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The Archbishop's Odyssey

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Len Sprague is a junior pursuing a B.S. in Chemistry with an Environmental concentration. This research began as an idea, buried among many others, that thrust itself to the forefront with its seemingly endless torrent of questions. Ten weeks of long days and nights were spent consumed by these questions, with conjectures forming from the numerous avenues of exploration. Such work was made possible with the guidance of his mentor, Dr. Ward Heilman (Mathematics), and with funding from an Adrian Tinsley Summer Research Grant. All results of this research are meant to open a new branch of mathematical inquiry, with the intention of revealing ever more useful patterns and solutions to those unending questions. This work was presented at the fall 2013 Mathematical Association of America (MAA) Northeastern Sectional Meeting.

For centuries, scholars have analyzed a collection of problems that, nowadays, has been defined as NP-complete. Currently, NP-complete problems have no known efficient solutions. The Clay Mathematics Institute has offered a reward of one million dollars for a solution. The problem of finding Hamilton paths and cycles has been shown to be in this category. Knight's tours, where the knight must visit every square of a chessboard exactly once, are examples of Hamilton paths and cycles.

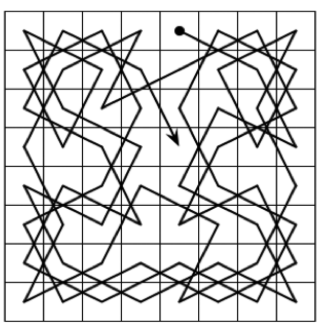
This research revolves around the creation of a new branch of the tour problems, through a new piece: the Archbishop. Chess Grandmaster José Capablanca created this piece, giving it the ability to move as either a Knight or a Bishop, to increase the complexity of Chess. Some of the questions investigated are: does the Archbishop have Hamilton cycles and paths on various size boards (not only 8x8 but 3x3, 4x4, ...)?, and, how many edges are there in the movement graphs of these boards?

One result used different counting arguments. The Archbishop, unlike the Knight, is not forced to "switch colors" on the checkered chess board, as it has the ability to move diagonally to a square of the same color. Therefore, it has the ability to tour a board with an odd number of squares. A second result was the equation for the number of edges the Archbishop movement graph has in relation to the size of the board: $6n^2 - 16n + 10$. With this finding also came the equation of a Bishop's number of edges: $2n^2 - 4n + 2$. Third, using graph theory, it was found that an Archbishop cannot complete a cycle on a 4x4 chess board. And fourth, using a cyclic solution of a 3x3 board, a solution for all $3n$x$3n$ boards was found by connecting the smaller solutions together. These findings suggest many new problems and present new opportunities for people to investigate.

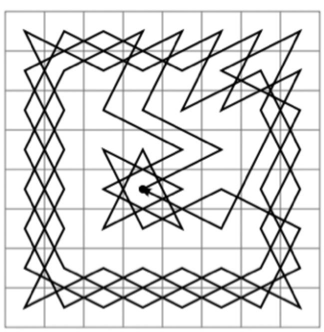
Introduction, Context, and Significance

Since the creation of the chess board in 9 A.D., its many pieces have given birth to new ideas and puzzles in mathematics. These puzzles were expansive enough to have sparked the creation of books specifically covering the topic of mathematics in chess and games similar to it (shogi, Go, checkers, etc…). One highly popular puzzle is the attempt at touring a chess board with a chess piece. This puzzle is described in Graph Theory as searching for a Hamilton path or cycle. Graph Theory is the study of graphs. A graph is defined by mathematicians as a set of objects, called vertices, and their connections, called edges, illustrated by lines linking them together. On a chess board, each square is a vertex, and the chosen piece's possible movements are the edges (Figure 1a & 1b, example edges; Figure 2, example vertices). Each vertex in a graph has a degree, which is the number of edges connected to it (a degree 5 vertex has 5 edges sprouting from it). Two vertices are said to be connected if there is an edge from one vertex to the other vertex. A Hamilton path is a sequence of connected vertices which contains every vertex exactly once, while a Hamilton cycle is a Hamilton path that, in addition, returns to the starting vertex. A graph with a Hamilton cycle is referred to as Hamiltonian. The Knight is the most studied of the chess pieces in this puzzle of touring, and is the most difficult one so far. The Knight became a piece of intrigue because its move differed so greatly from any of the other pieces, and became the object of greater study in the realm of Hamilton paths and cycles. In Figures 1a and 1b, examples of solved Knight's tours (a path and a cycle) are illustrated. Presently, there is no way of solving for a Knight's tour in an efficient amount of time, although strategies and theorems are known for determining the existence of a solution on a given board.

Figure. 1a & 1b



(Above left) Fig 1a-8x8 Knight's Path



(Above right) Fig 1b-8x8 Knight's Cycle

Note: Grid lines in later personal solutions have been removed to clear up the solution's image

One strategy is known as Warnsdorff's rule. The concept is simply to keep moving to squares with the least amount of possible future movements (therein removing the lowest degree vertices first, making travel later simpler). This is a well known strategy for solving Hamilton path and cycle problems. Two other examples of the strategies for determining if a cycle is possible are counting arguments, and the Rubber Band Theorem. A counting argument, described by John Watkins in his book *Across the Board*, consists of simply counting up of the number of white squares and black squares on a chess board, and comparing their values. Since the Knight must switch colors every move, the number of black and white squares must be the same in order to have the possible existence of a Hamilton cycle. This approach allows a quick and easy determination of a board of nxn size, without extensive work or trials: to have the same number of black and white squares, an mxn board must have mn even. That is, m and n cannot both be odd.

The Rubber Band Theorem involves taking the Knight's graph, and then slowly removing vertices and their connected edges. This may cause the graph to become disconnected and fall into a number of separate components. If the number of vertices removed is smaller than the number of components remaining, the graph is not Hamiltonian. *Introduction to Graph Theory* introduces this Theorem as Theorem 6.5, and illustrates its usefulness on larger problems (i.e., larger boards), and is therefore the second step in looking for a Hamilton cycle after use of the counting argument. The Theorem is as follows: if G is a Hamiltonian graph, then for every nonempty proper set S of vertices of G, $k(G - S) \leq |S|$ where $k(G)$ is equal to the components in graph G. The contrapositive is usually used to show that the graph is not Hamiltonian. In short, if we were to remove n vertices (from the set S), then in order to have a Hamilton cycle, the number of components after removal must be equal to or less then the number removed. This is similar to cutting a rubber band in two places, wherein we will have only a maximum of only two segments after the cutting. A rubber band cut in two places will not result in three separate segments, because the rubber band is a cycle. Similarly, if we cut out two vertices and are then left with three separate components, the original graph is not Hamiltonian. Along with these two strategies, one is also able to compare known solutions and known impossibilities to the problem at hand.

Puzzles such as the Knight's tour problem have gained enough interest to warrant a million dollar reward for a complete solution. These tour-problems fall into a category of problems called NP complete, which is specifically designated for problems lacking an efficient solution (defined as finding a solution in polynomial time). Problems such as route efficiency (which are either Hamiltonian or Eulerian questions) make up a portion of these NP complete problems, and if a pattern or calculation were known for one it can be transformed into a solution for other NP complete problems. The Clay Mathematics Institute has defined a set of seven problems, known as "The Millennium Prize Problems," each of which holds a million dollar reward for a fully fleshed out solution. Determining if NP complete problems have an efficient solution is one of these.

Despite all the interest in the Knight's tour problem and problems similar to it, there were some chess pieces left out of the math world's gaze that deserve attention as well. In the 1920's, José Capablanca, a chess grandmaster and, at the time, the World Chess Champion, extended the 8x8 chessboard (the standard chess board size) and created two new pieces, the Archbishop and the Chancellor, in order to increase the difficulty of the game of chess and prevent what he thought would soon be constant stalemates between Grandmasters. The Archbishop increased the complexity of the Knight's abilities since it could move as a Knight or a Bishop (Figure 2 illustrates this combination). When Capablanca lost the World Championship title the next year, his new pieces were almost forgotten. Yet these new abilities endowed to the Archbishop piece created a new puzzle filled with many new questions which should be explored.

George Polya's book *How to Solve it* was a very helpful tool and key factor in researching this rather complex problem. He focuses on the concept of breaking down larger problems into smaller questions and attempting to solve the smaller problems first. Research, therefore, began with a chess board of 3x3 squares instead of the traditional 8x8, to search for the Archbishop's Hamilton cycle.

Figure. 2

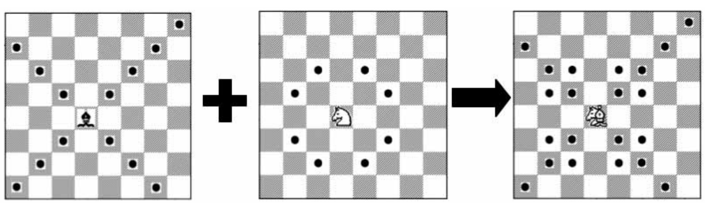


Figure 2. Bishop, Knight, and Archbishop Movement Patterns (Left to Right, respectively)

Findings

1. Smaller Solutions

Using a trial and error method, the following solutions of some Hamilton paths and cycles on boards of size 3x3, 4x4, 5x5, and 2x3 were found (Figures 3a and 3b). Larger solutions were found using a modified Warnsdorff's rule. Due to the exponential growth of the number of solutions a board will have, brute force methods become unfeasible, and abhorrent.

Figure. 3a & 3b

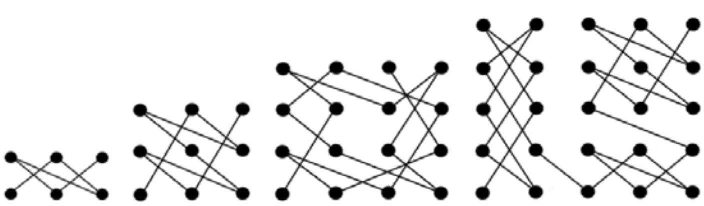


Figure 3a. 2x3, 3x3, 4x4, and 5x5 Archbishop Paths

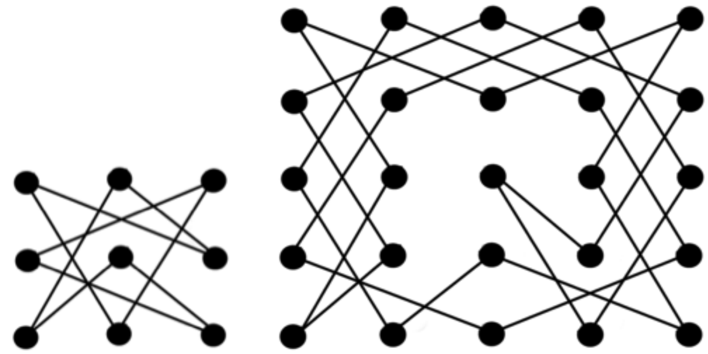
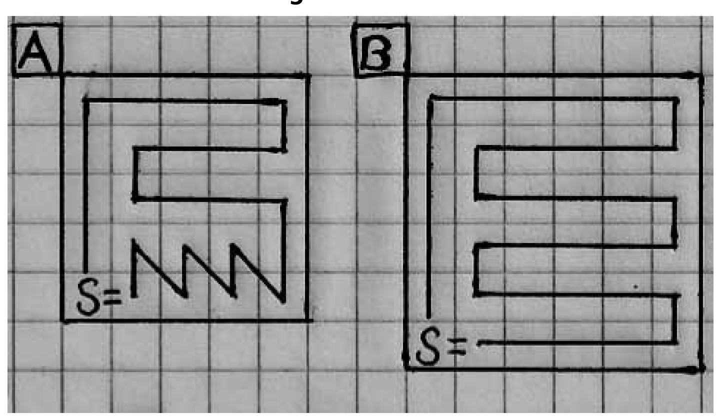


Figure 3b. 3x3 and 5x5 Archbishop Cycles

2. Larger solutions using smaller ones

With the above smaller solutions, larger ones can be solved using a sectioning method of a given board size, i.e., a 6x6 board is equivalent to connecting four 3x3 solutions together. In order to solve larger boards of 3*n*x3*n* size, the smaller solution can be used by breaking an edge of the cycle to make it a path, therein covering all the squares while allowing two squares to have a connection to other 3x3 sections of the 3*n*x3*n*. Each grid block in Figure 4a and 4b represents a 3x3 segment. Using all cycles (turned into the necessary path to continue movement to the next 3x3 segment), one can follow the directions shown in Figure 4a and 4b to complete a 3*n*x3*n* board, *n* being odd or even respectively.

Figure. 4a & 4b



(Above) Fig 4a and 4b-Odd/Even solution pattern

S = starting position

Starting from the corner 3x3 segment of a 3*n*x3*n* sized board, travelling the full length of segments in one direction, and then the full length 90 degrees from the first direction, weaving back and forth until returning to the original segment will complete the cycle. This pattern was discovered by following a modified Warnsdorff strategy, traveling around the edges of the board and working closest to the solved 3x3 grids to avoid missing a vertex.

3. Use of the counting argument in determining what size boards can be solved

Among the first and simplest methods of determining a chessboard's possibility of having a Hamilton cycle is through the aforementioned strategy of counting. To summarize, the idea is: when a Knight moves on the chess board with squares colored alternately black and white, it must switch to a different colored square every time (the Knight's only option is a black square when it moves from the white). Therefore, in order to visit every square only once and return to the start (creating a cycle), there must be an even number of black and white squares so that it can finish in a square that allows it to return to the starting square. Thus, there can be no Knight tours of $(2n+1)$x$(2n+1)$ boards.

However, since the Archbishop has the ability to move as a Knight or a Bishop, it does not have to switch colors every move (a Bishop movement keeps it on the same color). The Archbishop can, therefore, take up as many white or black squares in a row as the board will allow, meaning that the number of white and black squares overall does not have to be the same. So, the Archbishop has the ability to establish a Hamilton path or cycle on boards with an odd number of squares, as well as the possibility of one even if the numbers differ significantly. Figure 3b illustrates this with a cyclic route on a 5x5 board done with an Archbishop, whereas the Knight is unable to complete a cycle on a 5x5.

4. A 4x4 Has No Cycle (drawing graphs)

Graphs are useful in finding Hamilton paths and cycles in largely two ways. One, if the graph can be drawn such that the vertices and some edges create a circle (see Figure 6, with an Archbishop's graph on a 2x3 board), then it has a Hamilton cycle (exactly one, if no other edges exist that lie on the circle). The original graph may look like Figure 5, but the vertices upon shifting around can be manipulated into a circular shape if the cycle exists. On a 2x3 and a 3x3 grid, this circle is easily visible (Figure 6). And two, The Rubber Band Theorem uses graphs to quickly test for the lack of a Hamilton cycle.

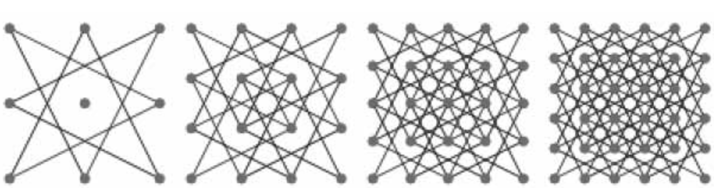


Figure 5. Example graphs of a Knight's possible movements

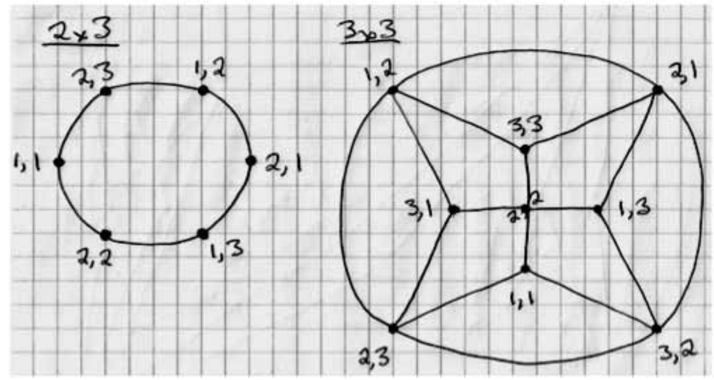


Figure 6. Manipulated forms of an Archbishop's graph: Left, 2x3; Right, 3x3.

Figure 7a shows the four vertices and their connected edges that were removed for the Rubber Band Test (circled). Figure 7b shows the five components of the graph that remain; one more component than the number of vertices removed. Therefore, the 4x4 Archbishop graph does not pass this Rubber Band Test, and therefore the Archbishop is unable to complete a Hamilton cycle on a 4x4 grid. However, as seen above in the 5x5 case, a Hamilton path is still possible. The Knight, however, also cannot complete a cycle on a 4x4 grid, and in turn the Archbishop may not be able to cycle every board size, but has the ability to cycle more than the Knight alone.

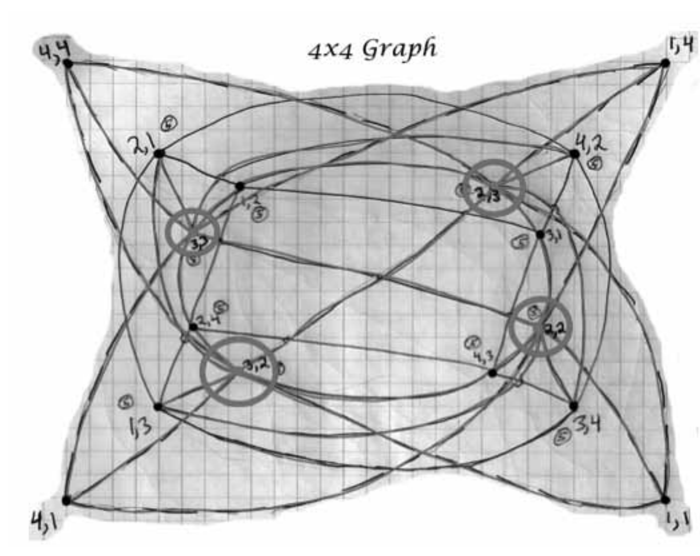


Figure 7a. Graph of 4x4 with vertices being deleted (Circled)

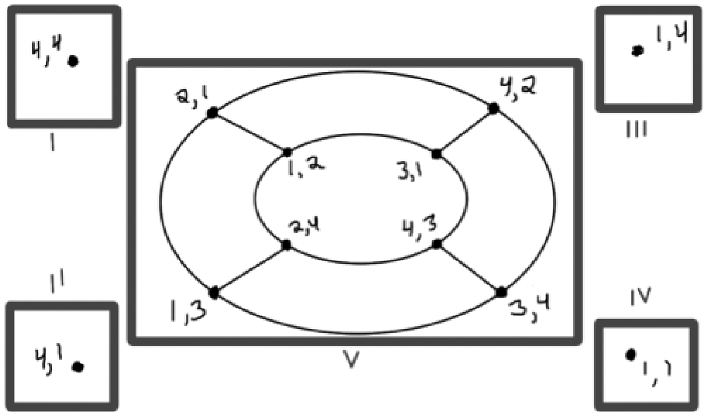


Figure 7b. 5 remaining components (Squared off) after 4 vertices have been removed. Test failed.

Archbishop Edge Equation

Apart from the above findings which focused on determining and solving for cyclic solutions of boards, a more Eulerian approach was applied, dealing with the number of edges a graph of size *n*x*n* would have. In Ian Parberry's article, *An efficient algorithm for the Knight's tour problem*, he gives the equation for a Knight's number of edges on an *n*x*n* board: $4n^2 - 12n + 8$. This led to the question: is there an equation for finding the number of edges an Archbishop's graph will have on a square board?

The top table in Figure 8 shows the results of hand counting the number of edges an Archbishop has on each *n* sized board for n = 1 to 9, producing evidence that there was indeed a trend. The Archbishop's board was then broken down in an attempt to discover its edge equation. Three segments were formed, nicknamed: the outer rim, mid lane, and core (the squares at the edges and corners of the board, those that are only one step in, and then the rest of the board, respectively).

However, knowing the equation of a Knight's edges and that the Archbishop is simply the combination of the two, this segmentation was applied to the Bishop's board, and each segment was generalized to a simple equation (with the Bishop's segments, the mid lane was a part of the core due to the range of the Bishop). It turned out that this breakdown was simpler to work with than the twelve possible movements of the Archbishop. With the outer rim (the vertices or squares along the very edge of the board), there are limits as to how many movements the Bishop has due not only to its own abilities but to its position on the board. For example, the corners of a board allow movement only towards the inside of the board since there are no vertices farther out than those, leaving only one possible movement (edge) for a Bishop (4 *corners* * 1 *edge/corner* = 4 edges). As for the points between the corners (still in the outer rim), these vertices are limited to two movements each, and the number of them can be calculated as: 4(*number of board sides*) * 2(*possible moves*) * (n-2). These two facts give the equation $4+(4*2(n-2))$, or, $8n-12$. Through similar breakdowns the second equation, for the core, was found: 4(*possible moves per point*) * $[n-2]^2$(*the number of points on the board without the outer rim*), or, $4n^2-16n+16$. This equation was added to the outer rim's to account for the whole board, and divided by two to account for edges counted twice, producing the equation $2n^2-4n+2$ for the Bishop's number of edges per n board.

By adding the Bishop and Knight equation together, the equation $6n^2 - 16n + 10$ was formed: the Archbishop's edge equation. The lower table of Figure 7 shows where the Excel spreadsheet plugged in the size, n, and used the equation to get the

vertices	n /	edges found by hand	Diff	Diff of Diff
9	3	16		
16	4	40	24	
25	5	80	40	16
36	6	130	50	10
49	7	192	62	12
64	8	266	74	12
81	9	352	86	12

Archbishop

$6n^2 - 16n + 10$

n	edges by equation	Diff	Diff of Diff
3	16	14	-36
4	42	26	-24
5	80	38	-12
6	130	50	0
7	192	62	12
8	266	74	24
9	352	86	36
11	560	110	60

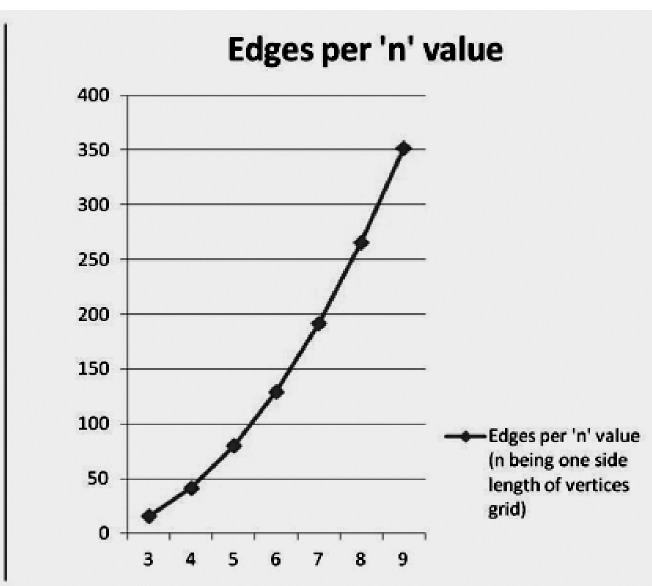


Figure 8.-Excel Spreadsheet

exact same answers as done by hand, therein validating these earlier calculations.

Partially Explored Conjectures

While working with the creation of the Bishop Edge equation, two approaches were created: the Bishop's diagonal movements are considered one square at a time, and are considered bound only by the board size diagonal moves (more than one square at a time). While the earlier equation was created using the first approach (what was originally nicknamed the *Limpin' Bishop*), the latter was also attempted for a short amount of time (nicknamed the *Unlimited Bishop*). Similar steps were taken to break down the *Unlimited Bishop's* movements into manageable equations, such as hand counting up to an 11x11 sized board and separating the board into sections, yet gave only a rough ability to estimate larger boards. By using excel's ability to form an equation from a set of graphed data points (the values calculated by hand for the number of edges up to an $n = 11$ board), this equation was estimated: .

$$\frac{2}{3}n^3 - n^2 + \frac{1}{3}n$$

By rounding up to the nearest whole number, the value this equation puts out matches the actual number of edges per n sized board. However, the catch, of course, is having to round up each output from the equation, which at some point will cause the values to skew, albeit at very high values of n. By having excel estimate using a fourth degree polynomial, this equation is formed:

$$((-3*10^{-14})n^4) + \frac{2}{3}n^3 - n^2 + \frac{1}{3}n - (3*10^{-9})$$

This fourth degree equation gives a much closer value output, but is slightly overestimating the number of edges by about .002 edges per increase of n by 1.

Recall that the Archbishop is unable to complete a cycle on a 4x4 board. This is true as long as the board is not linked from one side to the other in what is known as a wrap-around. A wrap-around allows for a connection to be made between all the side squares. Each square on one side is connected to a square on the other side. For example, if we were using a Bishop, and he was positioned on the lower left corner of a 4x4 board, not only could we move to the square up one and right one, we could move from there to: the top right square, the one above the bottom right corner square, and the one to the right of the top left square, in only one movement. If the Archbishop is placed on this wrap-around version of a 4x4 board, it turns out that it can, in fact, complete a cycle.

Conclusion

Two things embody the purpose behind research such as this: one, it is a new branch of study that holds many unanswered questions, which opens up a whole new set of possibilities; and two, good, pure, mathematical research inevitably leads to important applications. As a new branch of study, it opens up interesting lines of research on a whole new set of problems, involving questions such as: is there a limit to the number of diagonal (Bishop type) movements that can be made to complete a tour; and, is there a minimum number of Bishop movements needed to complete a cycle where Knight's moves alone cannot; or, is there a maximum? What is it? Do even or odd, square or rectangular boards have a higher maximum, or are they the same? These questions arose during only ten weeks of study on the Archbishop, allowing the creation of conjectures and side quests from the main problem. For example, after only a small amount of time exploring how many Bishop moves are needed and if there is a maximum allowed when completing an Archbishop cycle, the following conjectures were made: (i) it seemed that the number of movements must be odd in order for a cycle to be formed; (ii) it seemed that the minimum was three bishop movements in order to create a cycle; and (iii) it appears the movement of a Bishop will approach 50% traversal of the board, though never reach it as it cannot escape from a corner, and therefore less than half of the movements in a cycle must be a Bishop's. Many more of these questions and conjectures exist, opening many other fascinating and challenging lines of inquiry.

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