham, 2019). Conner (2016) states “Teachers know that engaged students are usually happier than disconnected ones who have isolated tasks to do, and research confirms that engagement activates more of the pleasure structures in the brain than do tasks of simple memorization” (Conner, 2016, p. 16). When students are engaged in their task, they experience more pleasant emotions. They are prone to getting caught up in their work, thus, making them less bored. When students are disengaged, they are usually not interested in their work. This makes them feel emotionally disconnected. It is important for students to emotionally respond to their learning so teachers can assess their level of engagement.
A second way students stay emotionally engaged is by gaining an identity throughout their time at school. Studies in this domain also investigate the students’ feelings of belonging and value through the students’ identification with their school (Nguyen et al., 2018). Some conceptualize emotional engagement as a feeling of identification. Identification is a sense of belonging, feeling like an important part of the school body and finding value in the school experience. From this perspective, the student identifies with a place or activity that may represent certain expectations, values, beliefs, and practices (Conner, 2016). However, researchers Jennifer Fredricks, Phyllis Blumenfeld and Alison Paris (2004) argued against this view. They argued that identification is usually general and not differentiated by domain or activity. Their reasoning for this is that it may be unclear where students are directing their positive emotions. They could be directed toward academic content, their friends, or the teacher (Fredricks et al., 2004). Although scholars may have different perspectives about identification, the overall concept of identity is still the same.

Identification is likely to occur over time when students are active participants during classroom and school-wide activities. It also occurs when students receive praise, rewards, and acknowledgement. An internalized sense of belonging can serve to perpetuate the student's active participation in class and school (Conner, 2016). Students need to feel affirmed and be assured they are valued (Camp, 2011). As students grow, they typically become more comfortable within the classroom and overall school atmosphere as they learn the ropes of what a structured educational environment looks and feels like. Being an active participant brings the feeling “I do belong here” and shows that they play an important role in their school. Praise, rewards, and acknowledgement always help to reinforce a student’s belongingness. No matter whom it comes
from, it reminds the student that they are noticed within their school. When students feel connected to their school, they are more likely to display positive emotions.

A third way students stay emotionally engaged is by building student teacher relationships. Human connection is the most important trait of humankind. Without it, none of us would learn, survive, or work. Therefore, the relationship between a student and a teacher is the foundation for learning. Teacher expectations can be very powerful and influence students’ attitudes and actions toward school. Teachers who show personal involvement with students show them that they are respected and cared about. Feelings of respect motivate and engage students toward positive productivity and academic achievement. Teachers who care about their students are remembered, create change, stimulate growth, and are more likely to succeed at teaching their students. As a result, students become more curious, self-directed, and empathetic (Camp, 2011). A teacher can learn a great deal about a student in a variety of ways. Some strategies include greeting students at the door, regular check-ins, and focusing on solutions instead of problems. Once a relationship is established, it is also necessary to maintain and restore it, so the bond does not break over the course of the school year (Terada, 2019). Conclusively, teachers need to challenge students to be the best they can be, while guiding and supporting them along the way.

Conner (2016) states “When children experience teachers as warm and affectionate [and] providing clear expectations, children feel happier and more enthusiastic in class” (Conner, 2016, p. 16). Students look for these connections because they learn best from people they like and get along with. They express a desire for engagement through relationships and want to be acknowledged as an important part of their school community. However, there is a limitation for older students’ emotional engagement compared to younger students. Conner (2016) found that
middle school and high school students rated their own emotional engagement within school significantly lower than the elementary students. The degree of emotional engagement depends on the quality, affiliation, and depth of the relationships. Elementary students are mostly in a self-contained classroom, meaning the students keep the same teacher for all the core subjects. The students have other teachers for physical education, music, art, and library, but most of each school day is spent with the same teacher and peers. This allows more time to build personal and emotional relationships between students and the classroom teacher. Contrarily, middle school and high school students change classes and have multiple teachers for core subjects. Those students have limited opportunities to build the emotional relationships since less time is spent with each teacher daily (Conner, 2016). Although middle and high schoolers have the limitation of time, teachers can implement strategies as previously discussed to get to know their students.

The third type of student engagement is behavioral engagement. Behavioral engagement encompasses the idea of active participation and involvement in academic and social activities. It is considered crucial for the achievement of positive academic outcomes and preventing dropout (Attard, 2012; Conner, 2016). Behavioral engagement is thought of as “engagement in the life of the school” because it captures the ways in which students interact within the school setting (Conner, 2016). When students are behaviorally engaged, they are doing what they are supposed to be doing.

It is often challenging for teachers to manage student learning and behavior at the same time. Research suggests that the majority of poor classroom behavior would not arise if students’ needs were successfully catered for in their learning environment. The enforcement of rules is an important classroom practice to manage classroom behavior. Teachers can reduce behavioral issues by modeling and reinforcing appropriate behavior. Teachers can also reduce behavioral
issues by maintaining high expectations while being calm, fair, and consistent with students (Pedler et al., 2020). The advantage of behavioral engagement is that it is measurable by classroom observations (Knight, 2019). There are four main components to behavioral engagement: student conduct, student participation, student interest, and active learning.

This domain focuses on student conduct in class, student participation in school-related activities, and student interest in their academics. Students can exhibit positive behaviors or negative behaviors. For example, positive behaviors include obeying school rules, attending class regularly, adhering to norms, and avoiding disruptive behaviors. These are indicators of higher student engagement. Negative behaviors include being disruptive in the classroom and disobeying an administrator. These are indicators of disengagement. Moreover, school-related activities consist of student participation within the classroom and the school. Examples would include classroom activities, clubs, and sports. Student's participation links to their motivation to be part of the school. In addition, students’ interest in their academic tasks refer to their willingness to engage in classroom activities, as well as overcoming challenging material. This includes persistence, focus, asking questions, and contributing to class discussions (Conner, 2016; Nguyen et al., 2018). The distinction between the three areas is helpful for organizing and distinguishing specific behaviors.

One specific aspect of behavioral engagement is active learning. Active learning is any instructional method, other than a lecture, that engages students in learning. Teaching is not pouring knowledge into a student’s head (Hyun et al., 2017). Students do not learn by just sitting in classes listening to teachers, memorizing assignments, and spitting out answers. Instead, active learning techniques (ALTs) encourage students to reflect on the materials, analyze, process, and prepare to discuss. All these strategies improve critical thinking skills and positively impact
students’ ability to retain and understand new material. Incorporating active learning techniques into the classroom creates opportunities for learning instead of allowing students to sit passively and learn by absorbing information (Camacho & Legare, 2015). There are many different active learning techniques including group discussion, journal writing, case study, and role play. These strategies create opportunities for student participation, engagement, and exploration of peer perspectives, which further develops learning. They also provide students the chance to self-assess and revise their own thinking process to better problem-solve. Active learning motivates students to remain engaged in the learning process.

3.6 Differentiated Learning

Differentiation is important to mention because students enter classrooms with various abilities, learning styles, and personalities (Gentry et al., 2013). Varying abilities can include English language learners, special education students, and students with attention disorders such as attention deficit hyperactivity disorder (ADHD) and attention deficit disorder (ADD). Differentiated instruction is a way of recognizing and teaching according to different student talents and learning styles (Morgan, 2014). Students of the same age differ in their readiness to learn, they move forward at different rates of speed, and they acquire different patterns of learning and thinking. Therefore, teachers must determine what students are ready for and to what degree. Effective teachers recognize differences among their students and plan academic enrichment opportunities to accommodate their differences. Implementing differentiation allows teachers to reach learners through a variety of methods and activities. Students are more successful when they are taught based on their own readiness levels, interests, and learning profiles (Camp, 2011). When instruction is tailored to meet students’ needs, it allows them to demonstrate what they know, understand, and are capable of doing (Gentry et al., 2013). The
overall hope is to maximize each student’s growth by meeting each student where he or she is (Logan, 2011).

3.7 Concrete, Representational, Abstract (CRA) Instruction

Concrete, representational, and abstract (CRA) instruction is an effective teaching method that began with educational theorist Jerome Bruner. Jerome Bruner (1915-2016) was a psychologist at Harvard University and formed the foundation for CRA instruction (Gibbs, 2014). Bruner developed three modes of representations: enactive, iconic, and symbolic. Modes of representation are the way in which information or knowledge are stored and encoded in memory (McLeod, 2019). In the enactive form, students develop mathematical concepts by manipulating concrete objectives. In the iconic form, students learn to represent a mathematical concept in a graphical or pictorial form. In the symbolic form, students learn to represent a concept with an abstract model or symbols (Kim, 2020). One of Bruner’s most famous phrases and thoughts on learning states, “We begin with the hypothesis that any subject can be taught effectively in some intellectually honest form to any child at any stage of development” (Bruner, 1960, p. 33). Bruner believed that a child of any age is capable of understanding complex information. This idea is referred to as the “spiral curriculum” where students revisit a topic several times throughout their schooling. The complexity of the topic is increased each time, so the new learning is connected to the old learning (Gibbs, 2014). The adaptation of Bruner’s model, CRA instruction will be explored.

Teachers matter in fostering engagement because they have the ability to shape the immediate learning environment. They have the most control over learning environments, content, and pedagogy. They play the key role in providing a stimulating and positive learning environment (Pedler et al., 2020). To gain a deep understanding of mathematical ideas, students
CRA instruction, also known as concreteness fading, is a process for teaching and learning mathematical concepts. It supports students in gradually fading from manipulating concrete materials to representing abstract concepts in mathematical learning (Kim, 2020). Concepts are developed through a progression of three distinct stages: concrete, representational, and abstract (Hurrell, 2018). Dr. Julie Jones, director of teacher education at Converse College, and second grade teacher Margaret Tiller state, “Children understand math better when they can see and touch it” (Jones & Tiller, 2017, p. 18). CRA instruction brings math to life and allows children to learn by doing, which strengthens active learning. It provides the foundation for using manipulatives to engage students in mathematical learning because they are used in the first stage. It also allows students to make connections from one stage of the process to the next with the purpose of gaining a thorough understanding (Jones & Tiller, 2017). CRA instruction is nationally recommended as an instructional strategy for the development of mathematical concepts. It is mostly used at the elementary level but can be used for other grade levels as well. It is also effective for students who have difficulties with mathematics and for students who have learning disabilities (Leong et al., 2015). All three stages will be further elaborated on.

The first stage of CRA instruction is the concrete stage and starts with manipulation of concrete materials such as base ten blocks, geoboards, and pattern blocks. The concrete stage has been the theoretical basis for the use of manipulatives in learning mathematics (Leong et al., 2015). This is where manipulatives are used in purposeful activity through senses of sight, touch and/or sound (Hand2mind, n.d.; Hurrell, 2018). Being introduced to and working with
manipulatives is a critical first step to develop students’ understanding of mathematical concepts. Manipulatives help students learn by allowing them to move from concrete experiences to abstract reasoning (Boggan et al., 2010). Instead of delving straight into an abstract concept, the use of hands-on interactions with manipulatives allows students to begin instruction on the same level, no matter what their mathematical ability may be (Jones & Tiller, 2017). The use of concrete materials is a strategy teachers can use to make a lesson more engaging through providing a hands-on experience. Manipulatives give students an effective way to represent their thinking and enables teachers to determine if there any misconceptions in the student’s understanding (Hurrell, 2018). When students manipulate objects, they are taking the first step toward understanding math processes and procedures. This can help students connect ideas and integrate their knowledge so they can perform certain math skills and truly understand concepts at the abstract level (Boggan et al., 2010; Jones & Tiller, 2017).

An example of this stage in action for young elementary students, such as first and second graders would be using two-color counters to calculate a simple addition problem such as 7+3. Other tools including place value mats and ten frames may be used in this stage as well to serve as another aid for the student (Jones & Tiller, 2017). A ten frame, seen in Figure 12, is a rectangle split into two rows, with ten equal spaces. If a student was using a ten frame, they would place seven red counters in seven spaces, and three yellow counters in the remaining three spaces. This would show the student that all ten spaces are occupied.

*Figure 12: Ten Frames.*

*Learning Advantage. (n.d.)*
An example of this stage in action for older elementary students, such as fifth and sixth graders would be using algebra tiles to factor polynomials. A polynomial is a series of mathematical expressions joined by addition and subtraction. Factoring polynomials is the inverse procedure of finding the product of polynomials. Students find different factors that can be multiplied together to form the given polynomial (Cabahug, 2012).

Algebra tiles are manipulatives that are used to introduce new algebraic concepts to students (Cabahug, 2012). There are three basic types of algebra tiles: a large square tile which represents \( x^2 \), a rectangular tile representing \( x \), and a small square tile representing a 1. In Figure 13, the \( x^2 \) tile on the left is green, the \( x \) tile in the middle is red, and the 1 tile is on the right in yellow.

In the concrete stage, a teacher would write an expression on the board such as \( 2x+8 \) for the students to represent using the algebra tiles. In this case, a student would use two “\( x \)” tiles, and eight “1” tiles to represent the expression (Cabahug, 2012). The process moves students to the representational stage next.

![Figure 13: Algebra Tiles. Edwards, 2010.](image)

The second stage of CRA instruction is the representational stage. Students transition to this stage after they have shown conceptual understanding towards the mathematical task using concrete manipulatives. Teachers use the same procedures of model, guide, and practice during
this phase (Hurrell, 2018). This moves students to using pictures such as tallies, dots, and stamps to replace the manipulative used in the previous stage (Jones & Tiller, 2017). Other visual representations that can be used in this stage are images, graphs, diagrams, and tables (Cabahug, 2012). In the representational stage, students should be comfortable with solving problems using visual representations. Students should also be able to demonstrate how they can both visualize and communicate the concept at a pictorial level (Hand2mind, n.d.). The purpose of this stage is to gradually move students away from relying on manipulatives and building up their skillset to reach the final stage of the process.

Continuing the prior example from the concrete stage, young students would be able to draw the counters with a type of writing utensil such as a marker in the representational stage. The seven red counters would become seven red circles drawn by the student. The three yellow counters would become three yellow circles drawn by the student. The ten frames can still be used in this stage, but students should be able to draw it themselves without a template. Students should also be able to recognize that there are still ten circles drawn altogether without the manipulative.

Older students move on to the representational stage by rearranging the algebra tiles into a rectangle. The total area of the tiles represents the product, and its length and width represent the factors. As seen in Figure 14, the tiles are rearranged into a table with two rows and five columns. The two “x” tiles create the factor of two for the width and the eight “1” tiles are split into four “1’s” in each row.

Figure 14: Representation Stage of 2x+8.

Cabahug, 2012, p. 94.
This creates the factor of $x+4$ for the length because there is one “$x$” and four “1’s” in each row (Cabahug, 2012). Students’ progress to the final stage once they have mastered using visual representations to represent manipulatives and show conceptual understanding.

The third, and final stage of CRA instruction is the abstract stage. Students transition to this stage by using abstract symbols with their drawing to explain their reasoning. Students at the abstract level no longer need pictures or manipulatives to solve the problem (Jones & Tiller, 2017). This transition is the most challenging aspect of the CRA sequence because students are required to generalize their understanding in succinct ways (Hurrell, 2018). Piaget suggested that children begin to understand symbols and abstract concepts only after experiencing the ideas on a concrete level (Moore, 2014). This is because students are more likely to perform math skills accurately, and truly understand concepts at the abstract level when they can first develop a concrete understanding. Instead of manipulatives or pictures, mathematical symbols (numerals, operation signs, etc.) are adopted and used to express the concept. In this stage, students demonstrate their understanding by using the language of mathematics (Hand2mind, n.d.). To conclude the examples, to add seven and three, a student would write $7+3=10$. This shows they are at the abstract level by writing numerals, operation signs (+), and the equal (=) sign to add the two numbers together. An older student would write $2x+8=2(x+4)$ using numbers and mathematical symbols to show they understand factoring. The abstract stage concludes the CRA process.

Teachers play a crucial role in helping students move through the three stages of CRA instruction successfully (Hand2mind, n.d.). Their role is to facilitate the student's progression through the three stages and ensure they are understanding the knowledge learnt (Cabahug, 2012). Furman (2017) states “When a teacher creates an environment in which the student
engages in doing, undergoing, and reflecting, the experience guides the student toward making sense” (Furman, 2017, p. 72). Students should partake in each stage of CRA instruction, so they arrive at a deep understanding of mathematical concepts. The reflection piece is often forgotten about but is needed to fully comprehend the concept. Students should not only actively engage with the manipulatives during the concrete stage, but should also have a chance to reflect on their experience after each stage. Teachers can prompt students with questions such as “How was the experience?”, “What did you learn?”, “What did you notice?”, and “What surprised you?” to generate and encourage mathematical discussion. These questions can show each student’s progression as they move to a new stage in CRA instruction. This provides insight for teachers as to whether students are above grade level, below grade level, or in the middle. All this information is useful for lesson planning and catering to different individual needs. Students can follow the three stages in chronological order to achieve success when CRA instruction is implemented into mathematics lessons.

The three stages of the CRA process are concrete, representational, and abstract. A summary of each stage can be seen in Table 1 below. The concrete stage utilizes manipulatives. The representational stage uses visual representations instead of manipulatives. Lastly, the abstract stage uses mathematical symbols such as numerals and operation signs. CRA instruction is encouraged to use in classrooms because when it is used in an appropriate manner, students are likely to advance in mathematical understanding. By following the sequence of CRA instruction, teachers assist students on their progression to each stage, and ensure they achieve the goal of demonstrating understanding at the abstract level.
This section explored the importance of student engagement and how it affects students’ educational experiences. Student engagement is a student’s interest and enthusiasm for school. The three types are cognitive, emotional, and behavioral.

Cognitive engagement draws on the idea of investment and a student’s willingness to learn. Emotional engagement encompasses positive and negative reactions to teachers, classmates, academics, and school. Behavioral engagement draws on the idea of participation, and a student’s involvement both inside and outside of the classroom. Differentiated instruction is when instruction is tailored to meet each student’s needs to provide the best classroom experience. CRA instruction allows students to begin their mathematical learning with manipulatives and build up their skillset to the abstract level. These three types of engagement, differentiated instruction, and CRA instruction create a multidimensional construct so students can experience a positive and well-rounded education.

4. PROBLEM-SOLVING

4.1 Definition

Manipulatives lead to student engagement by increasing success and comprehension of mathematics concepts. Problem-solving is the result of engagement, where students gain
confidence in their ability to solve challenging tasks. Manipulatives serve as a tool for students to utilize when challenging tasks arise. Problem-solving is a key skill to have in today’s world. It is a principle instructional strategy used to fully engage students in important mathematical learning at all grade levels (Kelly, 2006; DiMatteo & Lester, 2010). The main goal in teaching mathematical problem-solving is for students to develop an ability to solve real-life problems and apply mathematics to real-life situations (Gurat, 2018). It is an important part of mathematics instruction because students need a robust mathematical background to effectively solve problems. This is especially useful for students who pursue careers in fields such as business, medicine, engineering, and architecture because they are math heavy (DiMatteo & Lester, 2010). Problem-solving develops critical thinking skills, reasoning, deep understanding of concepts, and cooperative group work (Gurat, 2018). These skills are desired for all individuals because they prepare them for real problems and life situations that require effort and thought (Rigelman, 2007). What exactly causes a student to problem-solve? Challenging tasks lead to problem-solving, and students’ abilities to persist plays a major role in the problem-solving process.

4.2 Challenging Tasks

Students often need to problem-solve because they are faced with a challenging task to overcome. A challenge teachers face is introducing challenging tasks in a way that makes them accessible, rather than daunting (Cheeseman et al., 2016). Sullivan et al. (2011) identified the requirements of a challenging mathematical task. They require students to:

- Plan their approach, especially sequencing more than one step;
- Process multiple pieces of information, with an expectation that they make connections and see mathematical concepts in new ways;
- Choose their own strategies, goals, and level of accessing the tasks;
• Spend time on the task and record their thinking;
• Explain their strategies and justify their thinking to the teacher and other students;
• Extend their knowledge and thinking in new ways (Sullivan et al., 2011, p. 34).

Teachers can take these requirements and foster a classroom environment that promotes productive struggle by providing students with challenging tasks. Providing challenging tasks is sometimes discomforting for teachers because they see students struggling and want to intervene too soon. However, intervening too soon is detrimental to the learning taking place in the discomfort. Productive struggle enables students to work through challenging problems that they have not seen before. The tasks that teachers provide to challenge students are designed in a way that are accessible to all students. The expectation is that everyone will persist when solving challenging mathematical tasks. This expectation is because teachers anticipate that students will engage with productive struggle when exploring a task without their input. Students tend to struggle without guidance because they are attempting to solve a problem that they do not know how to solve yet. The task is also designed in a way where the answer is not obvious, and students are required to process multiple pieces of information to solve the problem (Livy et al., 2018).

Teachers can use manipulatives to assist their students through a challenging task. However, there is a decrease of manipulative usage in math curricula because teachers are over relying on mental math. An over reliance on mental math makes it harder for students to conceptually understand a concept. Since mental math is done in a student’s head, it may be easier to make a mistake due to the lack of visual representations of the problem. Without conceptual understanding, mathematical tasks are harder to learn and can result in barriers for the students. This results in increased math anxiety. It is not easy to determine the causes of math
anxiety, and in fact many scholars disagree about the possible causes because there are so many (Das & Das, 2013). Developing increased persistence is one way to help diminish students’ math anxiety (Furner, 2017). Since manipulatives can increase a student’s persistence, they may experience a decrease in math anxiety.

4.3 Persistence

To effectively overcome productive struggle, students must be able to persist throughout the problem-solving process. Persistence, also known as perseverance, is productive struggle in the moment while facing mathematical obstacles, setbacks, or discouragements. In other words, it is the ability to “stick with it” to reach a solution (DiNapoli, 2019). Persistence is also described as a category of student actions that include concentrating, applying themselves, believing that they can succeed, and making an effort to learn (Roche & Clarke, 2014). Enduring uncertainty and overcoming obstacles are recognized as key practices that support learning. Students make meaning through productive struggle as they engage with mathematical ideas that are within reach, but not yet well formed (DiNapoli, 2019). Since math requires an overt amount of persistence for every student, manipulatives are tools to not shy away from when it gets hard. The amount of persistence varies by degree and by topic area for each student, but manipulatives open a door for accessibility and get students to persist through those hard times.

One major influencer on the topic of perseverance is Angela Duckworth. Duckworth was a former math teacher and is currently a psychology professor and author. Her famous book *Grit: The Power of Passion and Perseverance* highlights exactly what perseverance is and why it matters. Duckworth defines grit as a combination of passion and perseverance for very long-term goals. Those who are highly successful are hardworking, determined, and have direction. Duckworth says that we need to make students grittier, and one way of doing that is by using
manipulatives in mathematics to persevere through problem-solving since the answer is not always obvious. Through conducting multiple studies, Duckworth found that what matters most in life is grit. Grit was the number one characteristic that emerged consistently as a significant predictor of success in her studies (Duckworth, 2016).

Grit is exceptionally important for students because students who have grit are more likely to graduate. It is even more important for those who are at risk for dropping out. Duckworth found that the best way to develop grit and a strong work ethic in children is to foster a growth mindset. A growth mindset is an idea developed by Carol Dweck, a psychology professor at Stanford University. A growth mindset is the belief that the ability to learn is not fixed but can change with effort. When children learn about how a growth mindset effects the brain, they tend to persevere because they do not see failure as a permanent condition (Duckworth, 2013). Having a strong work ethic and a growth mindset will produce the right attitude for success. Problem-solving with manipulatives allows students to reach a long-term goal and attain success by persevering. The teacher’s role in fostering persistence and what actions they can apply in the classroom that support students in the problem-solving process will be explored further.

4.4 The Teacher’s Role in Fostering Persistence

Teachers play a significant role in the problem-solving process by encouraging students to persist. The National Council of Teachers of Mathematics (NCTM) recommends that teachers need to provide opportunities for productive struggle. Productive struggle is essential to learning mathematics with understanding (Livy et al., 2018). Providing consistent opportunities for students to engage with unfamiliar mathematical tasks encourages problem-solving strategies. Consistent opportunities also encourages student metacognition, which facilitates more
independent thinking. Besides providing opportunities for students to engage in these challenging tasks, explicitly supporting them will nurture their persistence. When students are supported properly, persistence can improve over time (DiNapoli, 2019).

After providing challenging tasks, teachers can take a step back and allow for students to go through productive struggle. Teachers can emphasize that struggle is important because it stimulates brain growth and helps develop a growth mindset. A student with a growth mindset embraces challenges, persists, implements effort with every task, and learns from their mistakes (Suh et al., 2011). Teachers are encouraged to maintain challenge by encouraging students to persist, reinforce the connections between persistence and learning, and highlight student persistence when they see it (Ingram et al., 2016). Finding the balance between allowing students to struggle with mathematics and supporting their understanding is not easy. However, when teachers talk less, teach less, hold back from telling students how to solve problems, and give more time to think about and work on the task, it provides room for growth. As one student wrote in a written reflection: “We do learn more when we’re confused and we’ve got to work our way out of it” (Roche & Clarke, 2014, p. 7). As this student has stated, students do learn when they are confused even if it does not seem like it. Being confused is productive in mathematics, and means students need to critically think about the challenging task.

It is vital for teachers to allow time for confusion without giving away the answer too quickly. To ensure a strong mathematical focus within the classroom, students need to think for themselves. For this reason, teachers need to avoid the kind of relationships that encourage dependency. Students may be highly engaged in a problem when they seek help, but they may also be over-dependent on the teacher. Therefore, teachers are encouraged to allow students to enter a zone of confusion. Ingram et al. (2016) defines a zone of confusion as “A state of
confusion before a pathway for solving the problem has been identified” (Ingram et al., 2016).

The zone of confusion is key because an important part of maintaining the challenge of a task is for students to make decisions about how to approach the problem. In return, students learn to appreciate and evaluate different thinking when they undergo decision making since the mathematical reasoning is in their hands (Cheeseman et al., 2016). There are three strategies teachers can use to help students navigate through the zone of confusion. The strategies are: asking questions, using materials, and reflecting on their time in the zone of confusion (Ingram et al., 2016).

The first strategy teachers can implement to help students persist through the zone of confusion is asking questions. Students are sometimes “locked” in the zone of confusion for some time because of their limited knowledge. Teachers can consider what questions they might pose, and when they might do so instead of immediately relieving the students when they get stuck (Ingram et al., 2016). Their role is to motivate and clarify the problem rather than show students how to solve the problem (Cheeseman et al., 2016). Students find it helpful when teachers help them do the mathematics themselves by giving them advice, encourage them to keep going, and not give them the answer. The second strategy is using materials. Sometimes students need concrete materials to support their learning, and this is where manipulatives can be used. Manipulatives such as pattern blocks can jumpstart the thinking process for students so they can eventually form a conclusion. The third strategy is getting students to reflect on their time in the zone of confusion. The reflection piece is often overlooked or forgotten about, but equally important. Teachers can develop students’ reflection skills by first establishing that being persistent in the zone of confusion was a normal part of mathematics, and again remind them that
persistence is a key trait to have (Ingram et al., 2016). Allocating time for reflections allows students to reflect on their thinking and mistakes.

It is important for students to persist until they are certain that they have fully solved the problem and feel confident with their answer. Students can develop persistence as they work and discover multiple answers that satisfy the problem. Furthermore, when students acknowledge that there are different problem-solving strategies, they have more choices to work with. Some students are better at verbalizing their strategy and thinking processes, but others may use strategies such as tables or organized lists to show their thought process (Suh et al., 2011). Overall, students are encouraged to approach a problem believing that they can succeed and recognize that learning mathematics takes effort.

4.5 Polya’s Method

It is critical for students to persist through problem-solving, and for teachers to foster students’ persistence. Once students are met with a challenging task, they need to have an effective method to work through it. One strategy is the Polya method, named after George Polya. Manipulatives and Polya’s method can be used simultaneously to conquer a problem. In 1957, Polya succeeded in applying the mathematic model for solving problems (In’am, 2014). This model is called the Polya method and is a four-step process for teaching and assessing problem-solving in mathematics. The four steps are: understanding the problem, devising a plan, carrying out the plan, and looking back. Some teachers may use other variations of Polya’s method such as define the problem, develop a plan, implement the plan, and evaluate or plan, do, act, and check (Ortiz, 2016). The Polya method has been implemented to solve mathematical problems at elementary, secondary, and university levels. This method guides students to make steps in solving problems, and complete the result by looking back (In’am, 2014). Each step of
Polya’s method will be explored along with examples of manipulative implementation at elementary, secondary, and university levels.

The first step of the Polya method is to understand the problem. This is the step where students engage with the problem before attempting to solve it (Ortiz, 2016). Understanding the problem encompasses looking for information on the problem. By looking for information, students take a step forward to understand the problem to be solved (In’am, 2014). Tohir et al. (2020) lists some questions to guide students in understanding the problem:

- What is the unknown?
- What are the data?
- What is the condition?
- Is the condition sufficient, insufficient, redundant, or contradictory to determine the unknown? (Tohir et al., 2020, p. 1736).

The second step of the Polya method is to devise a plan. After identifying the problem, the next step is to plan appropriate strategies to solve the problem. Two strategies students may use are making an appropriate diagram and making an analogy with similar problems. It is important to note that different problems require different approaches, and not one strategy will be used to solve all problems (In’am, 2014). The student should try to find connections between the data and the unknown (Ortiz, 2016). Tohir et al. (2020) and Ortiz (2016) lists some questions to guide students in devising a plan:

- Have I seen the same, similar, or related problem before?
- Do I know a theorem that could be useful? (Tohir et al. 2020, p. 1736).
- Could I restate the problem?
• Do I know the vocabulary in the problem? (Ortiz, 2016, p. 6).

The third step of the Polya method is to carry out the plan. Understanding a problem, and then making a plan to solve it is not useful if the plan is not implemented. Therefore, the next step is to implement the strategy from the prior step to solve the problem (In’am, 2014). Students should check each step of the solution plan to make sure they clearly see each step is correct (Ortiz, 2016). Tohir et al. (2020) lists the following guiding questions for carrying out the plan:

• Can I see that the step is correct?
• Can I prove that it is correct? (Tohir et al., 2020, p. 1736).

The fourth and final step of the Polya method is to look back. It is important for students to examine and review the solution they obtain at the end of problem-solving. This step might be done by using the answer through inverse method. By using the inverse method, students can see whether the answer is appropriate with the expected solution. For example, a student may look back on a multiplication problem by using division (In’am, 2014). The following guiding questions are from Tohir et al. (2020) for looking back:

• Can I check the result/argument?
• Can I derive the result differently?
• Can I use the result, or the method, for some other problem? (Tohir et al., 2020, p. 1736).

A challenging task will often stump students to the point where they no longer want to engage in problem-solving. Polya’s method provides students a set of steps to follow, so they can obtain a solution to a challenging task. Understanding the problem, devising a plan, carrying out the plan, and looking back help students think like mathematicians. Students must ask themselves if they truly feel comfortable at any given step before moving on to the next one. As
mentioned, the Polya method is implemented at elementary, secondary, and university levels to engage students in problem-solving. One different example of interconnecting manipulatives with Polya’s method will be discussed at each level.

4.5.1 **Polya's method in action: elementary level.** Polya’s method can be interconnected with manipulatives to solve mathematics problems at all different levels. The following are examples of challenging tasks that utilize problem-solving strategies while using manipulatives. At the elementary level, students can use Polya’s method while using base ten blocks to learn numbers, operations, and place value. In Figure 15 to the right, a cube is in the upper left corner, flats are in the middle, rods are on the far right, and units are under the cube on the bottom left. For example, students may use base ten blocks to add 45 and 31.

*Figure 15: Base 10 Blocks.* Learning Resources Store. (n.d.).

In the first step, understand the problem, students must gather information before attempting to solve. At this step, students may recognize that it is an addition problem, and the two numbers are two digits each. Students could ask themselves:

- What is the problem asking?
- What operation do I need to perform?
- What is the place value of each number?

Students must know what the problem is asking, and what operation they need to use to solve the problem correctly. If students do not use the operation of addition, then the wrong solution will be obtained. It is also important for students to understand place value. To add correctly, they
must understand the first number occupies the tens place and the second number occupies the ones place.

In the second step, devise a plan, students must plan a strategy to use to solve the problem. Students may plan out which base ten blocks they will use and how many they will need. They may also think of using a place value mat if need be. Students could ask themselves:

- What specific base ten blocks do I need? (cubes, flats, longs, units)
- How many blocks do I need of each kind?

To correctly carry out this step, students need to decide what types of base ten blocks they are going to use to solve the problem. To represent 45, four longs and five units should be used. For 31, three longs and one unit should be used.

In the third stage, carry out the plan, students solve the problem and check each step of their plan. At this step, students would count the number of longs and the number of units to receive the final answer. In other words, adding four and three longs equals seven longs to represent seventy, and adding five units and one unit equals six units to represent six. This gives the final answer of 76. Students could ask themselves:

- Did I represent 45 and 31 correctly?
- Will I need extra blocks for carrying digits over?
- Can I prove that 76 is the right answer?

Students can check that they represented the two numbers correctly by making sure that they used the right blocks for the respective place value. Students should also pay attention to any digits that get carried over. In this question, there are no digits being carried over, but sometimes
if the answer has ten longs, that becomes a flat. Lastly, students can prove that 76 is the right answer by performing the problem by hand.

In the fourth stage, look back, students review their solution. Students may do this in a variety of ways. One way is by using the answer through inverse method, where the student would solve the problem by using subtraction. Students could ask themselves:

- Am I certain that I got the correct answer?
- What other ways can I check my answer?
- Can I explain my process and solution to a teacher and/or classmate?
- Can I use what I learned in another problem?

Students can check their answer by using the inverse method. If they subtract 31 from 76, they should receive 45, and if 45 is subtracted from 76, 31 should be the solution. Furthermore, students can take the extra step by checking their work on a calculator. After they are positive that they have the correct solution, students can explain their thinking.

4.5.2 Polya’s method in action: secondary level. At the secondary level, students can use Polya’s method while using dice to learn theoretical probability. Theoretical probability is the number of favorable outcomes divided by the total number of outcomes. For example, students may use dice to decipher the probability of rolling an even number. Figure 16 on the right shows the faces of a die. Having a physical die or a layout of a die may help students distinguish the number of favorable outcomes and total number of outcomes.  

Figure 16: Dice Faces.

Teachers Pay Teachers. (n.d.)
In the first step, students must understand that the question is asking about probability. Furthermore, the question specifically asking about rolling an even number may stand out as another important piece of information. Students could ask themselves:

- How many numbers are on the dice in total?
- How many even numbers are on the dice in total?
- What do I know about theoretical probability?

There are many different types of probability problems in mathematics, thus it is critical to know what information is needed to solve a theoretical probability problem. This specific problem is asking about dice, so students need to know how many numbers there are in total. The total amount of numbers would give them the total number of outcomes. To answer the question about rolling an even number, students must understand that there are three even numbers (2, 4, 6) to represent the number of favorable outcomes.

In the second step, students must strategize how to solve the problem. They may think about how they will exactly use the dice to find the probability of rolling an even number. Students could ask themselves:

- Do I understand the concept of theoretical probability?
- What other probability questions can I connect this to?
- What information about probability can I use to help me?

Once students know what information they need to solve the problem, they must understand it. Students must understand what the terms “favorable outcomes” and “total number of outcomes” mean and how it applies to the problem. They may also try and connect it to other probability
problems that ask a similar question. Students can ask themselves if they have seen a similar question with the same or different manipulative.

In the third step, students solve the problem. The first step would be to find the number of favorable outcomes, in this case the number of even numbers on a die (2, 4, 6). The second step would be to find the total number of outcomes, in this case the total amount of numbers on a die (1, 2, 3, 4, 5, 6). Since there are three favorable outcomes and six total outcomes, the probability of rolling an even number is 3/6 or ½. ½ can also be written as a decimal (0.5) or a percent (50%). Students could ask themselves:

- How can I prove that ½ is the correct answer?
- Did I count the number of favorable outcomes correctly?
- Did I count the total number of outcomes correctly?

These questions help students check their work, so they are confident with their answer. Once students solve, they must check that each step is correct. One way to accomplish this is by going back and making sure each step was calculated correctly. A second way is by thinking of ways to prove that the answer is correct.

In the fourth step, students look back on their solution. Reflecting on their solution is important so students are certain and confident that they obtained the correct answer. Students could ask themselves:

- How do I know ½ is the correct answer?
- What is my reasoning?
- What other probability problems can I apply this method to?
It is important for students to prove that their answer is correct, but even more important for them to understand why it is correct. To think like mathematicians, students must be able to explain their reasoning to the teacher and students. Students can further advance their reasoning skills by taking what they have learned and applying it to other problems.

4.5.3 **Polya’s method in action: university level.** At the university level, students can use Polya’s method while using solids of revolutions to find the volume of a solid figure. A solid of revolution is a technique to obtain the volume of a region rotated about an axis of revolution. On the top of *Figure 17* seen to the left is a continuous non-negative function on the interval \([a, b]\). On the bottom of the figure, the volume of the solid is formed by revolving the region bounded by the curve \(y=f(x)\) about the \(x\)-axis. *Figure 18* is a 3D printed manipulative of \(x\) rotated around the \(y\)-axis that was created as part of an Adrian Tinsley Program Grant at Bridgewater State University in the summer of 2020.

*Figure 17:* \(Y=f(x)\) on the Interval \([a, b]\).

*Figure 18:* \(X\) Rotated Around the \(Y\)-axis.

*Math24. (n.d.)*

*Monte & Ames, 2020.*
In the first step, students must understand what the problem is truly asking. They would need to have background knowledge of multiple topics to find the volume of a solid figure. Some topics that are connected to solids of revolution include operations, axes, functions, and integration. Students could ask themselves:

- What does the solid of revolution look like?
- What is the correct axis to rotate the region?
- What am I calculating?
- What would I like to obtain?
- What is the given information? (given function, regions, axis of revolution to be rotated around)
- How is the solid of revolution related to integration?

Students can use manipulatives to visualize the solid of revolution they are working with. It is important that students understand the information given to them, so they understand what exactly needs to be calculated. It is also important to understand that the application of integration is the application of the solid of revolution.

In the second step, students must take the information they are given and create a plan to solve the problem. They need to understand the information and their plan to solve the problem effectively, and correctly. Students could ask themselves:

- What is the formula to calculate the volume of solids of revolutions?
- Do I have conceptual knowledge and technical skills on how to evaluate the integral?

It is vital that students know the correct formula to calculate the volume of a solid of revolution. If students use the wrong formula, they will obtain the wrong answer and each step will be
incorrect. Another crucial aspect for devising a plan is to have conceptual knowledge and skill to evaluate the integral. This step is where students can evaluate their comfort with integration and think about reviewing material before moving on to solve.

In the third step, students take their plan and implement it to solve. At this step, students must have a strong foundation of integration skills to carry out the problem. Students could ask themselves:

- Are the steps to the solution easy to follow?
- Is the representation of the steps and solution visually appealing for the reader?

Since integration contains many steps, it is crucial that students display their work in an organized manner. If a student’s work is not legible, it is difficult for the reader to follow the steps and interpret the student’s thinking process. Therefore, it is best for students to arrange and present their work, so it is not messy, but visually appealing to the reader.

In the fourth step, students reflect on their problem-solving and solution. There are multiple details that can be reviewed after the final solution is calculated. Students could ask themselves:

- Did I obtain a positive or negative result?
- Did I keep the units at the end?
- Is there a different approach to calculating the volume of solids of revolutions?

These help students decipher if they have obtained the wrong or right solution. For instance, a negative solution would mean that the student had an error because volume cannot be negative. In this case, a negative solution would be noticed and then the student could go back and fix their calculations. Another aspect is making sure that the final answer is cubed because that is always
the units for volume. Lastly, a student may think about how they can calculate the volume using a different approach.

5. CONCLUSION

After completing a full literature review, more work must be done so students can reach their highest potential. Work may include future research by surveying teachers, studying virtual manipulatives, and creating professional development workshops. In addition to future research, some recommendations based off of the literature will be given, so teachers can adjust their mathematical teaching to create real-world problem solvers.

The next step is to perform studies in classrooms to see if the literature accurately reflects the effectiveness of manipulatives. Studies can be done by observing and surveying teachers at all grade levels. The following questions could be asked to survey teachers:

- Are manipulatives used in the classroom? Why or why not?
- If manipulatives are used, which ones specifically?
- When are manipulatives used?
- How are manipulatives used? (whole class lesson, small group work, center rotations)
- Is there a significant difference in test scores between using manipulatives and not using manipulatives?

By observing and surveying teachers, it would give the surveyor a sense of the structure of manipulative use pertaining to the classroom. The surveyor may seek to answer who, what, when, where, why, and how manipulatives are used in the classroom.

A second next step is to study virtual manipulatives and their benefits. Virtual manipulatives are relatively new, and becoming increasingly popular within schools because of
easier access to technology. Many of the manipulatives that were explored throughout this thesis have been turned into virtual manipulatives, such as base 10 blocks and pattern blocks. Since virtual manipulatives is a new area in mathematics, it warrants further research to explore their effectiveness. A surveyor could ask teachers the same questions listed above to see if teachers use physical manipulatives, virtual manipulatives, or both.

The literature shows that teachers play the most important role in the classroom when it comes to providing effective lessons that will engage students. The literature also shows that teachers need help and support to provide those types of lessons. One way teachers and teacher candidates can gain that support is by having professional development workshops on manipulative use in the classroom. Professional development workshops would benefit teachers and teacher candidates by helping them build up their skillset, confidence, and comfortableness while teaching with manipulatives.

The literature on manipulatives, student engagement, and problem-solving has shown that the approaches used to teach math could use some alterations. The literature recommends that teachers do the following:

- Re-evaluate how mathematics is taught.
- Incorporate manipulatives into classrooms.
- Use differentiated learning and CRA instruction to engage students in mathematics.
- Use the Polya method to engage students in problem-solving.
- Explore how manipulatives help other instructional methodologies such as project-based learning (PBL) and problem-based learning (PBL).
Mathematics is often seen as a subject that is all about memorizing facts, formulas, and procedures. Teachers need to take a step back and re-evaluate how that is hurting each student’s attitude and success towards mathematics. Manipulatives provide a wide variety of opportunities for students to engage and participate in their learning. By considering the recommendations listed above, teachers can help improve students’ attitudes towards mathematics and opportunities for success.

While a thesis is based on others research and it is common practice to not make the work personal, I would be remiss to not share how this process has helped me become a stronger teacher in the future. As a future teacher, this process has highlighted for my future practice that I want to use manipulatives in my classroom. The literature has shown me that manipulatives are valuable tools to use as a resource in the classroom. When they are implemented into lessons correctly, there are numerous benefits to them. I believe that manipulatives truly engage students by providing a visual, hands-on experience of a mathematical concept. I also believe manipulatives provide students with the skills needed to successfully problem-solve. Using manipulatives allow students to think for themselves, which empowers them to create their own solution pathways in mathematics. My hope is that more educators will see the benefits of manipulatives, and with some help and support, will adjust their teaching practices to incorporate them into their own classrooms.

If teachers want to see students’ attitudes change from “I hate math!” to “I love math!” then the re-evaluation of mathematical teaching must be viewed as an urgent matter and top priority. Students do not hate math. What they hate is being confused, intimidated, and embarrassed by math. With some re-evaluation on the education system’s part, teachers can
provide students with endless learning opportunities in mathematics when manipulatives and constructive instructional methodologies are used jointly.

6. REFERENCES


Figures

Figure 1: One Demonstration Clock and 24 Student Clocks.

https://www.amazon.com/Learning-Resources-Classroom-Clock-Kit/dp/B000F8T9AO/ref=sr_1_15?dchild=1&keywords=clock+manipulative&qid=1617643337&sr=8-15

Figure 2: An Ohaus School Balance.


Figure 3: (a) A Cube, (b) A Flat, (c) A long, (d) A Unit, (e) Place Value Mat, and (f) Students Using Base Ten Blocks with Place Value Mats.


https://www.amazon.com/ETA-hand2mind-Plastic-Blocks-Flats/dp/B01J6B1TGM/ref=sr_1_5?dchild=1&keywords=blue+plastic+base+ten+blocks+flat+only&qid=1614718664&sr=8-5

https://www.amazon.com/ETA-hand2mind-Blue-Plastic-Blocks/dp/B01J6BPP0S/ref=sr_1_7?dchild=1&keywords=blue+plastic+base+ten+blocks+flat+only&qid=1614718664&sr=8-7

https://www.amazon.com/ETA-hand2mind-Blue-Plastic-Units/dp/B01J6C3NOM/ref=sr_1_8?dchild=1&keywords=blue+plastic+base+ten+blocks+flat+only&qid=1614718664&sr=8-8
(e) Hand2mind Store. (n.d.). Paper Base Ten Place Value Mat. [Digital Image]. Amazon. https://www.amazon.com/ETA-hand2mind-Paper-Place-Package/dp/B01MT8H1/U/ref=sr_1_1_sspa?dchild=1&keywords=place+value+mat&qid=1614192365&sr=8-1-spons&psc=1&spLa=ZW5jcnlwdGVkUXVhbGlmaWVyPUEzRTkRXQlRYVzM2JmVuY3J5cHRlZEKrPUExMDE1NjI4MUyTUJVOkNCUDEwUCZlbmNvXFB0ZWRBZEkPUEwNDYwMzA3MVUxNzVVTkJHNkM2USZ3aWRnZXROYW1lPXNwX2F0ZiZhY3Rp249Y2xpY2tSZWRpcmVjdCZkb05vdExvZ0NsaWNrPXVydmU=


Figure 4: (a) Geoboards and (b) A Student Working with a Geoboard.


Figure 5: (a) Six Color Spinner and (b) Four Color Spinner.

(a) Hand2mind Store. (n.d.). Six Color Spinner. [Digital Image]. Amazon. https://www.amazon.com/ETA-hand2mind-Color-Spinners-Plastic/dp/B01N1Z9XZ3/ref=sr_1_15_sspa?dchild=1&keywords=a+math+spinner&qid=1614727630&sr=8-1-spons&psc=1&spLa=ZW5jcnlwdGVkUXVhbGlmaWVyPUEzRTM5WlZTQzhEMkJtJmVuY3J5cHRlZEKrPUExMDE1NjI4MUyTUJVOkNCUDEwUCZlbmNvXFB0ZWRBZEkPUEwNDYwMzA3MVUxNzVVTkJHNkM2USZ3aWRnZXROYW1lPXNwX2F0ZiZhY3Rp249Y2xpY2tSZWRpcmVjdCZkb05vdExvZ0NsaWNrPXVydmU=


Figure 6: Dice.

Figure 7: (a) Playing Cards and (b) Playing Cards with Storage Tote.


(b) Hand2mind. (n.d.). Playing Cards with Storage Tote. [Digital Image]. Amazon.  

Figure 8: The Zone of Proximal Development.


Figure 9: Two-Color Counters.

https://www.amazon.com/Learning-Resources-Color-Counters-Yellow/dp/B0017D9BDG/ref=sr_1_3?dchild=1&keywords=two+colored+counters+two+piece&qid=1615084265&sr=8-3

Figure 10: Cuisenaire Rods.

https://www.amazon.com/hand2mind-hand2mind-Foam-Cuisenaire-Introductory-Pieces/dp/B07KB41MJL/ref=sr_1_4_sspa?crid=3MX466U1G0E6U&dchild=1&keywords=cuisenaire+rods&qid=1615084265

Figure 11: (a) Pattern Blocks and (b) Students Working with Pattern Blocks.

https://www.hand2mind.com/glossary-of-hands-on-manipulatives/pattern-blocks

https://ewing45.wordpress.com/2020/03/09/hands-on-math/

Figure 12: Ten Frames.
https://www.amazon.com/Learning-Advantage-Giant-Magnetic-Frames/dp/B00YW63V3Q/ref=sr_1_13?&keywords=ten+frame+manipulatives&qid=1615164587&sr=8-13

Figure 13: Algebra Tiles.


Figure 14: Representation Stage of 2x+8.


Figure 15: Base 10 Blocks.

https://www.amazon.com/Learning-Resources-Giant-Magnetic-Base/dp/B004DJZ7H6/ref=asc_df_B004DJZ7H6/?tag=hyprod-20&linkCode=df0&hvadid=167141005679&hvpos=&hvnetw=g&hvrand=1343839923060886879&hvpos=&hvnetw=g&hvrand=1343839923060886879&hvpos=&hvnetw=g&hvrand=1343839923060886879&hvpos=&hvnetw=g&hvrand=1343839923060886879&hvdev=c&hvlocint=&hvlocphy=1018290&hvtargid=pla-313288134706&psc=1

Figure 16: Dice Faces.

Teachers Pay Teachers. (n.d.) Dice Faces. [Digital Image].

Figure 17: Y=f(x) on the Interval [a, b].

https://www.math24.net/volume-solid-of-revolution-disks-washers

Figure 18: X Rotated Around the Y-axis.


Tables:

Table 1: CRA Instruction Stages.