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Sean Koval

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On the Toughness of Some Johnson Solids

Sean Koval

Submitted in Partial Completion of the Requirements for Departmental Honors in Mathematics

Bridgewater State University

May 8, 2018

Dr. Ward Heilman, Thesis Advisor
Dr. Shannon Lockard, Committee Member
Dr. Rachel Stahl, Committee Member
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Sean Koval

Mentor Dr. Ward Heilman

May 8, 2018

Abstract

The Johnson solids are the 92 three-dimensional, convex solids (other than the Platonic and Archimedean solids) that can be formed with regular polygons. The purpose of this honor’s thesis work is to determine the toughness of some of the Johnson Solids and similar graphs. The Johnson solids can be broken up into classes of solids with certain characteristics. While there are only 92 Johnson solids in three dimensions, we can generate infinite classes of graphs in two dimensions with similar characteristics. We have identified some of these classes, studied the toughness of individual graphs and begun to analyze a few classes of graphs. Many different techniques from a variety of sources have been employed to explore the toughness of these graphs. We have achieved bounds for toughness in some of these classes and look to prove exact results.

1 Introduction

In Graph Theory, a graph $G$ consists of a finite nonempty set $V$ of objects called vertices and a set $E$ of 2-element subsets of $V$ called edges. The number of vertices of $G$ is the order of the graph, and the number of edges is called the size. If there is an edge adjoining two vertices on a graph, those vertices are said to be adjacent to one another, and we say that edge is incident to those vertices. For our purposes, when we say graph, we are discussing graphs with no loops (if $uv \in E(G)$ then $u \neq v$) or multiple edges (no two vertices have more than one edge incident to both of them). The degree of a vertex is the number of edges that a vertex is incident with. The minimum degree of $G$, denoted $\delta(G)$, is equal to the least degree of any of the vertices of $G$. The maximum degree of $G$, denoted $\Delta(G)$, is equal to the highest degree of any of the vertices of $G$. A path $P_n$ is a graph on $n$ distinct vertices with $E(P_n) = \{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_n \}$. A cycle $C_n$ is a graph on $n$ distinct vertices with $E(C_n) = \{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-2}v_{n-1}, v_{n-1}v_n, v_nv_1 \}$. A complete graph $K_n$ is a graph on $n$ distinct vertices where every vertex is adjacent to every other vertex. A graph $G$ is hamiltonian if it contains a cycle which visits every vertex exactly once, other than the vertex we start and end with. Let $T \subset V(G)$. We define the graph $G - T = G[V(G) \setminus T]$ as the subgraph of $G$ induced by deleting the vertices of $T$. When we delete vertices from a graph, the edges incident to those vertices are removed as well. A graph $G$ is a spanning subgraph of $H$ if $G \subseteq H$ and
\( V(G) = V(H) \). A subset \( S \) of \( V(G) \) is an independent set of \( G \) if no two vertices of \( S \) are adjacent in \( G \). The independence number, \( \beta(G) \), is defined as the number of vertices in a maximum independent set of \( G \).

A graph \( G \) is a connected graph if there exists a \( u - v \) walk along the edges of \( G \) between any two vertices \( u, v \) in \( G \). \( G \) is said to be a disconnected graph otherwise. A common problem when looking at graphs is vertex connectivity. A graph is said to be \( k \)-connected if it requires the removal of at least \( k \) vertices to become disconnected. We denote the connectivity of a graph \( G \) by \( \kappa(G) \). The graph below is an example of a 2-connected graph.

The graph is 2-connected because if we were to remove any one of the vertices, the graph would remain connected. However if we remove two vertices, 2 and 5 for example, the graph becomes disconnected. So \( \kappa(G) = 2 \).

This set of vertices \( \{2, 5\} \), or any set of vertices \( S \) such that \( G - S \) separates the graph into multiple components is called a vertex-cut of \( G \). Note that \( \kappa(G) \) is also defined as the size of the minimum vertex cut of \( G \). In this case, the graph became split into two components after removing two vertices. A component of a graph is a set of vertices such that all of the vertices of that set are connected. That is, for any two vertices \( x, y \) of a given component, there exists a path of edges in the component from \( x \) to \( y \). While connectivity is a useful definition, it doesn’t provide the full picture of the weakness of a graph. Consider the graph below.
The graph is 2-connected just like our last example. If we remove vertices 2 and 5 the same as we did for the last graph we find that the graph is split into three components.

This fragmentation is certainly worse than two components, yet connectivity tells us nothing about this phenomenon. For a deeper measure of connectivity, we consider the toughness of a graph. That is, how many components do we have when we remove a set of vertices. In 1972, V. Chvatal introduced a new invariant for graphs that addresses this [1]. For non-complete graphs, the toughness of a graph $G$ is defined as the minimum ratio of the size of a vertex-cut and the number of resulting components taken over all vertex-cuts $S$ of $G$. This can be expressed as

$$
\tau(G) = \min \frac{|S|}{\omega(G - S)}
$$

where $|S|$ is the cardinality of $S$ and $\omega(G - S)$ is the number of components in $G - S$. For $K_n$, $\tau(K_n) = \infty$. A graph $G$ is said to be $t$-tough for any real number $t$ where $t \leq \tau(G)$. So for our two different examples of 2-connected graphs, one has a toughness of $2/2 = 1$ while the other has a toughness of $2/3$. Since $2/3 < 1$, the second example is less tough than the first. So, in this sense, less connected and more vulnerable to attack. Thus toughness can provide a deeper measure of graph vulnerability.

Proposition 1.1 (Chvatal [1973]) If $G$ is a spanning subgraph of $H$ then $\tau(G) \leq \tau(H)$.

Proposition 1.2 (Chvatal [1973]) $\tau(G) \geq \frac{n(G)}{\mu(G)}$.

Proposition 1.3 (Chvatal [1973]) $\tau(G) \leq \frac{n(G)}{2}$ if $G$ is not complete.
Proposition 2.1 (Chvatal [1973]) Every hamiltonian graph is 1-tough.

Together, propositions 1.2 and 1.3 gives us lower and upper bounds on the toughness of a non-complete graph. That is, \( \frac{\kappa(G)}{\beta(G)} \leq \tau(G) \leq \frac{\kappa(G)}{2} \) for \( G \neq K_n \).

2 Known Toughness Results

Since the introduction of this measure of connectivity, several results have been proven for classes of graphs that provide insight into toughness and assist us in proving toughness results for new classes of graphs. The toughness of a path on \( n \) vertices is equal to \( \frac{1}{2} \) for \( n > 2 \). The toughness of a cycle on \( n \) vertices is equal to 1 for \( n > 3 \). Let \( W_n \) denote a wheel, a graph that is a cycle of length \( n - 1 \) with an additional vertex that is adjacent to every vertex of the cycle. Then \( \tau(W_n) = \lceil \frac{n}{2} \rceil \lceil \frac{n}{2} \rceil - 1 \) for \( n > 4 \). The toughness of a tree \( T \), which is an acyclic graph, is \( \frac{1}{\Delta(T)} \). The toughness of a hypercube is equal to 1. Results have also been found for the cross products of some of these classes of graphs. The Cartesian Product of two graphs, \( G \) and \( H \), denoted \( G \times H \), is the graph whose vertex set, \( V(G \times H) \), is \( V(G) \times V(H) \) and \((u,v)\in E(G \times H)\) if either \( u = v \) and \((v,z)\in E(H)\), or \( v = z \) and \((u,w)\in E(G)\).

Theorem 2.5 Goddard and Swart [1990]

(i) \( \tau(P_m \times P_n) = 1 \), for \( mn \) even, \( m, n \geq 2 \);

(ii) \( \tau(P_m \times P_n) = \frac{mn-1}{mn+1} \), for \( m, n \) odd, \( m, n \geq 3 \);

(iii) \( \tau(P_m \times C_n) = 1 \), for \( n \) even (\( m \) even or odd);

(iv) \( \tau(C_m \times C_n) = 1 \), for \( m, n \) even.

In 1999, W. Heilmann proved the conjectures of Goddard and Swart that,

(i) \( \tau(P_m \times C_n) = \frac{n}{n-1} \), for \( n \) odd, \( m \) even, \( m \geq 2, n \geq 3 \);

(ii) \( \tau(P_m \times C_n) = \frac{mn-1}{mn-m} \), for \( m, n \) odd, \( m, n \geq 3 \);

(iii) \( \tau(C_m \times C_n) = \frac{n}{n-1} \), for \( m \) even, \( n \) odd, \( m \geq 2, n \geq 3 \);

(iv) \( \tau(C_m \times C_n) = \frac{mn-1}{(m-1)(n-1)} \), for \( m, n \) odd, \( m, n \geq 3 \).

Some other commonly known graphs have toughness results as well. For example, the Petersen Graph shown below has a toughness of \( \frac{4}{3} \) by deleting vertices 1, 3, 8 and 10 to get three components \( \{2,5\}, \{4,9\} \), and \( \{6,7\} \).
Another known toughness result is on the Platonic solids. The toughness of the tetrahedron is infinite since it is the complete graph $K_4$. The toughness of the hexahedron is 1 since it is a bipartite graph with equal parts in the partitions. The toughness of the octohedron is 2, the toughness of the icosahedron is $\frac{5}{2}$, and the toughness of the dodecahedron is $\frac{3}{2}$.

3 Johnson Solids

In Geometry, we have known of the Platonic and Archimedean solids for centuries. Also, it is quite obvious that there are several other polyhedra that can be formed with regular polygons, however it was not always known just how many were possible. In 1966, Norman Johnson proposed that there are exactly 92 non-uniform, regular-faced polyhedra, and in 1969 Victor Zalgaller proved Johnson’s conjecture [2],[3]. Johnson classified all 92 of these objects in his conjecture with particular symbols, nomenclature, even by edges and dihedral angles. A table can be found in the appendix of [2]. For our purposes it will suffice to refer to them by their symbol, and provide their name and its corresponding graph.

4 Toughness of the Johnson Solids

All of the Johnson solids can be viewed as 2-dimensional graphs since we know all of their vertices and edges. Because of this, the Johnson solids are another class of graphs for which we can attempt to find toughness results. Here we will prove several toughness results on these solids.
Square Pyramid: $Y_4$

Theorem 1: $\tau(Y_4) = \frac{3}{2}$

*Proof.* $\kappa(Y_4) = 3$ and $\beta(Y_4) = 2$, so by Proposition 1.2 we have $\tau(Y_4) \geq \frac{3}{2}$. If we begin by removing $v_5$, we are then left with a cycle of length 4. We know cycles are 2-connected so we must remove at least two more vertices to disconnect the graph. Thus we remove three vertices from $Y_4$ to get $\tau(Y_4) \leq \frac{3}{2}$. Since $\tau(Y_4) \geq \frac{3}{2}$ and $\tau(Y_4) \leq \frac{3}{2}$, we conclude $\tau(Y_4) = \frac{3}{2}$. \hfill $\square$

Pentagonal Pyramid: $Y_5^*$

Theorem 2: $\tau(Y_5) = \frac{3}{2}$

*Proof.* $\kappa(Y_5) = 3$ and $\beta_0(Y_5) = 2$, so by Proposition 1.2 we have $\tau(Y_5) \geq \frac{3}{2}$. Next remove $v_6$. We are then left with a cycle which we know requires us to remove two more vertices to disconnect the graph. Thus we remove three vertices to get two components which yields $\tau(Y_5) \leq \frac{3}{2}$. Since $\tau(Y_5) \geq \frac{3}{2}$ and $\tau(Y_5) \leq \frac{3}{2}$, we conclude $\tau(Y_5) = \frac{3}{2}$. \hfill $\square$

*Alternatively, both $Y_4$ and $Y_5$ are wheels, so we can also use the rule mentioned earlier to get the same results or $\frac{3}{2} = \frac{5}{2} \leq \tau \leq \frac{3}{2} = \frac{5}{2}$.*
Theorem 3: $\tau(Q_3) = \frac{3}{2}$.

Proof. We know $1 \leq \tau(Q_3) \leq \frac{3}{2}$ since the graph is hamiltonian along with proposition 1.3. Furthermore we know it is pointless to remove either of the two cycles contained in this graph completely, since the resulting graph would still be connected and needing another two vertices removed from the other cycle to disconnect the graph. The resulting ratio would be much larger than $\frac{3}{2}$. So we know we will be removing at least one vertex from each cycle and never removing all of the vertices from either cycle. We begin by removing one vertex from the outer cycle to create our vertex-cut.

Case 1: Without loss of generality delete $v_1$. From here delete $v_7$. Next, if we delete $v_3$, we have disconnected the graph, and get a ratio value that is equal to that of proposition 1.3. From there we see if the removal of additional vertices yields a lower toughness bound. Obviously we won’t bother deleting $v_2$ as it is already its own component and its removal would just make the toughness bound we get for this case larger. The remaining five vertices we are concerned with then are $v_4, v_5, v_6, v_8,$ and $v_9$ which form a $C_5$ with one additional edge between two of the vertices. Hence we must remove at least two of them to create more components. By exhausting these cases we can see that at best we can remove two vertices to get two more components. This results in $\tau(Q_3) \leq \frac{5}{3}$, but $\frac{5}{3} > \frac{3}{2}$, so we already have a better bound.

Case 2: Delete $v_1$ and $v_7$. The other way to proceed from Case 1 is to remove a vertex other than $v_3$. Removing $v_2$ is of no use, since it is only adjacent to $v_3$, so we have five vertices to concern ourselves with here. The vertices $v_4, v_5, v_6, v_8,$ and $v_9$ form a $C_5$ with one additional edge between two of the vertices. So we know we must remove at least two more vertices to disconnect the graph. Removing $v_8$ and $v_4$ results in $\tau(Q_3) \leq 2$ which is no better bound than we have already. Therefore we know at least one of $v_4$ or $v_8$ must remain as we proceed in this case. If we remove $v_4$ and keep $v_8$, the best we can do is remove $v_9$ which yields $\tau(Q_3) \leq 2$. If we remove $v_8$ and keep $v_4$, the only way to disconnect the graph in this case is to remove $v_5$, which yields $\tau(Q_3) \leq 2$. The only other option in this case is now keeping the four vertices $v_2, v_3, v_4,$ and $v_8$. But from here the only option is removing $v_5$ and $v_9$ which
again yields $\tau(Q_3) \leq 2$.

Case 3: Delete $v_1$ and $v_8$. Suppose we delete $v_3$. We can’t delete $v_9$ otherwise we get cases isomorphic to Case 1 and Case 2. Also removing $v_2$ is also pointless due to the same reasons in the first cases. If we delete $v_4$, we only add to the toughness bound since the degree of this vertex is already one at this point, so it is of no use to delete it. Removing $v_7$ does nothing to disconnect the graph any better, since it is the last vertex adjacent to $v_2$ and we can leave it as the component $\{2,7\}$ and remove other vertices. Removing $v_5$ yields $\tau(Q_3) \leq 2$, and removing $v_9$ results in the same bound. If we remove $v_5$ and $v_9$ we get $\tau(Q_3) \leq 5/2$.

Case 4: Delete $v_1$ and $v_8$. We now know $v_2, v_3$, and $v_7$ must all remain for the last scenarios. Thus we can proceed by deleting either $v_4, v_5, v_6$, or $v_9$. If we remove $v_4$, we arrive at a graph identical to Case 3, so $v_4$ will not be in our cut-set either. If we remove only one of $v_5, v_6$, or $v_9$, the graph is still connected. So we must remove some combination of them, but not all three to disconnect the graph. If we remove $v_5$ and $v_6$ we are left with a $P_5$, which is clearly still connected. Removing $v_6$ and $v_9$ does the same. Lastly, removing $v_5$ and $v_9$ results in $\tau(Q_3) \leq 2$.

In no case is there a toughness ratio that is less than $3/2$. Hence we have shown that for all of these cases $\tau(Q_3) \geq 3/2$, and that the graphs of all other cases will be isomorphic to these cases. Also by proposition 1.3 we know $\tau(Q_3) \leq 3/2$. Thus $\tau(Q_3) = 3/2$.

\[\Box\]

Elongated Triangular Pyramid: $Y_3P_3$

Theorem 4: $\tau(Y_3P_3) = 3/2$

Proof. Since $\kappa(Y_3P_3) = 3$ and $\beta(Y_3P_3) = 2$, proposition 1.2 gives us $\tau(Y_3P_3) \geq 3/2$ and proposition 1.3 tells us that $\tau(Y_3P_3) \leq 3/2$. Therefore $\tau(Y_3P_3) = 3/2$.

\[\Box\]
Elongated Square Pyramid: $Y_4P_4$

![Graph Image]

**Theorem 5:** $\tau(Y_4P_4) = \frac{5}{4}$.

**Proof.** The graph is made up of two cycles both on four vertices where each distinct vertex of one cycle is adjacent to one distinct vertex of the other, which forms a cube. It also has an additional vertex $v_9$ which is adjacent to each vertex of one of the cycles. The graph is hamiltonian so by Proposition 2.1 it is at least 1-tough. So $1 \leq \tau(Y_4P_4)$. Also, we can see the case where if we remove $v_2, v_4, v_6, v_7,$ and $v_9$ we are then left with four components. Thus $\tau(Y_4P_4) \leq \frac{5}{4}$. It is impossible to remove five vertices and get five components (which would imply the toughness is 1), since the graph has only nine vertices. Since we know the toughness is bounded between 1 and 5/4, it will suffice to show that there are no three vertices we can remove which will split the graph into three components, and no four vertices we can remove which would split the graph into four components. If we remove the center vertex we are left with a cube, and we get the bound $\tau(Y_4P_4) \leq \frac{5}{4}$ which we already have. So we know we are only concerned with cases in which we remove vertices from the two cycles.

**Case 1:** Since $\delta(Y_4P_4) = 3$, in order to achieve a disconnected graph our cut-set must include three vertices that are all adjacent to the same vertex. Moreover, this same vertex must be one from the outer cycle since the outer cycle is where the vertices of degree 3 are located. Due to the symmetry of the graph we can see that whichever three vertices we remove to disconnect one of the vertices of the outer cycle from the rest of the graph, the rest of the graph is still connected which yields $\tau(Y_4P_4) \leq \frac{3}{2}$.

**Case 2:** If we remove all four vertices from the outer cycle, the graph is still connected. If we remove all four vertices from the inner cycle we get an upper bound of 2. So we know we must proceed by removing vertices from both of them. Suppose we remove three vertices from the outer cycle and one from the inner cycle. Without loss of generality remove $v_1, v_2, v_3$. If we remove $v_8$ we get $\tau(Y_4P_4) \leq 2$. If we remove any of the other vertices from the inner cycle instead of $v_8$ the graph is still connected. Next, suppose instead that we remove three vertices from the inner cycle. Without loss of generality remove $v_4, v_5,$ and $v_7$. If we remove $v_6$, we again
get $\tau(Y_4P_4) \leq 2$. If we remove any of the other vertices from the outer cycle instead of $v_6$ the graph is still connected.

Case 3: The only other way to proceed in finding a lower upper bound is to remove two vertices from each of the cycles. Due to the symmetry of the graph the only unique possibilities are including two adjacent vertices from the outer cycle or two non-adjacent vertices from the outside cycle in our cut-set. Without loss of generality remove $v_1$ and $v_2$. We could then remove $v_3$ and $v_8$ to get $\tau(Y_4P_4) \leq 2$. Other than that no matter which two vertices from the inner cycle we remove the graph is still connected. Next suppose instead we remove two non-adjacent vertices from the outer cycle, $v_1$ and $v_3$. If we then remove $v_4$ and $v_5$, we get $\tau(Y_4P_4) \leq 2$. If we remove $v_4$ and $v_7$, the graph is still connected. If we remove $v_4$ and $v_8$, we again get $\tau(Y_4P_4) \leq 2$. If we remove $v_5$ and $v_7$ we still get $\tau(Y_4P_4) \leq 2$. If we remove $v_5$ and $v_8$, we get $\tau(Y_4P_4) \leq \frac{3}{2}$ and $\frac{3}{2} \geq \frac{5}{4}$. Lastly if we remove $v_7$ and $v_8$ we get $\tau(Y_4P_4) \leq 2$.

For all of these cases the resulting ratios were all larger than $\frac{5}{4}$ or equal to $\frac{5}{4}$. Hence $\tau(Y_4P_4) \geq \frac{5}{4}$, and the graphs of all other cases will be isomorphic to these cases. Also from our cut-set mentioned earlier we know $\tau(Y_4P_4) \leq \frac{5}{4}$. Thus $\tau(Y_4P_4) = \frac{5}{4}$.

Elongated Pentagonal Pyramid: $Y_5P_5$

**Theorem 6:** $1 \leq \tau(Y_5P_5) \leq \frac{3}{2}$.

**Proof.** The graph is hamiltonian and $\kappa(Y_5P_5) = 3$, so by propositions 1.3 and 2.1 we get that $1 \leq \tau(Y_5P_5) \leq \frac{3}{2}$.

\[\square\]
Theorem 7: $\tau(Y_{3^2}) = \frac{3}{2}$.

Proof. The graph is a $K_3$ with two additional vertices $v_2$ and $v_4$ that are each adjacent to every vertex in the $K_3$, but are not adjacent to each other. Thus if we remove either one of these two additional vertices we are left with a $K_4$. If we remove both, a $K_3$. So we must remove at least one vertex from the $K_3$ to disconnect the graph. But since each vertex of the $K_3$ is adjacent to every other vertex, we must remove all three of these vertices to split the graph into two components and arrive at the result $\tau(Y_{3^2}) = \frac{3}{2}$.

Pentagonal Dipyramid: $Y_{5^2}$

Theorem 8: $\tau(Y_{5^2}) = 2$.

Proof. Similar to $Y_{3^2}$, the graph is a $C_5$ with two additional vertices $v_6$ and $v_7$ that are each adjacent to every vertex of the cycle. That is, for these two additional vertices $v_6$ and $v_7$, $(v_6, v_i) \in E(Y_{5^2})$ and $(v_7, v_i) \in E(Y_{5^2})$ for every $v_i \in C_5$. If we remove either one of the two additional vertices $v_6$ or $v_7$ then we are left with a wheel on six vertices, so $\tau(Y_{5^2}) \leq 2$. If we remove both of the additional two vertices we get a $C_5$ and again arrive at $\tau(Y_{5^2}) \leq 2$. Finally, if we keep both and therefore proceed to remove vertices from the $C_5$, since each of the additional two vertices are adjacent
to every vertex of the cycle, we must remove the entire cycle resulting in $\tau(Y_{52}) \leq \frac{5}{2}$.
Since there is no case in which we get a ratio of 2, we also know that $\tau(Y_{52}) \geq 2$
Thus the toughness is equal to 2.

Gyroelongated Square Pyramid: $Y_{4}S_{4}$

Theorem 9: $\tau(Y_{4}S_{4}) = 2$.

Proof. The independence number is 3 and the graph is 4-connected so by proposition 1.2 and 1.3 we can say $\frac{4}{3} \leq \tau(Y_{4}S_{4}) \leq \frac{4}{2} = 2$. To attain equality on this upper bound it suffices to show we can not make a cut-set of five vertices that results in three components. If we can’t achieve three components by removing five vertices then we certainly can not with less vertices, and the smallest ratio that can be achieved by removing six vertices is 2. Also, while a toughness upper bound of $\frac{5}{4}$ is certainly less than 2, if we can not even get three components by removing five vertices, we won’t be able to get four components. If we remove all four vertices from the inner cycle, then the last of the five vertices removed is either the center vertex or any of the outer vertices. Removing the inner one just takes away one of our already made components, and the outer cycle is still connected if we only remove one vertex from it. Hence we can not include the entire inner cycle when creating our cut-set of five vertices. Similarly, we wouldn’t include the entire outer cycle in our cut-set. Then the only cases are where we remove the center vertex along with four other vertices, where the vertices come form both cycles, or we leave the center vertex and remove five vertices from the cycles. Also, notice that we can only remove two from one cycle and three from the other and vice versa since removing more would take away a whole cycle.

Case 1: Suppose we remove the center vertex, we are left with a 4-regular graph with an independence number of 2. By proposition 1.2 and 1.3 we get that the toughness of this subgraph would be 2, and thus if we are removing the center vertex the minimum toughness ratio of our original solid would be $\frac{2n+1}{\alpha} > 2$ for some natural
number $a$. Since we know by proposition 1.3 that $\tau(Y_4S_4) \leq 2$, it is clear we must not include the center vertex $v_9$ in our cut-set.

Case 2: Suppose we remove two vertices from the inner cycle that are not adjacent. Due to the symmetry of the graph, without loss of generality remove $v_4$ and $v_7$. Then we must include three vertices from the outer cycle in our cut-set. By symmetry, removing any three vertices from the outer cycle leaves in a connected graph.

Case 3: Suppose we remove two adjacent vertices form the inner cycle. Without loss of generality remove $v_4$ and $v_5$. Then we must remove three vertices from the outer cycle again. If we remove $v_1$, $v_2$, and $v_3$, the graph is still connected. If we remove $v_1$, $v_2$, and $v_6$, we get $\tau(Y_4S_4) \leq \frac{5}{2}$. If we remove $v_1$, $v_3$, and $v_6$, the graph is still connected. If we remove $v_2$, $v_3$, and $v_6$, the graph is again still connected.

Case 4: Suppose we remove three vertices from the inner cycle. Without loss of generality remove $v_4$, $v_5$, and $v_7$. Then we must remove two vertices from the outer cycle. If we remove $v_1$ and $v_2$, the graph is still connected. If we remove $v_1$ and $v_3$, we get $\tau(Y_4S_4) \leq \frac{5}{2}$. If we remove $v_1$ and $v_6$, we again get $\tau(Y_4S_4) \leq \frac{5}{2}$. Removing $v_2$ and $v_3$ does not disconnect the graph. Removing $v_2$ and $v_6$ results in $\tau(Y_4S_4) \leq \frac{5}{2}$. If we remove $v_2$ and $v_3$, the graph is still connected. Finally, removing $v_3$ and $v_6$ again leaves a connected graph.

We have shown that for all possible cases, $\tau(Y_4S_4) \geq 2$, and we know $\tau(Y_4S_4) \leq 2$. Therefore $\tau(Y_4S_4) = 2$.

Gyroelongated Pentagonal Pyramid: $Y_5S_5$

Theorem 10: $1 \leq \tau(Y_5S_5) \leq 2$.

Proof. The graph is hamiltonian and is 4-connected. So by propositions 1.3 and 2.1 we see that $1 \leq \tau(Y_5S_5) \leq 2$. 

\[ \Box \]
Pentagonal Orthobirotunda: $R_5^2$

Theorem 11: $1 \leq \tau(R_5^2) \leq 2$.

Proof. The graph is hamiltonian so by proposition 2.1 we know it is at least 1-tough. $S = \{v_1, v_3, v_4, v_5, v_{10}, v_{11}, v_{12}, v_{13}, v_{16}, v_{17}, v_{20}, v_{21}, v_{22}, v_{23}, v_{25}, v_{26}, v_{28}, v_{29}\}$ is a cut-set of this graph which yields $\tau(R_5^2) \leq 2$. Thus $1 \leq \tau(R_5^2) \leq 2$.

5 References


