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Tyler Martell
Bridgewater State University

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A Survey of Dark Matter Candidates and Relations to Particle Physics and General Relativity

Tyler Martell

Submitted in Partial Completion of the
Requirements for Departmental Honors in Physics

Bridgewater State University

May 14, 2019

Dr. Edward Deveney, Thesis Advisor
Dr. Thomas Kling, Committee Member
Dr. Jeffrey Williams, Committee Member

List of Abbreviations

SM	**S**tandard **M**odel of Particle Physics
GR	**G**eneral **R**elativity
QFT	**Q**uantum **F**ield **T**heory
SUSY	**Su**per**sy**mmetry
CP	**C**harge-**P**arity
SR	**S**pecial **R**elativity
MOND	**MO**dified **N**ewtonian **D**ynamics
MACHO	**M**assive **A**strophysical **C**ompact **H**alo **O**bject
EL-EoM	**E**uler **L**agrange **E**quations **o**f **M**otion
EMF	**E**lectro**m**agnetic **F**orce
WNF	**W**eak **N**uclear **F**orce
SNF	**S**trong **N**uclear **F**orce
QED	**Q**uantum **E**lectro**d**ynamics
QCD	**Q**uantum **C**hromo**d**ynamics
WIMP	**W**eakly **I**nteracting **M**assive **P**article
GUT	**G**rand **U**nified **T**heory
DM	**D**ark **M**atter

Contents

1 Abstract

Cosmological observations of certain galaxies suggest that the amount of known, measured matter accounted for by the Standard Model of Particle Physics (SM) in those systems is insufficient to account for galactic mechanics (orbital paths and velocities). These observations have led physicists to believe that either General Relativity (GR) is incomplete, or that there exist new sources of yet-to-be detected matter, that may or may not be consistent with SM, called dark matter. Neither GR nor the SM can alone be considered complete theories of the universe for GR is not quantum mechanical and the SM does not include GR. Modifications to GR are actively being considered especially since it is manifestly classical and not quantum mechanical. Extensions to the SM are also actively being researched both to include gravity and to fill voids in the SM such as the Strong CP (Charge-Parity) Violation Problem as well as the Hierarchy Problem. However, at the galactic scale, GR's successes have led many other research groups to conclude that the answer is instead dark matter. Leading theories suggest that dark matter could be a particle of some kind that would have to be heavy and weakly interacting (which is yet to be observed). Different theories of what this particle could be have been proposed as extensions to the SM introducing new particles such as the Axion and the Weakly Interacting Massive Particle (WIMP).

My thesis will focus on surveying the leading dark matter particle candidate searches mainly by discussing research done to reproduce velocity distribution curves of galaxies and galaxy clusters and highlighting important characteristics of Quantum Field Theory (QFT) and the SM studied this semester, both studied this semester. Overall, my thesis allows me to explore the leading theoretical physics theories and study some of the most compelling researchers of our time working on this problem, with impact from particle physics and the SM to galactic physics and GR - all of which I hope to pursue in graduate school following Bridgewater State University.

2 Introduction

The SM is a relativistic quantum-gauge field theory that accounts for three of the four known forces in nature - electromagnetic, weak, and strong (excluding gravity) - as well as all the particles and matter currently observed. GR, on the other hand, is a completely classical theory, with no quantum mechanics involved, describing everything from the mechanics of matter to the curvature of space-time as a result of energy (and matter), that completely describes all astrophysical phenomena known that are measured and compared to theory. A complete theory of nature, must unify both GR and SM. Physicists have yet to come up with such a Grand Unified Theory (GUT) that would eloquently sew together the ideas of gravity into the fully quantum mechanical formalisms of the SM.

Using standard GR of the known, measured galactic SM matter, discrepancies appear. With increasing data, accuracy, and precision it became clear that either GR was incomplete or that particle physicists needed a new unseen and difficult-to-measure particle whose gravitational influence was at the root of the galactic mechanics (i.e. Some new physics beyond the SM). Not only was a new particle needed, if we were to preserve GR, but this potential form of matter had to outweigh ordinary matter about six to one and strikingly incorporate no electromagnetic interactions because all electromagnetically interacting particles emit detectable (visible, or in some other region of the spectrum) light energy and we cannot see this proposed particle (missing mass) at all. Thus the search has been on for the dark, massive matter (dubbed dark matter) that accounts for an estimated, what would now be, about a quarter of the matter in the entire universe and whose gravitational influence using standard GR could change the way we study mechanics on a galactic scale.

Several modifications to GR have been suggested and my thesis will include some of the most compelling theories. However, a broader community and number of research programs have focused on running experiments with the assumption of dark matter being a particle. Theoretical evidence for different dark matter candidates comes from the SM as well as theories beyond the Standard Model. The emphasis of my thesis will be to look into those new dark matter particle theories and what years of experiments have to offer both in favor and opposition of the different dark matter candidates. As an outcome of my work, three things could happen: 1) There may or may not be a preference of a specific dark matter candidate, 2) Some candidates may possibly be ruled out, or 3) We may gravitate toward a new form of GR. The overall outcome will be a deeper understanding of the theoretical frameworks based on evidence of GR and the SM with this new hunt bridging the two and possibly toward a future unification of relativity and quantum mechanics.

3 The Big Picture

If physicists are to move forward and discover the nature of dark matter, it is important to learn about its history. How did we come to know that there is dark matter in the first place? Getting an idea of how we first discovered dark matter's existence might give some insight as to what the dark matter actually is.

3.1 Experimental Evidence

Evidence for dark matter began popping up in the 1930's with Dutch astronomer J.H. Oort, who observed unusually high star velocity distributions in the Milky Way Galaxy. Oort calculated their velocities using their Doppler shifts and had realized that these stars should be moving quick enough to overcome gravity and escape from orbit[1]! This forced him to conclude that there must be more galactic mass unaccounted for.

An informative tool in astronomy is the idea of photometry, which takes advantage of the luminosity of an object in space to predict its mass using a ratio of the two properties ($\frac{M}{L}$). Using the $\frac{M}{L}$ ratio of the sun,

$$\frac{M_\odot}{L_\odot} = 1 \tag{1}$$

we can predict masses of cosmic objects as a direct proportional mass of the sun (mass of the sun times some scalar). Now, we don't use the sun's real $\frac{M}{L}$ ratio because that would be unnecessarily messy to work with. It is much simpler to use the previous equation. This technique was utilized in the 1930's by Swiss astronomer Fritz Zwicky, who expanded upon these velocity distribution studies of Oort to galaxies within a cluster (specifically, the Coma Cluster). He used the Virial Theorem of classical mechanics to come up with a cluster mass.

$$\langle T \rangle = -\frac{1}{2}\langle U \rangle \tag{2}$$

$\langle T \rangle$ is the average kinetic energy and $\langle U \rangle$ is the average potential energy. Zwicky found that the mass he calculated using the traditional $\frac{M}{L}$ ratio was only 2% of the value he calculated using the Virial Theorem. So, like Oort, Zwicky inferred some non-luminous (dark) mass to be present within his galaxy cluster that he had not accounted for[1]. It is important to distinguish luminous mass from 'dark' mass in order to understand the significance of what Zwicky and Oort found. Luminous mass refers to mass that interacts with the electromagnetic spectrum (charged particles). We can see all the luminous mass as long as it lies on the spectrum, however, all the electrically charged (luminous) particles have been accounted for. Thus, we call this extra mass 'dark'.

After Zwicky, about forty years later, astronomer Vera Rubin conducted an extensive study of sixty isolated galaxies, utilizing more sophisticated methods, and came up with the same results for the velocity distributions as Oort and Zwicky. She concluded that the luminous mass present within these galaxies could not possibly account for the matter required to reproduce the velocity distributions[1]. Additionally, modern gravitational lensing also supports the idea of missing mass within galaxies. Using GR and the results from the gravitational lensing measurement, a mass can be inferred. This inferred mass suggests that dark matter is required to produce the measurements seen with the lensing.

3.2 Derivation of Velocity Curves

3.2.1 The Newtonian Approach

Let us figure out the mathematics of these velocity curves of objects within galaxies (or by extension, galaxies within a cluster - the same math will apply). Consider a simple model of an object in a galaxy rotating about its luminous, massive center (Figure 1).

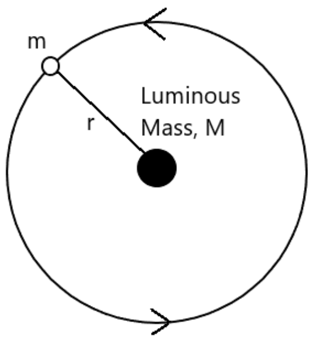


FIGURE 1: Object Orbiting a Galaxy

Mass m feels a centripetal force. Using Newton's second law and assuming the outward radial direction from M is positive, we can write:

$$\sum F_{rad} = -\frac{mv^2}{r} \tag{3}$$

$$-\frac{GMm}{r^2} = -\frac{mv^2}{r} \tag{4}$$

$$v(r) = \sqrt{\frac{GM(r)}{r}} \tag{5}$$

$$v(r) \propto \frac{1}{\sqrt{r}} \tag{6}$$

We call this the Keplerian velocity prediction[2] (Figure 2). According to this result, as mass m gets farther away from the mass M in the center, its velocity should decrease. Similarly, if it is very close, the speed should increase.

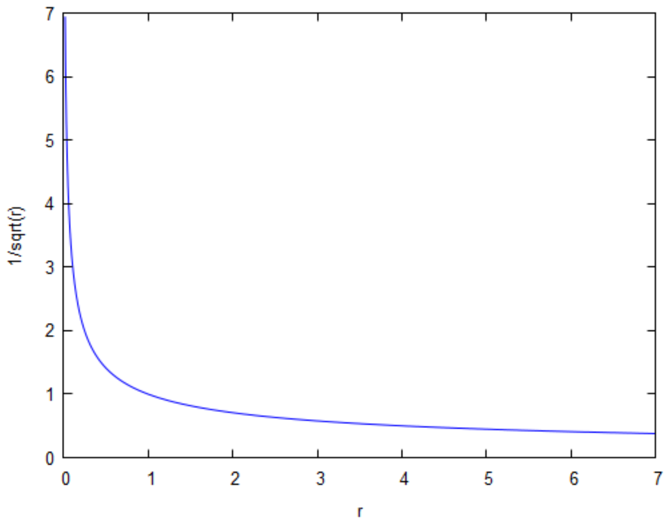


FIGURE 2: Keplerian Velocity Distribution Prediction

This is just a quick calculation using Newtonian mechanics to give us an approximation of the situation, but we know Einstein's GR is what governs spacetime and the motions of objects in the universe. It will give us the full picture. But, how can we find the velocity distribution curve of this situation using GR? This is a difficult problem to solve (called the Freeman problem) and much too complicated to go into detail, so I will just skip right to the answer.

3.2.2 The General Relativity Approach

First, let's write down Einstein's field equations for GR:

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{7}$$

From these, we solve for the metrics that describe the geometries of different spacetimes that will lead us to the velocity distributions of objects within galaxies. James O'Brien, a professor at Springfield College, was kind enough to assist my mentor and me with this problem and provided a paper of his where he lays out the solution to the Freeman problem[3,4].

$$v_{gr}(r) = \sqrt{\frac{N^* \beta^* c^2 r^2}{2{r_0}^3}\left[I_0\left(\frac{r}{2r_0}\right)K_0\left(\frac{r}{2r_0}\right) - I_1\left(\frac{r}{2r_0}\right)K_1\left(\frac{r}{2r_0}\right)\right]} \tag{8}$$

Here, I_0 and I_1 are modified Bessel functions of the first kind and K_0 and K_1 are modified Bessel functions of the second kind. N^* is the estimated number of stars in our given galaxy, β^* is the Schwarzschild radius and r_0 is the galactic scale length. Now, we have a more complete answer as to what the velocity distribution should look like for an object orbiting a galaxy (Figure 3).

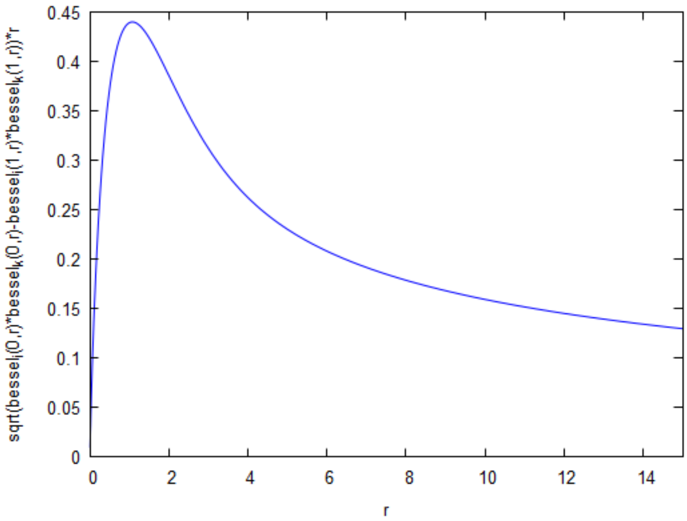


FIGURE 3: GR Luminous Mass Velocity Distribution Prediction

For any good theory in physics, if you are to expand to new realms, you must encompass the already known in your theory, much like how Einstein's relativity encompasses Newton's theory of gravity when it reduces to its simplest form. So, as $r \to \infty$, the fact that Newton's prediction is contained within the GR velocity distribution solution makes sense. For an object to stay bound in orbit, there must be sufficient gravitational force to keep it contained. As r increases, the force of gravity weakens, therefore objects far away must not be moving very fast, otherwise they would escape. So, the fact that the Newton and GR prediction for large r (i.e. being far away from the massive center) goes to zero makes sense, but this is extraordinary considering that the results presented by Oort, Zwicky, and Rubin all conclude the opposite, that the velocity distributions (whether it be of stars within galaxies or galaxies within clusters) do *not* tend to zero.

3.2.3 Extraordinary NGC3198 Galaxy Cluster Velocity Distribution Data

Interestingly, modern experimental data from the NGC3198 galaxy cluster[4] also concurs with the measurements and observations made by Oort, Zwicky, and Rubin (Figure 4).

At big r, the data shows that the velocity distribution does not tend to zero at all. Compared to the GR result, there is a sharp contrast similar to the prediction that led to the Ultraviolet Catastrophe in classical physics of black body radiation that tended towards infinity in theory, but measurements showed it actually tending to zero. The catastrophe was one of the catalysts that led to the discovery of quantum mechanics.

Because the velocities are so high, these objects (a planet, a star, or some other massive thing) should have enough speed to escape from the gravitational pull of the galaxy. Galaxies should be flying apart in a chaotic maelstrom! But, we know they don't do that. So, something in our analysis must be wrong, mainly that we need to incorporate dark matter into our theoretical model. The reason why astronomers believe we need dark matter (and dark energy) is because standard GR alone can only take you so far in distance and mass to explain things in the cosmos. Figure 5 alludes to this.

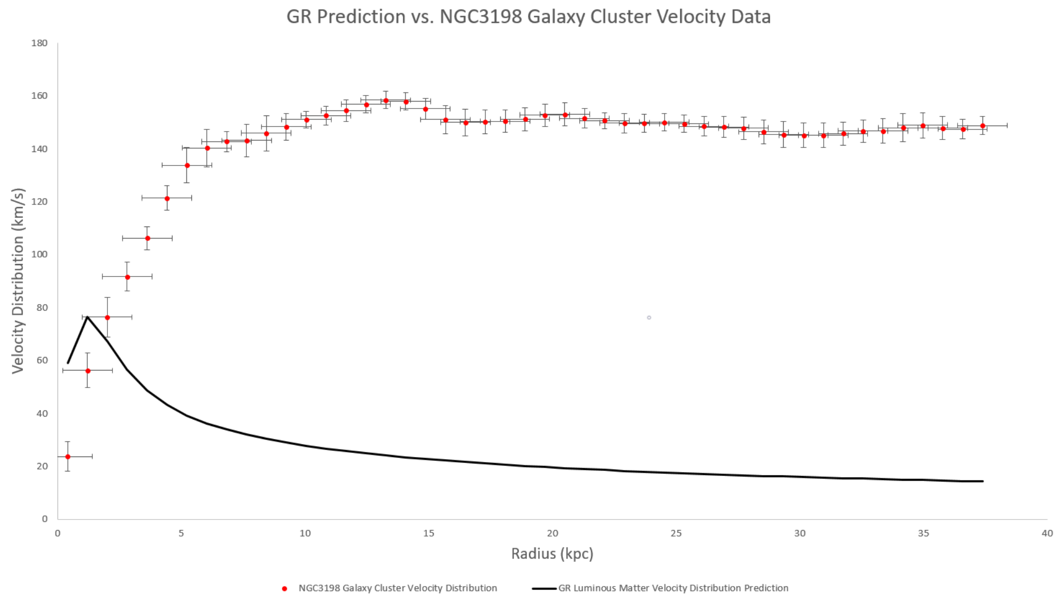


FIGURE 4: NGC3198 Galaxy Cluster Velocity Distribution Experimental Data vs. Theory

System	Scale	Experiment	Theory
Planet Size	Distance to GPS Satellites	GPS Satellites	Standard GR
Planet Size	Distance to Low Earth Orbit	Gravity Probe B	Standard GR
Sun	Distance from Earth to Sun, Sun's Mass	Radio Delays	Standard GR
Pulsar Timing	Solar Mass of Black Hole	Pulsar Timing	Standard GR
Black Hole	Galactic Black Hole Mass	Recent Radio Astronomy	Standard GR
Galaxy Velocity Curves	Galaxy Mass and Radius	Vera Rubin	Standard GR + DM
Cluster Velocity Curves	Cluster Mass and Radius	Ref. 5	Standard GR + DM
Radio Astronomy	Galaxy to Cluster Scale	Ref. 5	Standard DM
Gravitational Lensing	Galaxy Scale	Ref. 5	Standard GR + DM
Gravitational Lensing	Cluster Scale	Ref. 5	Standard GR + DM + Dark Energy
Supernova 1a	Cosmic Scale Distance	Perlmutter	Standard GR + DM + Dark Energy

FIGURE 5: The Need for Dark Matter and Dark Energy

It explains that standard GR alone cannot explain the physics on certain high mass and distance scales. In our particular case, when we look at galaxy and cluster velocity curves, we need dark matter for GR cannot stand by itself. Beyond 10^{45} kg mass and 10,000 light years distance, we need standard GR plus dark matter plus dark energy to get out to even further scales. It's like perturbation theory in quantum mechanics - we just keep adding new terms on so we can get to a more accurate description of our universe. This is why astronomers hold the view that Lambda-Cold-Dark Matter is the answer to explain the cosmos because it is one complete set of ideas that can explain things over very large scales of both mass and distance[5].

3.2.4 A Dark Matter Halo Model

Equation 5 provides some nice insight. I will write it here again for reference.

$$v(r) = \sqrt{\frac{GM(r)}{r}} \tag{9}$$

Velocity distribution depends on the mass profile of the galaxy as well as the radial distance. We can account for the radius pretty easily, so that must mean our mass profile is wrong (or rather incomplete). So now, there must be more non-luminous mass out there in galaxies that we have not accounted for. Following the ideas and procedures of Spooner[2] and Sofue[6], we theorize this non-luminous mass to take the shape of a halo (the simplest model), enveloping the galaxy.

Returning to our theoretical outline of the simple model of an object orbiting a galaxy's massive, luminous center, we add this "dark matter halo" (figure 6).

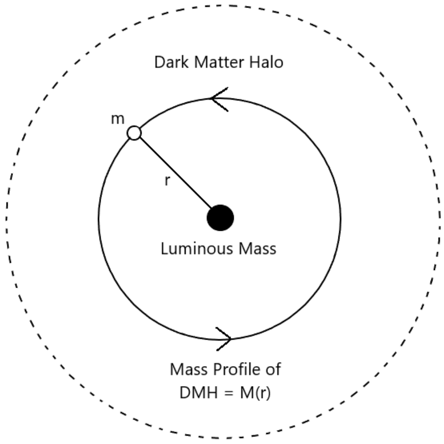


FIGURE 6: Object Orbiting a Galaxy w/ Dark Matter Halo

Consider this scenario of mass m orbiting about a galactic center of luminous matter, except this time we have a dark matter halo surrounding the galaxy that contributes to the mass profile significantly. So let's see what the velocity distribution looks like this time. Equation 5 reads:

$$v(r) = \sqrt{\frac{GM(r)}{r}} \tag{10}$$

We are now in a scenario where density and volume play a major role because we have a spherical halo, so let us rewrite mass in terms of those two.

$$v(r) = \sqrt{\frac{G}{r}\int_{Vol} \rho \, dV} \tag{11}$$

where ρ is the density profile and dV is the volume of the halo. Expanding this integral, we get:

$$v(r) = \sqrt{\frac{G}{r}\int_0^r \int_0^{2\pi} \int_0^{\pi} \frac{\rho_o r_o{}^2}{R^2 + r_o{}^2} R^2 sin\theta \, d\theta \, d\phi \, dR} \tag{12}$$

Here, I make the radii in the integral big R, not to confuse you with little r outside the integral. Also, the density profile takes on this mathematical form so we do not get a zero in the denominator. Usually, it goes like $\frac{1}{r^2}$, but getting an infinite density is a problem. This way, we allow the radius to go to zero, but still keep a very small, non-infinite density[2], ρ_o. The dθ and dϕ integrals reduce very easily to 4π. So, we can simplify to:

$$v(r) = \sqrt{\frac{4\pi G \rho_o r_o{}^2}{r}\int_0^r \frac{R^2}{R^2 + r_o{}^2} \, dR} \tag{13}$$

$$v(r) = v_o\sqrt{1 - \frac{r_o}{r} arctan(\frac{r}{r_o})} \tag{14}$$

where $v_o = \sqrt{4\pi\rho_o G {r_o}^2}$. If we add this onto our GR prediction, we come up with the velocity distribution that matches quite well with the experimental data from the NGC3198 galaxy cluster (Figure 7).

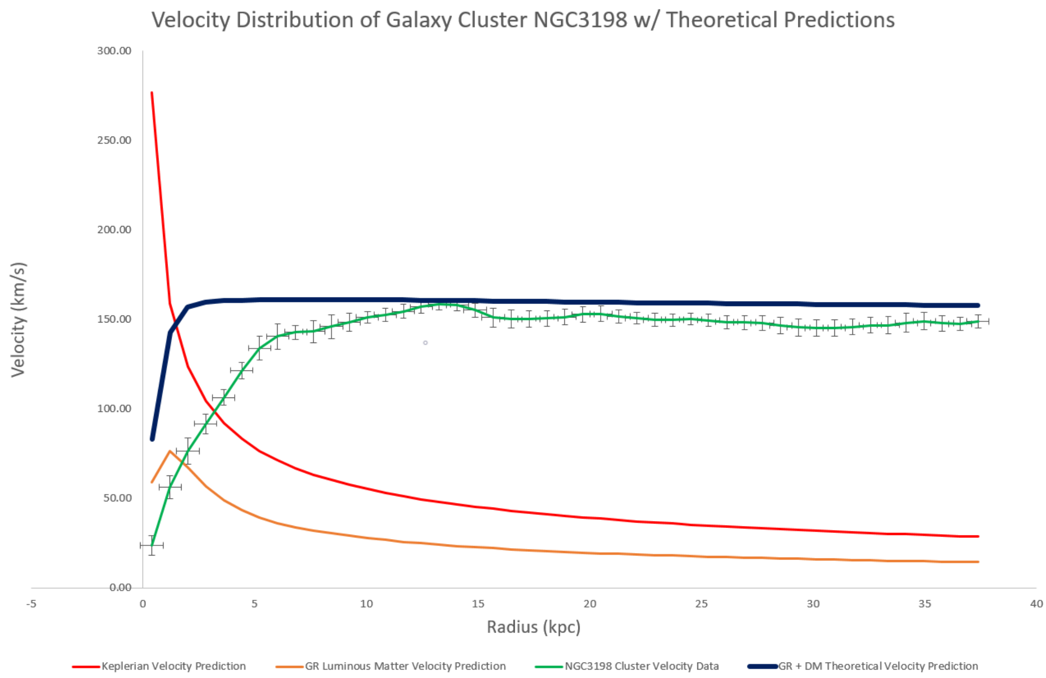


FIGURE 7: GR + DM Prediction w/ previous theories

I had to scale the theoretical predictions to closely match the experimental data numbers, which is fine because, at the moment, the main emphasis is on the functional form and ideas.

Let's take a step back and quickly recap what we just did. Before, we were under the assumption that there was only luminous baryonic matter at the center of the galaxy and that accounted for the majority of the mass. We applied Newton's laws and - voila! - we get a good approximation that the velocity goes like $\frac{1}{\sqrt{r}}$. We did even better by producing a function for the velocity distributions that obey GR using some fancy modified Bessel functions[4]. But, according to experimental data like the NGC3198 galaxy cluster velocity distributions, we noticed that the velocities do not obey the GR equation, so we inferred there to be more mass that is non-luminous (because we swore up and down that we accounted for all the luminous mass) and now we ended up here: an equation combining the GR prediction with a dark matter (DM) prediction. We still used the same physics, however the diagram of our physical situation changed, causing our velocity distribution to change.

Now that we know there is definitely dark matter out there, we can move on to a different question. Since we suspect dark matter to be a particle of some kind, it is equally important to talk about GR's 'partner' - the Standard Model of Particle Physics. The Standard Model is understood through relativistic, gauged Quantum Field Theory (QFT) that is based on the foundation of the fundamental symmetries of our universe. So, if we are to understand particles (dark matter), we must understand QFT. That is the next step in our journey.

4 The Standard Model of Particle Physics

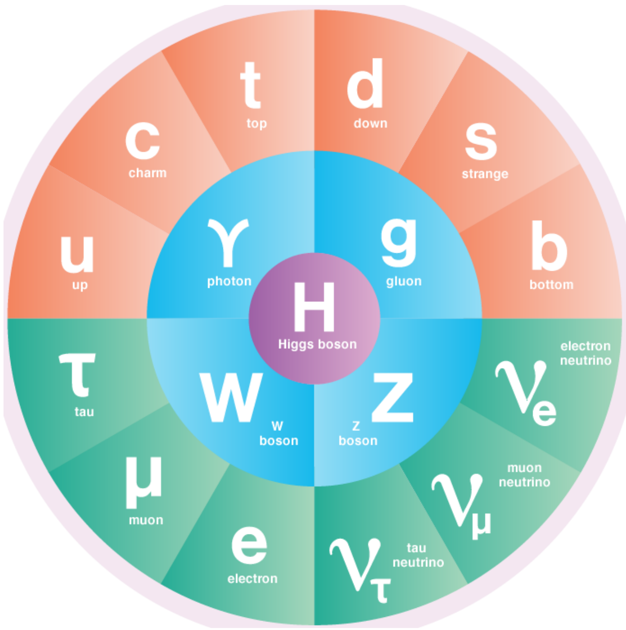


FIGURE 8: Standard Model of Particle Physics

The Standard Model of Particle Physics[7] (Figure 8) encompasses all the known matter, energy, and forces (excluding gravity) into one nearly-complete, beautiful theory of a manifestly covariant combination of gauged quantum symmetry fields with Poincare spacetime symmetry. It is by far the most successful physical theory out there, but it does not tell the whole story. If we are to compile a complete theory of everything that includes all the particles and their interactions with the known forces of the universe, gravity must be included. There are additional problems with the SM such as the strong CP violation related to baryogenesis as well as the Hierarchy Problem. Nevertheless, it is still the most accurate and precise theoretical tool to be developed. However, if we ever want a complete SM one day, dark matter must be understood. But, because we suspect dark matter to be a particle of some kind, we must understand the current establishments of the SM.

4.1 1^{st} to 2^{nd} Quantization

4.1.1 The Simple Harmonic Oscillator in Quantum Mechanics

The well-known Schr*ö*dinger equation for a simple harmonic oscillator is written as follows:

$$\hat{H}\psi_n = E_n\psi_n \tag{15}$$

$$(\frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x})\psi_n = \hbar\omega(n + \frac{1}{2})\psi_n \tag{16}$$

where $\hat{p}$ and $\hat{x}$ are the momentum and position operators, respectively. ψ_n is the wave function solution.

$$\psi_n(\xi) = \frac{1}{\sqrt{2^n n!}}(\frac{m\omega}{\pi\hbar})^{\frac{1}{4}} H_n(\xi) e^{-\frac{\xi^2}{2}} \tag{17}$$

Here, $H_n(\xi)$ is a Hermite polynomial (with n = 0,1,2...). This is the 1^{st} quantization answer for the simple harmonic oscillator; it looks very wave-like and rather complicated. And keep in mind - this is only one dimension. Luckily, there is an easier way to solve this problem. It involves transitioning from the wave-like 1^{st} quantization (with $\hat{H}(\hat{x},\hat{p})$ and ψ wave-function) to a particle-like 2^{nd} quantization (using $\hat{H}(\hat{a}^\dagger,\hat{a})$ and $|n\rangle$ particle-like eigenstates).

4.1.2 Creation and Annihilation Operators

The first step to transition to 2^{nd} quantization is introducing some new operators, $\hat{a}$ and $\hat{a}^\dagger$.

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p}) \tag{18}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p}) \tag{19}$$

One particularly important reason why we like these operators lies in their commutation relationship.

$$\left[\hat{a}^\dagger, \hat{a}\right] = 1 \tag{20}$$

This seems so much more natural and satisfying than the commutation relationship between the position and momentum operators.

$$\left[\hat{x}, \hat{p}\right] = i\hbar \tag{21}$$

The $\hat{a}^\dagger$ and $\hat{a}$ commutation relationship also encodes the uncertainty principle because the operators are built from $\hat{x}$ and $\hat{p}$, so we have preserved an important physical element in our abstraction! Although it is not clear yet, these operators correspond to bosons. For fermions, we use an anti-commutation relationship.

$$\{\hat{c}_i^\dagger, \hat{c}_j^\dagger\} = 0 \tag{22}$$

where the curly brackets indicate anti-commutation. Operating $\hat{a}^\dagger$ on $\hat{a}$ and doing some algebra, one can eventually state the following.

$$\hat{H}_{H-O} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) \tag{23}$$

If you do not trust that this is the correct result and want to do the calculation yourself, by all means, go ahead. However, I am skipping the algebraic steps in between to not only save time and also to illustrate what is more important about this whole story. Since this is true, however, the following can be stated:

$$\hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})\psi_n = \hbar\omega(n + \frac{1}{2})\psi_n \tag{24}$$

$$\hat{a}^\dagger\hat{a}\psi_n = n\psi_n \tag{25}$$

We have essentially identified this operation - $\hat{a}^\dagger\hat{a}$ - to be a number operator (and thus we've defined $\hat{H}$ to be a number operator too, hinting at its particle-like nature) that can pull out an n from ψ_n. Because of this, we can now build an abstraction from ψ_n and write:

$$\hat{a}^\dagger\hat{a}\,|n\rangle = n\,|n\rangle \tag{26}$$

If you apply $\hat{a}^\dagger$ to the abstract energy eigenstates $|n\rangle$, you get the following:

$$\hat{a}^\dagger\,|n\rangle = N\,|n+1\rangle \tag{27}$$

where N is just a normalization constant. This tells us that this operator *raises* the eigenstate by one. Similarly, $\hat{a}$ lowers the eigenstate by one.

$$\hat{a}\,|n\rangle = N\,|n-1\rangle \tag{28}$$

So essentially, $\hat{a}^\dagger$ creates a particle in the energy eigenstate $|n\rangle$ and $\hat{a}$ annihilates or destroys a particle in energy eigenstate $|n\rangle$. The original Schr*ö*dinger equation

$$\hat{H}(\hat{x},\hat{p})\psi_n = E_n\psi_n \tag{29}$$

can now be written as the following:

$$\hat{H}(\hat{a}^\dagger,\hat{a})\,|n\rangle = E_n\,|n\rangle \tag{30}$$

This means we have transitioned to 2^{nd} quantization (particle-like)! First, we were using wave functions with those ugly looking Hermite polynomials. Now, using our abstraction, we've brought it back to particles. We're able to create and annihilate particles in particular energy eigenstates, but where exactly is the particle? Can we create one and know where its position is? In fact, we can. To expand upon our abstraction, we have to look at the commutation relationship between the momentum operator and the Hamiltonian.

4.1.3 Field Operators

The commutation relationship between the Hamiltonian and momentum operator is:

$$\left[\hat{p},\hat{H}\right] = 0 \tag{31}$$

Because these two commute, that means they share the same eigenstates. So, if we wanted to (and we do), we can write the following:

$$\hat{p}\,|n\rangle = p_n\,|n\rangle \tag{32}$$

What have we done? What does this mean? It means we can write the energy eigenstates as momentum eigenstates. Why would we want to do this, though? For the same reason why we switched from wave functions to eigenstates - because it makes things easier. Let's go a step further. There's another relationship we know of between momentum and wave number.

$$p = \hbar k \tag{33}$$

So now, we can have *wave number* eigenstates. The reason we move to a k space is so we can use an inverse Fourier transform to switch to x space. Doing this, we can generate a function that creates a particle at a specific, known location (x) by summing over infinite momentum eigenstates (We're in k space, yes, however, k and p are proportional to each other, so it is more important for me to state that we are summing over *momentums* because it relates to the uncertainty principle, which has more physical meaning).

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{-ikx} dk \tag{34}$$

$\tilde{f}(k)$ is the wave function in k space and f(x) is the wave function in x space. Since $\hat{a}^\dagger$ and $\hat{a}$ create momentum eigenstates, we can replace the k space wave function with our creation operator in our 2^{nd} quantization abstraction. This gives us an equation for a *field operator* in momentum space.

$$\hat{\psi}_p^\dagger(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{a}_p^\dagger e^{-ikx} dk \tag{35}$$

So what exactly does this 'field operator' do? It creates one particle with momentum p right at position x as evident and detailed in Lancaster's *Quantum Field Theory for the Gifted Amateur* (Example 4.1)[8].

4.1.4 Generalizing the Hamiltonian

We currently have in our quantum mechanical arsenal the power to write the Hamiltonian in 1^{st} quantization in differential form as a function of $\hat{a}^\dagger$ and $\hat{a}$ for the single particle harmonic oscillator.

$$\hat{H} = f(\hat{a}^\dagger, \hat{a}) \tag{36}$$

But we don't just want the Hamiltonian for a single problem. We want to extend to *all* problems. So how can we generalize the Hamiltonian in 2^{nd} quantization? It involves transitioning from using differential formalism of 1^{st} quantization to matrix formalism in 2^{nd} quantization. The differential form for the kinetic energy term of the Hamiltonian is:

$$\hat{E}_k = \frac{\hat{p}^2}{2m} = \frac{\left(i\hbar \frac{\partial}{\partial x}\right)^2}{2m} \tag{37}$$

Because $\hat{p}$ commutes with itself, $\hat{E}_k(\hat{p})$ will be diagonal. So, the matrix representation is written as the following:

$$\hat{E}_k = \sum_{\alpha,\beta} E_\alpha \hat{n}_\alpha \delta_{\alpha\beta} = \begin{pmatrix} E_1\hat{n}_1 & 0 \\ 0 & E_2\hat{n}_2 \end{pmatrix} \tag{38}$$

where E_n is the energy eigenvalue we're familiar with[8]. If you recall, it was stated before that $\hat{a}^\dagger\hat{a}$ is equivalent to the number operator, $\hat{n}$. So, we can substitute our creation and annihilation operators in and get:

$$\hat{E}_k = \sum_{\alpha,\beta} E_\alpha \hat{a}_\alpha^\dagger \hat{a}_\alpha \delta_{\alpha\beta} = \begin{pmatrix} E_1\hat{a}_1^\dagger\hat{a}_1 & 0 \\ 0 & E_2\hat{a}_2^\dagger\hat{a}_2 \end{pmatrix} \tag{39}$$

To reiterate, this is our kinetic energy term of our generalized Hamiltonian operator represented in matrix form that is true for all *single* particle systems.

What about the potential energy term? We will use much of the same process as we did with the kinetic energy. In the 1^{st} quantization differential form, the expectation value for the potential energy operator looks like:

$$\langle \hat{E}_p \rangle = \int_{Vol} \psi^*(x)\hat{E}_p\psi(x)\,dV \tag{40}$$

where $\hat{E}_p$ is the potential energy operator. Here, we can utilize the field operators we derived to essentially replace the wave functions in the integral.

$$\int_{Vol} \psi^*(x)\hat{E}_p\psi(x)\,dV \rightarrow \int_{Vol} \hat{\psi}^\dagger(x)\hat{E}_p\hat{\psi}(x)\,dV \tag{41}$$

Doing some math (which I won't do here), you can get a 2^{nd} quantization potential energy operator term[8].

$$\hat{E}_p = \hat{V} = \frac{1}{2}\sum_{p_1,p_2,q} \hat{V}_q \hat{a}^\dagger_{p_1+q}\hat{a}^\dagger_{p_2-q}\hat{a}_{p_2}\hat{a}_{p_1} \tag{42}$$

We have generalized the Hamiltonian using matrix formalism, making our job a whole lot easier. Studying QFT took up a majority of my efforts when conducting this research. But it was studying well spent because it is important for understanding the foundation upon which our universe is based on.

4.2 Fermions & Bosons

From section 4.1, when 2^{nd} quantization is applied with the field operator formalism, we see that particles emerge as local, spacetime excitations of fundamental quantum field operators as seen in equation 35. Thus, we come to understand an amazing and important property of quantum mechanics and that is that all particles are identical and indistinguishable. One can think of these created particles to simply be excitations of this fundamental quantum field. We also know that the study of particle physics requires energy scales large enough to create particle-antiparticle pairs in a vacuum. This suggests a fascinating inherent feature, stating that particle physics is actually *many*-particle physics.

The natural thing to do next is to incorporate many-particle formalism into our quantum mechanical description of the world. Because particles are identical and indistinguishable, the 'labels' that we might attribute to classical particles (being able to distinguish the particles easily - particle 1 and particle 2) can be switched if attached to quantum particles and the particle wave function remains unchanged. We use an operator to define this exchange of particles, $\hat{P}_{1\Longleftrightarrow 2}$, which commutes with the Hamiltonian operator.

$$\left[\hat{P}_{1\Longleftrightarrow 2}, \hat{H}\right] = 0 \tag{43}$$

This means they share the same eigenstates (Hilbert space and solutions). The eigenstates of the $\hat{P}_{1\Longleftrightarrow 2}$ include both a symmetric and anti-symmetric solution under the exchange of the particle label. Therefore, quantum solutions, like the wave function, ψ, for many particles must come in two forms: symmetric and anti-symmetric. This revelation tells us something that no classical theory could: the particle-like excitation of a single quantum field (where the particles are identical and indistinguishable) tells us that the quantum mechanical description of nature requires two types of particles: Fermions and Bosons. Fermions (with half-integer spin, $\frac{1}{2}n$) possess anti-symmetric many-particle eigenfunctions under the exchange of particle labels resulting in these particles having to obey Fermi-Dirac statistics (including the

Pauli-Exclusion Principle). Bosons (with integer spin, n) produce symmetric many-particle eigenfunctions under the exchange of particle labels resulting in these particles having to obey Bose-Einstein statistics (most notably, Bose-Einstein condensates - the work achieved in this field was awarded the Physics Nobel Prize in 2001)[9].

4.2.1 Fermions: Leptons & Quarks

Fermions themselves are divided into two categories (Leptons and Quarks) and three *generations* or *families* of particles (Figure 9)[10].

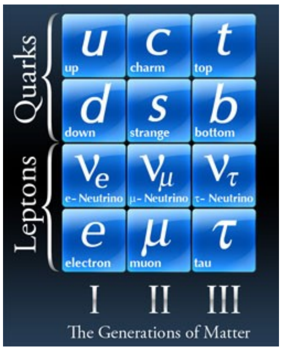


FIGURE 9: Generations of Fermions

The first family includes the electron and electron-neutrino leptons as well as the up and down quarks. This generation is the lightest of them all in terms of mass and also the most abundant. The second family features the muon and muon-neutrino particles, the second heaviest generation. The quarks in this category are the charm and strange. The tau and tau-neutrino are the heaviest of these families and least abundant. This generation is home to the top and bottom quarks.

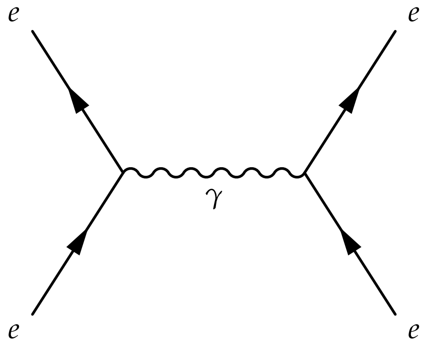
Both the leptons and quarks have electric charge associated with them. Everyone knows that the electron has an electric charge of -1, but so too do the muon and tau since they are just heavier versions of the electron. Their neutrino counterparts are neutral and have no electric charge. The quarks are a little funkier because they possess fractional charge. The up, charm, and top quarks have $+\frac{2}{3}$ charge, and the down, strange, and bottom quarks have $-\frac{1}{3}$ charge. Fractional charge might seem like an odd property to have, but they combine nicely to form the particles we are all familiar with, including the proton, neutron, and most fundamental baryons.

4.2.2 Bosons & the Four Fundamental Forces

Bosons are the force mediators in this relationship. These particles act as a sort of bridge between the fundamental, universal forces of nature and fermions. Light particles, photons, act as the mediator between fermions that possess electric charge, such as the electron or proton (it is okay to call the proton a fermion because its essential building blocks - quarks - are fermions). A feynman diagram illustrates this nicely.

Here, with the time arrow pointing up, we have an electron and positron interacting, creating a photon which then creates another electron-positron pair. The interaction is as follows: $e^- + e^+ \rightarrow e^- + e^+$, with the photon particle (boson responsible for the electromagnetic force) acting as the mediator in the interaction[11].

Another boson, $W^\pm$ and Z^o, is the mediator of the Weak Nuclear Force (WNF) (responsible for things like radioactive decay) interacting with fermions that have



weak charge. Just as electric charge is necessary for electromagnetic interaction, weak charge is necessary for interaction with the WNF.

The gluon is the boson mediator of the Strong Nuclear Force (SNF), responsible for keeping quarks, and therefore nuclei, held together. It interacts with fermions that have the property of color charge, such as quarks.

The last force we have to deal with is gravity. According to the Standard Model, all the fundamental forces have a particle mediator associated with them, so shouldn't gravity? It turns out that there is a predicted gravitational boson called the graviton. What underlying property must the fermions have in order to have an interaction with this force? Mass charge. Sounds strange, but it makes the most sense.

4.3 SU(3) $\otimes$ SU(2) $\otimes$ U(1) and the Fundamental Symmetries of our World

The beautiful mathematics of Group Theory have allowed us to identify that the spacetime we live in according to Einstein's Relativity is $SO(1,3)$, standing for one temporal dimension and three spacial dimensions. But, we now know that there is an even deeper underlying symmetry within and it takes on the form of $SU(2) \otimes SU(2)$. You can go even further than that to get to the symmetries that truly make up our universe. But to do that, one must study Lagrangians. The general equation for the Euler-Lagrange Equations-of-Motion (EL-EoM) is:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \tag{44}$$

And the action integral, S, is written as the following:

$$S = \int L\, dt \tag{45}$$

However, this is for when we are dealing with classical mechanics only. We want to transition to a 2^{nd} quantization formalism - how do we do this? We upgrade our Lagrangian, L, to a super Lagrangian that depends on the field operators as well as the four-vector derivative (∂_μ).

$$L \to \mathcal{L}\Big(\hat{\psi}^\dagger(\hat{a}^\dagger), \hat{\psi}(\hat{a}), \partial_\mu\Big) \tag{46}$$

Now the EL-EoM looks like:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0 \tag{47}$$

where ϕ is a scalar field generated (with spin = 0). Other fields available to us from SO(1,3) symmetry are the ψ spinor field (spin = $\frac{1}{2}n$) and $\vec{A}$ vector field (spin = n)[12] (where n = 1,2,3...). What are the 2^{nd} quantization field operators that create these?

$$\phi \rightarrow \hat{\phi}(\hat{a}^{\dagger}, \hat{a}) \tag{48}$$

$$\psi \rightarrow \hat{\psi}(\hat{c}^{\dagger}, \hat{c}) \tag{49}$$

$$\vec{A} \rightarrow \hat{A}(\hat{a}^{\dagger}, \hat{a}) \tag{50}$$

where $\hat{a}^{\dagger}$ and $\hat{a}$ are the bosonic creation and annihilation operators and $\hat{c}^{\dagger}$ and $\hat{c}$ are the fermionic creation and annihilation operators from before. We already know from earlier that these field operators have to take on the form of an inverse Fourier Transform.

$$\hat{\phi}\,\&\,\hat{A} \rightarrow \hat{A}^{\dagger}(x) = \frac{1}{\sqrt{V}} \int \hat{a}^{\dagger} e^{-ipx}\, dp \tag{51}$$

$$\hat{\psi} \rightarrow \hat{\psi}^{\dagger}(x) = \frac{1}{\sqrt{V}} \int \hat{c}^{\dagger} e^{-ipx}\, dp \tag{52}$$

$\hat{\phi}$ and $\hat{A}$ are both fields associated with bosons, while $\hat{\psi}$ is the field associated with fermions. Now, if we plug these field operators into the EL-EoM, we should get our world - with all the bosons, fermions, scalars, and interactions! After all, we derived these fields from our SO(1,3) symmetry which is what our universe is based on. But, it turns out we don't get our world. To navigate around this, we must look to see if there is some even deeper underlying symmetry, and add it onto our SO(1,3). The new group we will add on is U(1) - a type of abstract quantum symmetry[12]. So we get:

$$Our\,World \rightarrow U(1) \otimes SO(1,3) \tag{53}$$

We can represent U(1) as an exponential using a generator and a parameter.

$$U(1) = e^{i\hat{J}_{U_1}\theta_1} \tag{54}$$

However, θ_1 is not a function of t, x, y, or z. It is called a *global* parameter, but this is problematic because that would mean it is everywhere all at once and that violates SR. If we want U(1) to represent a force of the universe, it cannot happen everywhere. Instead, it must operate under a fundamental field moving less than the speed of light. So, naturally, the solution to this problem is to make the parameter *local* by making it depend on spacetime coordinates so it stays consistent with SR[12]. Now U(1) takes on a different definition.

$$U(1) = e^{-i\hat{J}_{U_1}\theta(x^{\mu})} \tag{55}$$

where $x^{\mu} = (ct, x, y, z)$. Now our new field operators are:

$$\hat{\phi} = e^{-i\hat{J}_{U_1}\theta(x^{\mu})}\hat{\phi}(\hat{a}^{\dagger}, \hat{a}) \tag{56}$$

$$\hat{\psi} = e^{-i\hat{J}_{U_1}\theta(x^{\mu})}\hat{\psi}(\hat{c}^{\dagger}, \hat{c}) \tag{57}$$

$$\hat{A} = e^{-i\hat{J}_{U_1}\theta(x^{\mu})}\hat{A}(\hat{a}^{\dagger}, \hat{a}) \tag{58}$$

If you plug these new field operators into the EL-EoM, we should be good to go, right? Unfortunately, we run into another problem. It turns out that the EL-EoM are all different, meaning they are non-covariant. To fix this covariance that we expect from Einstein's original idea of covariant, fundamental laws of physics derived from symmetries, we need to add a gauge field through a covariant four-vector derivative, replacing the regular four-vector derivative.

$$D_\mu = d_\mu + iq\vec{G}_\mu(x) \tag{59}$$

where D_μ is the covariant four-vector derivative and $\vec{G}_\mu(x)$ is the added gauge field. Now the Lagrangian takes on the following functional form:

$$\mathcal{L}(\partial_\mu) \rightarrow \mathcal{L}(D_\mu) \tag{60}$$

Now we have made things covariant, so do the EL-EoM work? They do! From U(1), you get all of Quantum Electrodynamics (QED) accounting for the Electromagnetic Force (EMF) and the particle mediator, the photon. What about the other forces? Do they work as well? Indeed, they do. Adding on the symmetry SU(2) gives the WNF as well as its particle mediators, the $W^\pm$ and Z^0 gauge bosons. For the SNF, we slap on a SU(3) abstract quantum symmetry and get all of Quantum Chromodynamics (QCD), including the color-charged SNF mediators: gluons. So a better equation for our world, now, would be:

$$Our\ World \approx SU(3) \otimes SU(2) \otimes U(1) \otimes \{SO(1,3)\} \tag{61}$$

From this, we get all the known particles (bosons, fermions) and interactions (EMF, WNF, SNF) of the SM (Figure 10)[1,12].

Fermions

Generation 1			Generation 2			Generation 3		
Particle	Mass (MeV)	Charge	Particle	Mass (MeV)	Charge	Particle	Mass (MeV)	Charge
up quark (u)	2.55	$+\frac{2}{3}$	charm quark (c)	1270	$+\frac{2}{3}$	top quark (t)	171200	$+\frac{2}{3}$
down quark (d)	5.04	$-\frac{1}{3}$	strange quark (s)	104	$-\frac{1}{3}$	bottom quark (b)	4200	$-\frac{1}{3}$
electron (e^-)	0.511	-1	muon (μ^-)	105.7	-1	tau (τ^-)	1776.8	-1
e neutrino (ν_e)	$< 2.0 \times 10^{-6}$	0	μ neutrino (ν_μ)	< 0.19	0	τ neutrino (ν_τ)	< 18.2	0

Gauge Bosons

Particle	Force	Acts through	Acts on	Mass (MeV)	Charge
Photon (γ)	Electromagnetic	Electric charge	Electrically charged particles	$< 1\times10^{-24} \approx 0$	0
Z boson (Z)	Weak nuclear	Weak interaction	Quarks and leptons	91188	0
$W^\pm$ bosons ($W^\pm$)	Weak nuclear	Weak interaction	Quarks and leptons	80398	±1
Gluon (g)	Strong nuclear	Color charge	Quarks and gluons	0	0
Higgs boson (H^0)	Higgs force	Higgs field	Massive particles	> 114400	0

FIGURE 10: SM Particles & Properties

One last ingredient for the SM that is not the scope of this work is the introduction of a Higgs field which gives mass to all these fundamental particles.

4.4 Beyond the Standard Model & Supersymmetry (SUSY)

Evidence from Big Bang Nucleosynthesis (BBN) - the phases in the early universe where nuclei other than regular hydrogen began to produce - and the Cosmic Microwave Background (CMB) - the first electromagnetic radiation that was emitted in the early stages of the universe - both force dark matter to adopt restrictive prerequisites. One obvious requirement is that dark matter must be electrically neutral. Since we cannot see dark matter, that means it does not emit light of any kind, having no interaction with the electromagnetic spectrum. Additionally, according to BBN and CMB, dark matter must be cold and non-relativistic. It cannot be moving very fast, nor have a lot of energy. Also from these concepts, we can rule out some of the neutral SM particles including the neutrino, photon, Z^0 boson, and Higgs boson to be dark matter[1].

Additionally, there was some early evidence that dark matter might be beyond the SM when the experimental data from the NGC3198 galaxy cluster did not match our theoretical GR prediction. If we think dark matter is a particle in the SM, then GR would be correct. But, we know from the data that the GR prediction did not match the observed velocity curves, so there must be a new, undiscovered particle out there that we have failed to incorporate (or perhaps a better theory of gravity is required).

However, there is no need to despair for there is a good amount of evidence that there is some new physics beyond the SM that we have yet to discover. Two main reasons we suspect this to be true are based upon the Hierarchy Problem and the Strong CP Violation. The Hierarchy Problem is the unusual circumstance in which there is a massive energy disparity (about 10^{16} GeV energy difference) between the points where SM particles start to behave quantum mechanically and gravity starts to come into effect at the atomic level and behave quantum mechanically. The Strong CP Violation brings up the question of why there are more particles over antiparticles if there were an equal amount of them produced in the early universe. What could have caused this difference? A new type of particle proposed by particle physicists is called the axion which has the properties that would help solve this problem, along with others in the particle physics world, so it is a popular candidate[1].

Theories for extensions beyond the Standard Model are already being put forth. Supersymmetry (SUSY) is a favorite among particle physicists. This doubles the amount of known particles by giving each particle a superpartner. Fermions would get bosonic partners and bosons would get fermionic partners. For example, an electron would get a bosonic superpartner called a selectron (add a prefix of 's-') and the Higgs boson would get a fermionic superpartner called a Higgsino (add a suffix of '-ino'). The reason this extension beyond the Standard Model is useful (if you haven't already guessed) is that we have several more particles to look at as potential dark matter candidates, widening the search field.

5 Dark Matter Candidates

Figure 11 offers a table of SUSY particles[1].
Of these, we looked specifically at the Neutrilinos and also the Axion (not on here), as they were the most intriguing. Neutrilinos are supersymmetric, superposition particle states of the photino, zino, and Higgsino superpartners of the neutral gauge bosons[1]. Axions are non-SM particles that solve the strong CP violation. A number of experiments are being conducted looking for them, including one by Dr. Deveney's colleague at Yale University (Professor Steve Lamoreaux). Experiments are also being done at the University of Florida to detect the axion[13]. Other potential DM particle candidates we did not study in as much detail are listed (Figure 12).

To complete the survey, we considered alternative theories of gravity such as MOND (Modified Newtonian Dynamics)[1,14] and the more drastic Conformal Gravity, studied by Professor James O'Brien at Springfield College[3]. However, to date, there are astonishingly no reproducible results with any indication of a measurement

Sfermions

Generation 1			Generation 2			Generation 3		
Particle	Mass (GeV)	Charge	Particle	Mass (GeV)	Charge	Particle	Mass (GeV)	Charge
up squark ($\tilde{u}$)	> 379	$+\frac{2}{3}$	charm squark ($\tilde{c}$)	> 379	$+\frac{2}{3}$	top squark ($\tilde{t}$)	> 92.6	$+\frac{2}{3}$
down squark ($\tilde{d}$)	> 379	$-\frac{1}{3}$	strange squark ($\tilde{s}$)	> 379	$-\frac{1}{3}$	bottom squark ($\tilde{b}$)	> 89	$-\frac{1}{3}$
selectron ($\tilde{e}$)	> 73	-1	smuon ($\tilde{\mu}$)	> 94	-1	stau ($\tilde{\tau}$)	> 81.9	-1
e sneutrino ($\tilde{\nu}_e$)	> 95	0	μ sneutrino ($\tilde{\nu}_\mu$)	> 94	0	τ sneutrino ($\tilde{\nu}_\tau$)	> 94	0

Gauginos

Particle	Mass (GeV)	Description
Neutralinos ($\tilde{\chi}^0_{1-4}$)	> 46	Mixture of photino ($\tilde{\gamma}$), zino ($\tilde{Z}$), and neutral higgsino ($\tilde{H}^0$)
Charginos ($\tilde{\chi}^{\pm}_{1,2}$)	> 94	Mixture of winos ($\tilde{W}^{\pm}$) and charged higgsinos ($\tilde{H}^{\pm}$)
Gluinos ($\tilde{g}$)	> 308	Superpartner of the gluon

FIGURE 11: Proposed SUSY Particles

Other Possible Dark Matter Candidates
WIMP's
MACHO's
Kaluza-Klein Particles
Gravitinos
WIMPzillas

FIGURE 12: Other Dark Matter Candidates

of any DM candidates, even with very sensitive detectors such as the IceCube detector at the IceCube Neutrino Obseratory in Antarctica. One of the main functions of this observatory is to search for dark matter (the more general WIMPs (Weakly Interacting Massive Particles)) by detecting neutrinos produced from self-annilhilating dark matter. A remarkable feat it has achieved as far as sensitivity is reducing the heavy background noise of incoming atmospheric muons by about seven orders of magnitude, increasing the precision of the measurements. It is reported now that the mass range in which the detector is now searching under is about 10 to 100 GeV[15]. However, as mentioned, there has been no data that directly confirms a detection of dark matter with this experiment, nor with any other. And so the mystery continues.

Appendix A

Relativistic Kinematics & GR

A small, but important portion of my research was studying relativistic kinematics. Since almost all particles travel at high speeds, we must consider and incorporate SR into our discussion. According to SR, the laws of physics are preserved and are the same in all inertial reference frames. In other words, the laws of physics are the same whether you are at rest (sitting on a park bench) or moving at a constant velocity (going on a bus ride across the country). The Lorentz Transformation, the spacetime rotation that keeps all the laws of physics the same when switching to different coordinate systems (different frames) comes about from SR and we represent it with the following matrix.

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.1}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ - the relativistic figure of merit - and $\beta = \frac{v}{c}$. Four-vector notation was also studied as it is an essential to understanding the notation of particle physics. For example, a spacetime four-vector can be written as:

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) \tag{A.2}$$

It can also be written as a vertical matrix.

$$x^{\mu=0,1,2,3} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{A.3}$$

where we have the temporal component, *ct*, and the spacial components, x, y, and z. We can represent our Lorentz Transformation using indices and combine it with x^μ to create a new four-vector in a different reference frame!

$$x^{\mu'} = \Lambda^\mu_\nu x^\nu \tag{A.4}$$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{A.5}$$

This index notation is the heart and soul of the kind of mathematics you see in GR and is littered in Einstein's Field Equations.

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{A.6}$$

The reason index notation was important to learn is because it shows up in relativistic kinematics (because we must incorporate SR), which has a direct connection to particle mechanics and motion. Several examples on relativistic kinematics from Griffith's *Introduction to Elementary Particles* were done including Examples 3.1, 3.2, 3.3, and 3.5[10].

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