Bridgewater State University

Virtual Commons - Bridgewater State University

Honors Program Theses and Projects

Undergraduate Honors Program

5-9-2019

Modified Ramsey Numbers

Meaghan Mahoney
Bridgewater State University

Follow this and additional works at: https://vc.bridgew.edu/honors_proj

Part of the Mathematics Commons

Recommended Citation

Mahoney, Meaghan. (2019). Modified Ramsey Numbers. In *BSU Honors Program Theses and Projects.* Item 385. Available at: https://vc.bridgew.edu/honors_proj/385
Copyright © 2019 Meaghan Mahoney

This item is available as part of Virtual Commons, the open-access institutional repository of Bridgewater State University, Bridgewater, Massachusetts.

Modified Ramsey Numbers

Meaghan Mahoney

Submitted in Partial Completion of the
Requirements for Departmental Honors in Mathematics

Bridgewater State University

May 9, 2019

Dr. Shannon Lockard, Thesis Advisor
Dr. Rachel Stahl, Committee Member
Dr. Stephen Flood, Committee Member

Abstract

Ramsey theory is a field of study named after the mathematician Frank P. Ramsey. In general, problems in Ramsey theory look for structure amid a collection of unstructured objects and are often solved using techniques of Graph Theory. For a typical question in Ramsey theory, we use two colors, say red and blue, to color the edges of a complete graph, and then look for either a complete subgraph of order n whose edges are all red or a complete subgraph of order m whose edges are all blue. The minimum number of vertices needed to guarantee one of these subgraphs is the Ramsey number, $R(n, m)$. Ramsey's Theorem shows that $R(n, m)$ exists for every n and m greater than one, yet very few Ramsey numbers are known. There are many interesting modifications of the original problem such as looking for subgraphs other than complete graphs. For this thesis, we will consider modified Ramsey numbers for star graphs instead of the classical Ramsey number $R(n, m)$. We will prove a general formula for the modified Ramsey number of two star graphs and begin exploring modified Ramsey numbers of a star graph and a path.

1 Introduction

Suppose you want to throw a party but there's a catch; you want to invite the minimum number of people to ensure there will be a group of three mutual friends or three mutual enemies, given any two people are either friends or enemies. Since you want there to be a group of three friends or three enemies, there must be at least three people invited to the party. But if you invite three people, there could easily be a situation where two people are friends while the other is an enemy. So you must invite more than three people. The same happens when looking at four or five people at the party though; there can be a situation where there is not a group of three friends or three enemies. Now let's consider inviting six people. If there are six people at the party, then each person will have a relationship (whether it be friends or enemies) to five other people. Let's look at one person's, say Lisa's, relationships with the others at the party. If Lisa has no friends at the party, then she will be enemies with five other people. If Lisa only has one friend at the party, then she will be enemies with four other people. If she has two friends at the party, she will be enemies with three other people. Otherwise, Lisa will have three or more friends at the party. Therefore, Lisa will always either have at least three friends or at least three enemies at the party. Now let's consider the case when Lisa has at least three friends and look at Lisa's friends' relationships. If any two of Lisa's friends are friends with one another, then there is a group of three friends at the party (the same goes for when she has two enemies that are enemies with one another). If none of Lisa's three friends are friends with one another, then those friends create a group of three enemies (the same goes for when Lisa has three enemies if they are all friends with one another). No matter what, there will always be a group of three mutual friends or three mutual enemies, and so we must invite at least six people to the party to ensure this occurrence.

This situation is known as The Party Problem. The Party Problem is a classical example of a field of mathematics called Ramsey theory. Ramsey theory is all about finding the smallest configuration of objects so that a specific structure must occur among those objects [4].

1.1 Graph Theory Notation

Ramsey theory problems are often solved using techniques of Graph Theory. To continue this discussion on Ramsey theory, it will be helpful to know the following definitions from Graph Theory.

Definition 1. *[4] A* ***graph*** $G = (V(G), E(G))$ *is a pair of sets, a vertex set,* $V(G)$*, and an edge set,* $E(G)$*. A* ***vertex*** v *is drawn as a point and an* ***edge*** $e = uv$ *is drawn as an arc connecting the vertices* u *and* v*.*

A graph G is shown in Figure 1.

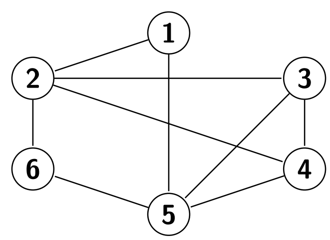


Figure 1: Graph G

In Figure 1, the vertices of G are labeled $1, 2, 3, 4, 5$ and 6. Thus, the vertex set is $V(G) = \{1, 2, 3, 4, 5, 6\}$. The edges are the lines connecting the vertices. For example, since there is a line connecting vertex 1 to vertex 2, the edge 12 will be in the edge set of G. For the graph G, the edge set is $E(G) = \{12, 15, 23, 24, 26, 34, 35, 45, 56\}$.

Definition 2. *[4] The number of vertices of a graph* G *is called the* ***order*** *of* G*, while the number of edges is its* ***size****.*

In Figure 1, the order of G is 6 since there are six vertices. The size of the graph is 9 since there are nine edges.

Definition 3. *[4] For a graph* G*, two vertices* u *and* v *are said to be* ***adjacent*** *if* $uv \in E(G)$*.*

Definition 4. *[4] For a graph* G*, a vertex* u *and an edge* e *are said to be* ***incident*** *if* $e = uv$ *is an edge in* G *for some vertex* $v \in V(G)$*.*

Vertices are adjacent if there is an edge between them. For example, in Figure 1, vertex 2 is adjacent to vertex 4 since 24 is in the edge set. Vertices 2 and 4 are incident to edge 24.

Definition 5. *[4] The **degree of a vertex** v in graph G is the number of edges incident with v and is denoted by* $\deg v$.

In Figure 1, vertex 5 has degree 4, vertex 1 has degree 2, and vertex 4 has degree 3.

Next, we will look at some results from graph theory that will be helpful when finding Ramsey numbers.

Theorem 1. *[The First Theorem of Graph Theory][4] If G is a graph of size m, then* $\sum_{v \in V(G)} \deg(v) = 2m$.

The First Theorem of Graph Theory states that for a graph with m edges, the sum of the degrees of all vertices is equal to $2m$. This means that the sum of the degrees of all the vertices is always an even number. The following Corollary stems from this result.

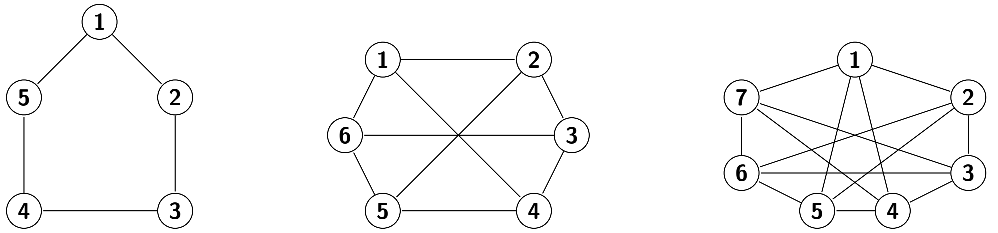
Corollary 1.1. *Every graph has an even number of odd degrees.*

Since the sum of all of the degrees of all vertices in the graph is even, we can conclude that if we have odd vertices, there must be an even number of them. If we had an odd number of odd vertices, the sum of the degrees would be odd, which contradicts Theorem 1.

Next, we will see some specific graphs that will be useful when finding Ramsey numbers.

Definition 6. *[4] If* $\deg v = r$ *for every vertex v of graph G with order n, where* $0 \leq r \leq n-1$, *then G is **r-regular**.*

In Figure 2, there is a 2-regular graph, a 3-regular graph, and a 4-regular graph. We could have, for example, made a 2-regular graph in Figure 2b by not including edges 14, 25 and 36. Through this example we see that there are different r-regular graphs based on the order of the graph. We also want to note that we can not make a 3-regular graph on 5 vertices in Figure 2a, or a 5-regular graph on 7 vertices in Figure 2c. This comes from our previous result in Corollary 1.1. This leads us to our next result.



(a) 2-regular graph on 5 vertices

(b) 3-regular graph on 6 vertices

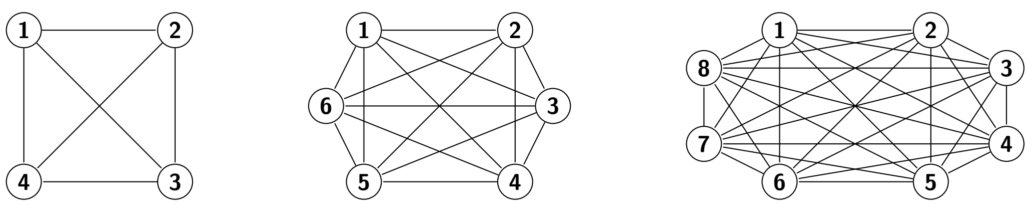
(c) 4-regular graph on 7 vertices

Figure 2: r-regular graphs

Theorem 2. *[4] Let r and n be integers with $0 \le r \le n-1$. Then there exists an r-regular graph of order n if and only if at least one of r and n is even.*

Theorem 2 states that there will only be an r-regular graph on n vertices if r or n is even. We will now look at another graph called the complete graph.

Definition 7. *[4] A graph G is* ***complete*** *if every two distinct vertices of G are adjacent. We denote a complete graph by K_n where n is the number of vertices.*



(a) K_4: Complete graph of order 4

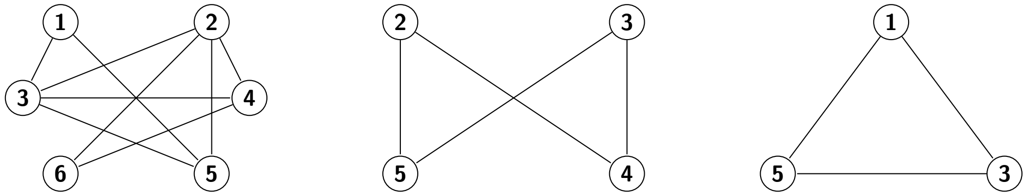
(b) K_6: Complete graph of order 6

(c) K_8: Complete graph of order 8

Figure 3: Complete Graphs

In each of the graphs in Figure 3, every vertex is adjacent to all other vertices of the vertex set. Thus, all of these graphs are complete graphs. K_4 is the complete graph on 4 vertices, K_6 is the complete graph on 6 vertices, and K_8 is the complete graph on 8 vertices.

Definition 8. *[4] A graph H is called a* ***subgraph*** *of a graph G, written $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.*



(a) Graph G (b) Subgraph H (c) Subgraph K

Figure 4

In Figure 4, we see a graph G and two subgraphs, H and K. Subgraph H is a subgraph of G since $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Also, subgraph K would be a subgraph since $V(K) \subseteq V(G)$ and $E(K) \subseteq E(G)$. In other words, all of the vertices in H and K are also in G and all of the edges in H and K are also in G. We note that K is also the complete graph of order 3, K_3, also called a **triangle**.

Definition 9. *[8] An* ***edge-coloring*** *of a graph is an assignment of a color to each edge of the graph. A graph that has been edge-colored is called a* ***monochromatic graph*** *if all of its edges are the same color. An edge coloring that uses k colors is also called a* ***k-edge coloring*** *or a* ***k-coloring****.*

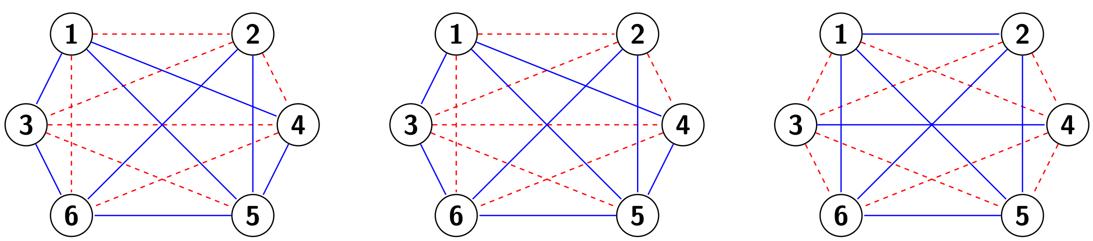


Figure 5: Three different 2-edge colorings of K_6

There are many different ways to color the edges of a graph. In Figure 5, we see three different ways to color the edges of K_6 using two colors. We can also say these are three red-blue colorings of K_6.

2 Ramsey Theory

2.1 History

Ramsey theory is named after the mathematician Frank Plumpton Ramsey. He was born February 22, 1903 and impressed many scholars at a young age. Ramsey went to Trinity College in Cambridge at the age of sixteen where he drew the interest of one of the most famous economists at the time, John Maynard Keynes. Even though Ramsey was interested in a wide range of subjects, with the encouragement of Keynes, most of Ramseys publications focused on mathematics, mathematical economics, and logic. Ramsey theory came from a result published in one of these publications. In his 1930 paper *On a Problem of Formal Logic*, Ramsey aimed to understand and build upon the ideas about logic from David Hilbert and from the ideas of Bertrand Russell and Alfred Whitehead in *Principia Mathematica*. The theorem that now carries Ramsey's name was in this paper as just a lemma! Unfortunately, Ramsey died at the age of 26, before the paper and the lemma that is now known as Ramsey Theorem was even published [5].

2.2 Ramsey's Theorem

After Ramsey died, many mathematicians started exploring more about this lemma published in his book. This work grew into the field of Ramsey theory. To begin exploring Ramsey theory, we will be looking at 2-colorings of complete graphs.

Definition 10. *A* ***(classical) Ramsey Number*** $R(p, q)$ *is defined to be the smallest integer* n *for which any 2-coloring of* K_n *in red and blue contains a monochromatic red* K_p *or a monochromatic blue* K_q*.*

Recall the Party Problem discussed in an earlier section. In this problem, we found that the minimum number of people to invite to a party to ensure there will be a group of three friends or three enemies was 6 people. We can model this problem and solution through graphs. In this graph, we are treating the vertices as people and the edges as their relationships, a red dotted line representing friends and a blue solid line representing enemies. In Figure 6, we see a red-blue coloring of K_5 that shows there is an instance where we do not get a group of 3 friends or 3 enemies when we invite 5 people. In this graph, we cannot find a group of three friends, in other words, we can't find a red dotted complete graph of order 3. In addition, we can't find a group of three enemies, that is, a blue solid complete graph of order 3. This means there is a red-blue coloring of K_5 with no red K_3 subgraph and no blue K_3 subgraph. So we know $R(3, 3) > 5$.

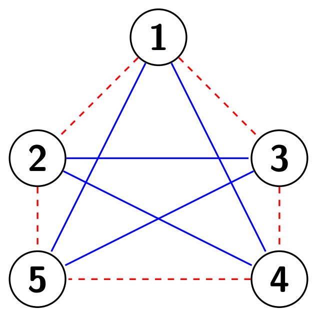


Figure 6: Two coloring of K_5

We also showed that when you invite 6 people to a party, there will always be a group of 3 friends or 3 enemies. The graph in Figure 7 gives us an example of a 2-coloring of K_6.

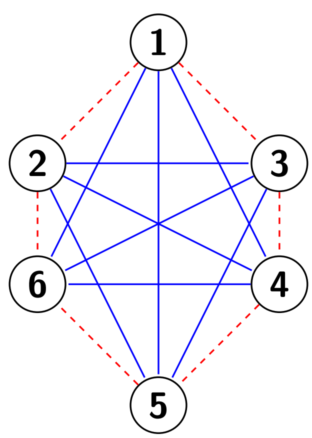


Figure 7: Two coloring of K_6

This is just one example of a 2-coloring of K_6. As we can see in this graph, there is more than one blue solid complete subgraph of order 3, for example, the subgraph containing vertices 1,4, and 6. Thus, this graph shows that there is at least one group of three enemies at the party. In fact, we can show that every 2-coloring of K_6 will contain either a red K_3 subgraph or a blue K_3 subgraph using reasoning similar to what we saw in the Party Problem. This implies $R(3,3) \leq 6$. Since we also have $R(3,3) > 5$, we know $R(3,3) = 6$. We see a formal proof that $R(3,3) = 6$ below.

Theorem 3. $R(3,3) = 6$.

Proof. First, we will show $R(3,3) > 5$. Consider the two coloring of K_5 in Figure 6. With this coloring, we will not have a red triangle or a blue triangle. Thus, $R(3,3) > 5$. Next, we will show $R(3,3) \leq 6$. Consider a two-coloring of K_6 and one vertex, say v_1. By the Pigeonhole Principle, we know that at least 3 vertices, say v_2,v_3, and v_4, are connected to v_1 with red edges or blue edges. Without loss of generality, suppose that edges v_1v_2, v_1v_3, and v_1v_4 are colored red. If any of the edges v_2v_3, v_2v_4, or v_3v_4 are colored red, then we have found a red K_3. If none of these edges are colored red, then they must be colored blue. Thus, we have found a blue K_3. So, for any two-coloring of K_6, there will always be a red K_3 subgraph or a blue K_3 subgraph, Hence, $R(3,3) \leq 6$.

So, since $R(3,3) > 5$ and $R(3,3) \leq 6$, we can conclude that $R(3,3) = 6$. □

In this proof, we see a common technique for proving Ramsey numbers. In general, we will prove the Ramsey number $R(n,m) = k$ by first showing $R(n,m) > k-1$ and then showing $R(n,m) \leq k$. We show $R(n,m) > k-1$ by giving a counterexample of a red-blue coloring of K_{k-1} that does not contain either a red K_n or a blue K_m. We show $R(n,m) \leq k$ by supposing every red-blue coloring of K_k does not contain either a red K_n or a blue K_m and reach a contradiction. Since every complete graph of greater order than k will contain K_k, all complete graphs of order k or greater will contain a red K_n or a blue K_m. So, by showing this always happens with K_k, we have showed $R(n,m) \leq k$.

The next logical question after finding one Ramsey number is if we can find other Ramsey numbers. Even though these numbers are difficult to find, Ramsey's Theorem states that every classical Ramsey number does in fact exist.

Theorem 4 (Ramsey's Theorem for Two Colors [8])**.** *Let $n, m \geq 2$. There exists a least positive integer $R = R(n, m)$ such that every edge-coloring of K_R, with the colors red and blue, admits either a red K_n subgraph or a blue K_m subgraph.*

This theorem states that every Ramsey number for two colors does exist. This theorem also expands to a more general version that states that Ramsey numbers exist even when we use more than 2 colors. For this thesis, we will focus on two colorings. However, even though $R(n, m)$ exists, the values of very few Ramsey numbers are actually known.

Even though finding Ramsey numbers are difficult, we can find some values and formulas for small values of n and m. We will prove that $R(2, q) = q$ for any $q \geq 2$.

Theorem 5. *[4] $R(2, q) = q$ for all $q \geq 2$.*

Proof. Let $q \geq 2$.

Consider a red-blue edge-coloring of a complete graph of order $q - 1$ where all edges are colored blue. Then, we have neither a red K_2 or a blue K_q. Thus, $R(2, q) > q - 1$.

Consider a red-blue edge-coloring of K_q such that it includes at least one red edge. Then, we have a complete red subgraph of order 2. On the other hand, consider a red-blue coloring that contains no red edges. Then, all edges must be blue and we have a complete blue subgraph of order q. Thus, K_q will contain either a red K_2 or a blue K_q. Hence, $R(2, q) \leq q$.

Thus, since we showed $R(2, q) > q - 1$ and $R(2, q) \leq q$, we can conclude $R(2, q) = q$ for all $q \geq 2$. ∎

2.3 Known Ramsey Numbers and Bounds

So far, we have given proofs that showed $R(3,3) = 6$ and $R(2,q) = q$ for all $q \geq 2$. Figure 8 gives a list of the known values and bounds for $R(n,m)$ for $3 \leq n, m \leq 10$. The bottom is not filled out because $R(n,m) = R(m,n)$.

n\m	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28	36	40-42
4		18	25	36-41	49-61	59-84	73-115	92-149
5			43-48	58-87	80-143	101-216	133-316	149-442
6				102-165	115-298	134-495	183-780	204-1171
7					205-540	217-1031	252-1713	292-2826
8						282-1870	329-3583	343-6090
9							565-6588	581-12677
10								798-23556

Figure 8: [9] Table of Values and Bounds for $R(n,m)$ for $3 \leq n, m \leq 10$

To illustrate how hard finding Ramsey numbers are, Paul Erdös famously said,

"Aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshall the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack" [6].

3 Modified Ramsey Numbers for Star Graphs

As we have seen, classical Ramsey numbers are extremely difficult to find. An interesting change is to look at different types of subgraphs other than complete graphs. We will investigate modified Ramsey numbers involving star graphs.

3.1 Notation

To aid in our understanding of modified Ramsey numbers, we will first look at a few more definitions from Graph theory.

Definition 11. *[8] Given two graphs G and H, a **modified Ramsey number**, denoted $R(G, H)$, is the smallest value of n such that any 2-coloring of the edges of K_n contains either a red copy of G or a blue copy of H.*

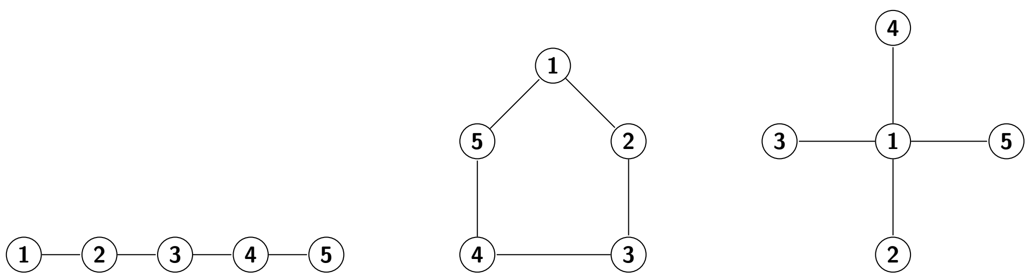
The classical Ramsey number $R(p, q)$ would in this context be written as $R(K_p, K_q)$.

Definition 12. *[8] A **path** in a graph G is a sequence of distinct vertices $v_1, v_2, ..., v_k$ such that $v_i v_{i+1} \in E(G)$ for $i = 1, 2, .., k - 1$. The **path graph** P_n is a path on n vertices.*

In Figure 9a, we can see P_5 is the path graph on 5 vertices.

Definition 13. *[4] If the vertices of a graph G of order $n \geq 3$ can be labeled $v_1, v_2, ..., v_n$, so that its edges are $v_1v_2, v_2v_3, ...v_nv_1$, then G is called a **cycle** and is denoted C_n.*

In Figure 9b, we see a cycle of order 5, denoted C_5.



(a) P_5 (b) C_5 (c) S_5

Figure 9: Path, Cycle, and Star graphs

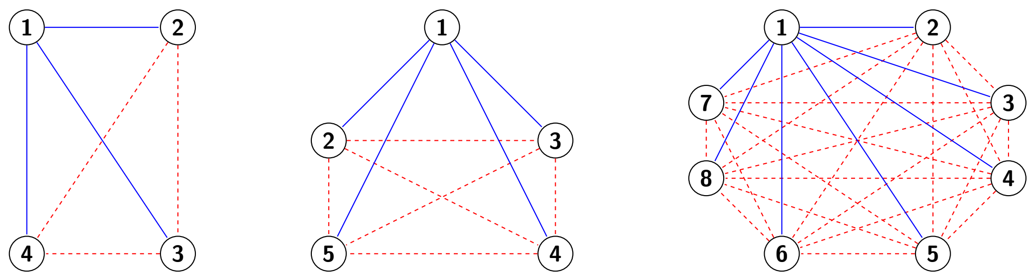
Definition 14. *[4] A **star graph**, denoted S_n is a graph with n vertices with one node having degree $n - 1$ and the other $n - 1$ nodes having degree 1.*

In Figure 9c, we have S_5, the star graph of order 5.

Now that we have these graphs, we will look for subgraphs of these types instead of complete graphs.

3.2 $R(S_n, S_m)$

The modified Ramsey number $R(S_n, S_m)$ is the smallest integer a such that any two-coloring of K_a in red and blue contains either a red S_n or a blue S_m. In Figure 10, we can see a few examples of complete graphs with blue star subgraphs.



(a) Two-colored K_4 with blue S_4

(b) Two-colored K_5 with blue S_5

(c) Two-colored K_8 with blue S_8

Figure 10: Different 2-edge colorings with star subgraphs

We will first prove a general formula for the modified Ramsey number $R(S_3, S_m)$.

Theorem 6. *For $m \geq 2$,*

$$R(S_3, S_m) = \begin{cases} m & \text{if } m \text{ is odd} \\ m+1 & \text{if } m \text{ is even.} \end{cases}$$

Proof. Suppose $m \geq 2$.

Case 1: m is odd

Suppose we have a complete graph of order $m-1$. Consider a 2-coloring for which every edge of the graph is blue, and so, every vertex will be incident to $m-2$ blue edges. Thus, we have found a 2-coloring of K_{m-1} that does not have a red S_3 or a blue S_m. Thus, $R(S_3, S_m) > m-1$.

Suppose we have a complete graph of order m. Suppose, by means of contradiction, that there is a two-coloring of K_m that has no red S_3 subgraph and no blue S_m subgraph. Consider one vertex, v_1. Since there is no red S_3 subgraph, v_1 must be incident to at most one red edge. Thus, v_1 will be incident to at least $m-2$ blue edges. Since there is no blue S_m subgraph, v_1 must be incident to at most $m-2$. Hence, we have found that v_1 must be incident to exactly one red edge and $m-2$ blue edges. Since there is no red S_3 subgraph and no blue S_m subgraph, every vertex must be incident to exactly one red edge and $m-2$ blue edges. Consider the subgraph consisting of all the blue edges. Since m is odd, and this subgraph has m vertices of degree $m-2$, we get a contradiction because we cannot have a graph with an odd number of odd vertices, Corollary 1.1. So at least one vertex will either be incident to 0 red edges or incident to 2 or more red edges, ensuring a blue S_m subgraph or a red S_3 subgraph respectively. Thus we have reached a contradiction and so $R(S_3, S_m) \leq m$.

Thus, since $R(S_3, S_m) > m-1$ and $R(S_3, S_m) \leq m$, we have found that $R(S_3, S_m) = m$.

Case 2: m is even.

Suppose we have a complete graph of order m. Color the edges so that each vertex has exactly one incident red edge. Thus, each vertex is incident to one red edge, and is incident to $m-2$ blue edges. Therefore, we have found a two-coloring of K_m that does not have either a red S_3 or a blue S_m. Hence, $R(S_3, S_m) > m$.

Consider the complete graph K_{m+1}. Suppose by means of contradiction that there is a 2-coloring of K_{m+1} that has no red S_3 subgraph and no blue S_m subgraph. Consider one vertex, v_1. Since there is no red S_3 subgraph, v_1 is incident to at most one red edge. Then, v_1 is incident to at least $m-1$ blue edges. Thus, there is a blue S_m, a contradiction. Hence $R(S_3, S_m) \leq m+1$.

Thus, since $R(S_3, S_m) > m$ and $R(S_3, S_m) \leq m+1$, we have found that $R(S_3, S_m) = m+1$. □

We have proved a general formula for $R(S_3, S_m)$ so next we will move on to look at the modified Ramsey number when one graph is S_4. We will next find a value for $R(S_4, S_m)$.

Theorem 7. *For $m \geq 2$, $R(S_4, S_m) = m + 2$.*

Proof. Let $m \geq 2$.

Consider a red-blue coloring of K_{m+1} so that the red subgraph is C_{m+1}. Thus, each vertex is incident to 2 red edges and $m - 2$ blue edges. Therefore, we have found a two-coloring of K_{m+1} that does not have a red S_4 or a blue S_m, so $R(S_4, S_m) > m + 1$.

Now, suppose by means of contradiction that there is a 2-coloring of K_{m+2} that has no red S_4 subgraph and has no blue S_m subgraph. Consider one vertex v_1. Since there is no red S_4 subgraph, v_1 must be incident to at most 2 red edges. But, that means v_1 will be incident to at least $m - 1$ blue edges, which gives a blue S_m subgraph. Hence we have reached a contradiction and so, $R(S_4, S_m) \leq m + 2$.

We have showed that $R(S_4, S_m) > m + 1$ and $R(S_4, S_m) \leq m + 2$ and so we have found that $R(S_4, S_m) = m + 2$. □

Next, we will prove a general formula for $R(S_5, S_m)$.

Theorem 8. *For $m \geq 2$,*

$$R(S_5, S_m) = \begin{cases} m + 2 & \text{if } m \text{ is odd} \\ m + 3 & \text{if } m \text{ is even.} \end{cases}$$

Proof. Suppose $m \geq 2$.

Case 1: m is odd

Consider a red-blue coloring of K_{m+1} so that the red subgraph is C_{m+1}. Thus, each vertex is incident to 2 red edges and $m - 2$ blue edges. Therefore, we have found a two-coloring of K_{m+1} that does not have a red S_5 or a blue S_m, so $R(S_5, S_m) > m + 1$.

Consider the graph K_{m+2}. Suppose by means of contradiction that there is a 2-coloring of K_{m+2} that does not have a red S_5 subgraph and does not have a blue S_m subgraph. Consider one vertex, v_1. Since there is no red S_5 subgraph, v_1 is incident to at most 3 red edges. Thus, v_1 is incident to at least $m - 2$ blue edges. But, since there is no blue S_m subgraph, v_1 can be incident to at most $m-2$ blue edges. Therefore, v_1 must be incident to exactly $m-2$ blue edges and exactly 3 red edges. Since this 2-coloring has no red S_5 and no blue S_m, all vertices must be incident to exactly $m-2$ blue edges and exactly 3 red edges. Consider the red subgraph. Note that the degree of each vertex in the subgraph is 3. Since m is odd, the red subgraph has an odd number of odd vertices, which is not possible by Corollary 1.1. Hence we have reached a contradiction and so $R(S_5, S_m) \leq m+2$.

Thus, since $R(S_5, S_m) > m+1$ and $R(S_5, S_m) \leq m+2$, we have found that $R(S_5, S_m) = m+2$.

Case 2: m is even

Consider a red and blue coloring of K_{m+2} so that every vertex is incident to 3 red edges and $m-2$ blue edges. The red subgraph makes up a 3-regular graph on m vertices and the blue subgraph makes up a $(m-2)$-regular graph on m vertices. So, by Theorem 2, we are able find this 2-coloring of K_{m+2}. Thus, we have found a two-coloring of K_{m+2} that does not have a red S_5 or a blue S_m, so, $R(S_5, S_m) > m+2$.

Now, consider the graph K_{m+3}. Suppose by means of contradiction that there is a 2-coloring of K_{m+3} that does not have a red S_5 subgraph and does not have a blue S_m subgraph. Consider one vertex, v_1. Since there is no red S_5 subgraph, v_1 is incident to at most 3 red edges. Hence, v_1 is incident to at least $m-1$ blue edges. But we have reached a contradiction because this guarantees a blue S_m subgraph. So, $R(S_5, S_m) \leq m+3$.

Thus, since $R(S_5, S_m) > m+2$ and $R(S_5, S_m) \leq m+3$, we have found that $R(S_5, S_m) = m+3$. □

We have proved three general formulas for modified Ramsey numbers with one of the graphs being fixed. By looking at patterns emerging in these results, we are able to generalize a formula to give us the value for any modified Ramsey number of two star graphs. Now, we will prove the general formula for $R(S_n, S_m)$.

Theorem 9. *If $n, m \geq 2$,*

$$R(S_n, S_m) = \begin{cases} n+m-3 & \text{if } n \text{ and } m \text{ are both odd} \\ n+m-2 & \text{if at least one of } n \text{ and } m \text{ is even} \end{cases}$$

Proof. Let $n, m \geq 2$.

Case 1: n and m are both odd

Consider a red-blue coloring of K_{n+m-4} so that every vertex is incident to $n-2$ red edges and $m-3$ blue edges. Since n and m are both odd, we have that $n+m-4$ is even. So, the red subgraph is a $(n-2)$-regular graph on an even number of vertices, and the blue subgraph is a $(m-3)$-regular graph on an even number of vertices. So, by Theorem 2, we are able to find this 2-coloring of K_{n+m-4}. Hence, we have found a two-coloring of K_{n+m-4} that does not have a red S_n subgraph or a blue S_m subgraph, so, $R(S_n, S_m) > m+n-4$.

Consider the graph K_{n+m-3}. Suppose by means of contradiction that there is a 2-coloring of K_{n+m-3} that does not have a red S_n subgraph and does not have a blue S_m subgraph. Consider one vertex, v_1. Since there is no red S_n subgraph, v_1 is incident to at most $n-2$ red edges, and thus, at least $m-2$ blue edges. Since there is no blue S_m subgraph, v_1 must be incident to at most $m-2$ blue edges, and at least $n-2$ red edges. Thus, to have neither a red S_n or a blue S_m, v_1 must be incident to exactly $n-2$ red edges and $m-2$ blue edges. Similarly, all of the vertices must be incident to the same amount of red and blue edges to avoid having a red S_n or a blue S_m. Consider the red subgraph. Note that the degree of each vertex in the subgraph is $n-2$, which is an odd number since n is an odd number. The number of vertices in this subgraph is $n+m-3$ which is also odd since n and m are both odd. But, we have reached a contradiction because we cannot have a graph with an odd number of odd vertices by Corollary 1.1. Hence, $R(S_n, S_m) \leq m+n-3$.

Thus, when n and m are both odd, $R(S_n, S_m) = m+n-3$.

Case 2: At least one of n and m is even

Consider a red and blue coloring of K_{n+m-3} so that every vertex is incident to $n-2$ red edges and $m-2$ blue edges. If exactly one of n and m is even, then $n+m-3$ is even. So, the red subgraph is a $(n-2)$-regular graph on an even number of vertices and the blue subgraph is a $(m-2)$-regular graph on an even number of vertices. If both n and m are even, then $n-2$ and $m-2$ are even. So, the red subgraph is a $(n-2)$-regular graph on $n+m-3$ vertices and the blue subgraph is a $(m-2)$-regular graph on $n+m-3$ vertices. Thus, by Theorem 2, we are able to find this 2-coloring of K_{n+m-3}. Hence, we have found a two-coloring of K_{n+m-3} that does not have a red S_n subgraph or a blue S_m subgraph, so $R(S_n, S_m) > m+n-3$.

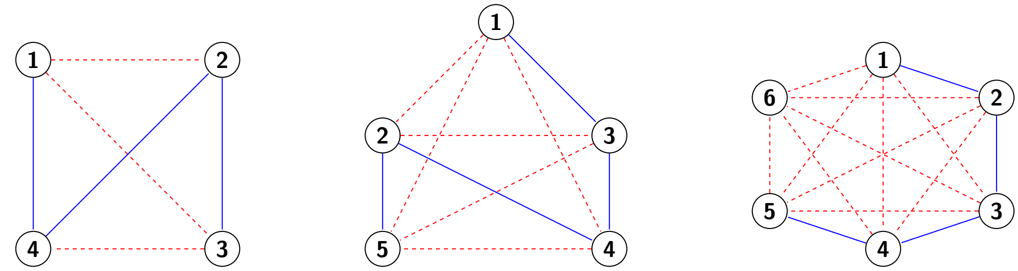
Consider a 2-coloring of the graph K_{n+m-2} and one vertex, say v_1. Suppose v_1 is incident to $(n-1)$ or more red edges. Then, we have a red S_n. Now suppose v_1 is incident to $(n-2)$ or fewer red edges. Then, v_1 will be incident to $m+n-3-(n-2) = m-1$ or more blue edges. Then we have a blue S_m. Therefore, every two-coloring of a complete graph of order $m+n-2$ will have either a red S_n or a blue S_m.

Thus, we have showed when at least one of n and m is even, $R(S_n, S_m) = m+n-2$. □

We have now proved a general formula to find the modified Ramsey number for two star graphs of any order. Next, we will consider the modified Ramsey number of two different graphs - a star graph and a path graph.

3.3 $R(S_n, P_m)$

We have explored classical Ramsey numbers $R(K_n, K_m)$ and the modified Ramsey numbers for two star graphs, $R(S_n, S_m)$. These were two examples of Ramsey numbers where we were looking for the same type of subgraph. We will now explore the use of two different types of graphs - the star graph and the path. The modified Ramsey number $R(S_n, P_m)$ is the smallest integer a such that any two-coloring of K_a in red and blue contains either a red S_n or a blue P_m. In Figure 11, we see a few examples of complete graphs with red star subgraphs and blue path subgraphs.



(a) Star and Path subgraphs in K_4

(b) Star and Path subgraphs in K_5

(c) Star and Path subgraphs in K_6

Figure 11: Star and Path Subgraphs

In Figure 11, graph (a) contains a red S_3 and a blue P_4. In graph (b), we can find a red S_4 and a blue P_5. Also, graph (c) contains a red S_5 and a blue P_5.

We will now look at proofs for the values of a few modified Ramsey with a star subgraph and a path subgraph.

Theorem 10. *For $n \geq 2$, $R(S_n, P_2) = n$.*

Proof. Let $n \geq 2$.

Suppose we have a complete graph of order $n-1$ with every edge colored red. Then we have found a complete graph of order $n-1$ that does not contain a red S_n or a blue P_2 and so, $R(S_n, P_2) > n-1$.

Suppose we have a complete graph of order n. Suppose, by means of contradiction, that K_n has no red S_n subgraph and no blue P_2 subgraph. Consider one vertex, say v_1. Since there is no red S_n subgraph, v_1 must be incident to at most $n-2$ red edges. Since there is no blue P_2 subgraph, we know v_1 must be incident to 0 blue edges. But since v_1 is incident to $n-1$ edges, we have reached a contradiction with v_1 being incident to at most $n-2$ red edges and no blue edges. Hence, $R(S_n, P_2) \leq n$.

Thus, since $R(S_n, P_2) > n-1$ and $R(S_n, P_2) \leq n$, we have found that $R(S_n, P_2) = n$. □

By the definition of a star graph and a path graph, we have that S_2 is isomorphic to P_2. Thus, we should see that $R(S_n, P_2) = R(S_n, S_2)$, which we can confirm using Theorem 9.

Next, we will show that $R(S_3, P_3) = 3$.

Theorem 11. $R(S_3, P_3) = 3$.

Proof. Suppose we have a complete graph of order 2 with all edges colored red. Thus, we have found a complete graph of order 2 that does not contain a red S_3 or a blue P_3 and so, $R(S_3, P_3) > 2$.

Next, we will show $R(S_3, P_3) \leq 3$. Suppose we have a complete graph of order 3. Suppose we color all edges red or all edges blue. Then we have a red S_3 or a blue P_3 respectively. Suppose we color one edge red and two edges blue. Then we have a blue P_3. Suppose we color one edge blue and two edges red. Then, we have found a red S_3. Since those are all of the different ways to use two colors to color K_3, we have found that $R(S_3, P_3) \leq 3$.

Thus, since $R(S_3, P_3) > 2$ and $R(S_3, P_3) \leq 3$, we have found that $R(S_3, P_3) = 3$. $\square$

Since we have that S_3 is isomorphic to P_3, we should see that $R(S_3, P_3) = R(S_3, S_3)$, which we can confirm using Theorem 9.

We will now show that $R(S_4, P_4) = 5$.

Theorem 12. $R(S_4, P_4) = 5$.

Proof. Consider the complete graph of order 4 in Figure 12.

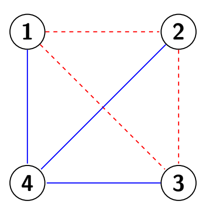


Figure 12: Red-blue coloring of K_4

Thus, we have found a two-coloring of K_4 such that there is no red S_4 and no blue P_4. Hence, $R(S_4, P_4) > 4$.

Suppose we have a complete graph of order 5. Suppose by means of contradiction that there is a red-blue coloring of K_5 that does not contain a red S_4 or a blue P_4. Since this two-coloring contains no red S_4, each vertex is incident to at most 2 red edges. Consider a vertex v_1.

Case 1: Suppose v_1 is incident to two red edges. Thus, v_1 is incident to exactly two blue edges. Without loss of generality, assume v_2 and v_3 are adjacent to v_1 with a blue edge. If either of these vertices are adjacent to v_4 or v_5 with a blue edge, then we have a blue P_4. Thus, all of the edges incident to v_2 and v_3, besides v_2v_3, must be colored red. Now, if we look at either of the remaining two vertices, say v_4, we will see v_4 is connected to v_1, v_2, and v_3 with red edges. Thus we have found a red S_4.

Case 2: Suppose v_1 is incident to one red edge. Thus, v_1 is incident to exactly three blue edges. Without loss of generality, assume v_2, v_3, and v_4 are adjacent to v_1 with a blue edge. If any of these vertices are adjacent to v_5 with a blue edge, then we have a blue P_4. Thus, all edges incident to v_5 must be red, which gives us a red S_4.

Case 3: Suppose v_1 is incident to no red edges. Thus, v_1 is incident to exactly four blue edges. Let's look at the four vertices, say v_2, v_3, v_4, and v_5, that are connected to v_1 with a blue edge. Consider the edges between v_2, v_3, v_4, and v_5, and suppose at least one of these edges is colored blue. Without loss of generality, suppose v_2v_3 is a blue edge. Then, $v_4v_1v_2v_3$ is a blue P_4. Thus, all edges incident to these four vertices must be colored red. Now, v_2 is incident to three red edges which gives a red S_4.

So, we have reached a contradiction. Hence, $R(S_4, P_4) \leq 5$.

Thus, since $R(S_4, P_4) > 4$ and $R(S_4, P_4) \leq 5$, we have that $R(S_4, P_4) = 5$. □

We need other techniques to prove modified Ramsey numbers for larger star and path graphs, so we will state no general formula right now.

4 Conclusion

In this paper, we have shown the following results.

1. $R(S_n, S_m) = \begin{cases} n+m-3 & \text{if } n \text{ and } m \text{ are both odd} \\ n+m-2 & \text{at least one of } n \text{ and } m \text{ is even} \end{cases}$

2. $R(S_n, P_2) = n$

3. $R(S_3, P_3) = 3$

4. $R(S_4, P_4) = 5$

For these results, we used the same general techniques used in proving classical Ramsey numbers. To prove a Ramsey number $R(P, Q) = k$, for some graphs P, Q and for some number k, we first show that the Ramsey number is greater than $k - 1$, and then show $R(P, Q)$ is less than or equal to k. We show $R(P, Q) > k - 1$ by finding a complete graph of order $k - 1$ that does not contain either a red P or a blue Q. Next, we show $R(P, Q) \leq k$ by contradiction or by checking every coloring. However, the proofs get more difficult as the subgraphs change and grow in size and order which often cause them to need different proof techniques. Modified Ramsey numbers are a natural progression from classical Ramsey research and these results will add to the growing literature and research of Ramsey theory.

References

[1] Angeltveit, V., McKay, B., $R(5,5) \leq 48$, Pre-print arXiv.1703.068786, (2017).

[2] Buschur, K., *Introduction to Ramsey Theory*, unpublished manuscript, (2011).

[3] Chartrand, G., Benjamin, A., Zhang, P., How Math Puzzles Help You Plan the Perfect Party, *Scientific American*, (2017).

[4] Chartrand, G., and Zhang, P., *A First Course in Graph Theory*, Dover Publications. Inc., Mineola, 2012.

[5] Dickson, J., *An Introduction to Ramsey Theory*, Master's Thesis (2011).

[6] Graham, R., and Spencer, J., Ramsey Theory, *Scientific American*, (1990).

[7] Harris, J., Hirst, J., and Mossinghoff, M., *Combinatorics and Graph Theory*, Springer Science+Business Media, LLC, New York, 2008.

[8] Landman, B., and Robertson, A., *Ramsey Theory on the Integers*, American Mathematical Society, 2004.

[9] Radziszowski, S., Small Ramsey Numbers, *The Electronic Journal of Combinatorics*, Dynamic Surveys Ds1, (2017).