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On the Toughness of Some Johnson Solids

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Abstract

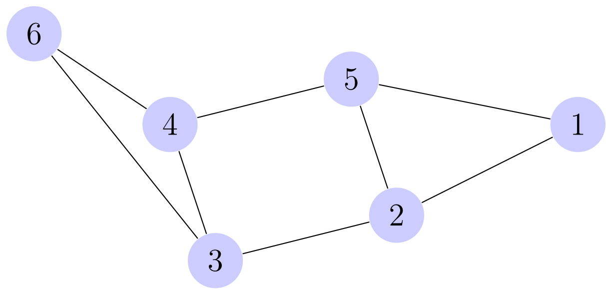
The Johnson solids are the 92 three-dimensional, convex solids (other than the Platonic and Archimedean solids) that can be formed with regular polygons. The purpose of this honor's thesis work is to determine the toughness of some of the Johnson Solids and similar graphs. The Johnson solids can be broken up into classes of solids with certain characteristics. While there are only 92 Johnson solids in three dimensions, we can generate infinite classes of graphs in two dimensions with similar characteristics. We have identified some of these classes, studied the toughness of individual graphs and begun to analyze a few classes of graphs. Many different techniques from a variety of sources have been employed to explore the toughness of these graphs. We have achieved bounds for toughness in some of these classes and look to prove exact results.

1 Introduction

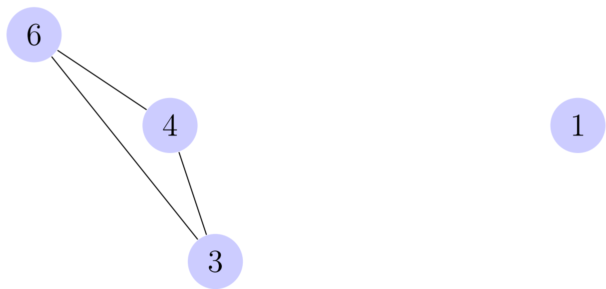
In Graph Theory, a graph G consists of a finite nonempty set V of objects called *vertices* and a set E of 2-element subsets of V called *edges*. The number of vertices of G is the *order* of the graph, and the number of edges is called the *size*. If there is an edge adjoining two vertices on a graph, those vertices are said to be adjacent to one another, and we say that edge is *incident* to those vertices. For our purposes, when we say graph, we are discussing graphs with no loops (if $uv \epsilon E(G)$ then $u \neq v$) or multiple edges (no two vertices have more than one edge incident to both of them). The *degree* of a vertex is the number of edges that a vertex is incident with. The *minimum degree* of G, denoted $\delta(G)$, is equal to the least degree of any of the vertices of G. The *maximum degree* of G, denoted $\Delta(G)$, is equal to the highest degree of any of the vertices of G. A *path* P_n is a graph on n distinct vertices with $E(P_n) = \{v_1v_2, v_2v_3, v_3v_4, ..., v_{n-2}v_{n-1}, v_{n-1}v_n\}$. A *cycle* C_n is a graph on n distinct vertices with $E(C_n) = \{v_1v_2, v_2v_3, v_3v_4, ..., v_{n-2}v_{n-1}, v_{n-1}v_n, v_nv_1\}$. A *complete graph* K_n is a graph on n distinct vertices where every vertex is adjacent to every other vertex. A graph G is *hamiltonian* if it contains a cycle which visits every vertex exactly once, other than the vertex we start and end with. Let $T \subset V(G)$. We define the graph $G - T = G[V(G) \setminus T]$ as the *subgraph of* G induced by deleting the vertices of T. When we delete vertices from a graph, the edges incident to those vertices are removed as well. A graph G is a *spanning subgraph* of H if $G \subseteq H$ and

$V(G) = V(H)$. A subset S of $V(G)$ is an *independent set* of G if no two vertices of S are adjacent in G. The *independence number*, $\beta(G)$, is defined as the number of vertices in a maximum independent set of G.

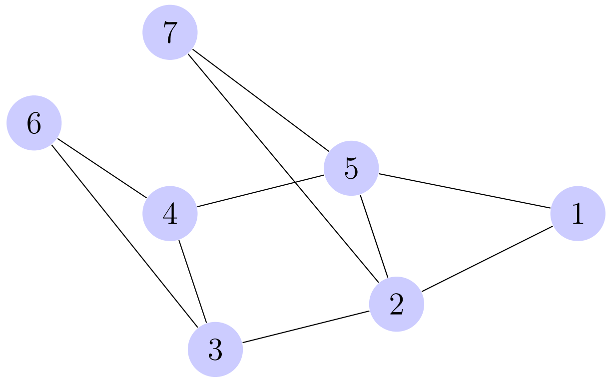
A graph G is a connected graph if there exists a $u - v$ walk along the edges of G between any two vertices u, v in G. G is said to be a disconnected graph otherwise. A common problem when looking at graphs is *vertex connectivity.* A graph is said to be *k-connected* if it requires the removal of at least k vertices to become disconnected. We denote the connectivity of a graph G by $\kappa(G)$. The graph below is an example of a 2-connected graph.



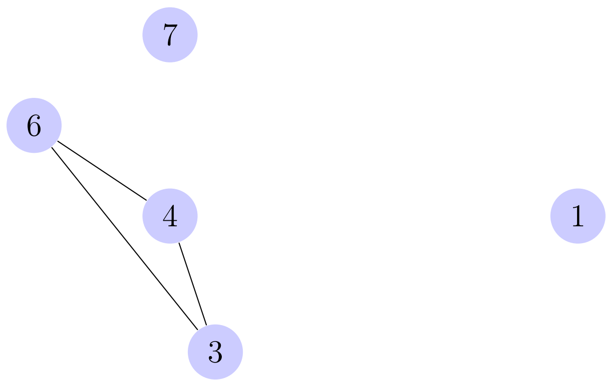
The graph is 2-connected because if we were to remove any one of the vertices, the graph would remain connected. However if we remove two vertices, 2 and 5 for example, the graph becomes disconnected. So $\kappa(G) = 2$.



This set of vertices {2,5}, or any set of vertices S such that G-S separates the graph into multiple components is called a *vertex-cut* of G. Note that $\kappa(G)$ is also defined as the size of the minimum vertex cut of G. In this case, the graph became split into two components after removing two vertices. A component of a graph is a set of vertices such that all of the vertices of that set are connected. That is, for any two vertices *x,y* of a given component, there exists a path of edges in the component from x to y. While connectivity is a useful definition, it doesn't provide the full picture of the weakness of a graph. Consider the graph below.



The graph is 2-connected just like our last example. If we remove vertices 2 and 5 the same as we did for the last graph we find that the graph is split into three components.



This fragmentation is certainly worse than two components, yet connectivity tells us nothing about this phenomenon. For a deeper measure of connectivity, we consider the toughness of a graph. That is, how many components do we have when we remove a set of vertices. In 1972, V. Chvatal introduced a new invariant for graphs that addresses this [1]. For non-complete graphs, the *toughness* of a graph G is defined as the minimum ratio of the size of a vertex-cut and the number of resulting components taken over all vertex-cuts S of G. This can be expressed as

$$\tau(G) = min\frac{|S|}{\omega(G-S)} \tag{1}$$

where $|S|$ is the cardinality of S and $\omega(G-S)$ is the number of components in $G-S$. For K_n, $\tau(K_n) = \infty$. A graph G is said to be *t-tough* for any real number t where $t \leq \tau(G)$. So for our two different examples of 2-connected graphs, one has a toughness of $2/2 = 1$ while the other has a toughness of $2/3$. Since $2/3 < 1$, the second example is less tough than the first. So, in this sense, less connected and more vulnerable to attack. Thus toughness can provide a deeper measure of graph vulnerability.

Proposition 1.1 (Chvatal [1973]) If G is a spanning subgraph of H then $\tau(G) \leq \tau(H)$.
Proposition 1.2 (Chvatal [1973]) $\tau(G) \geq \frac{\kappa(G)}{\beta(G)}$.
Proposition 1.3 (Chvatal [1973]) $\tau(G) \leq \frac{\kappa(G)}{2}$ if G is not complete.

Proposition 2.1 (Chvatal [1973]) Every hamiltonian graph is 1-tough.

Together, propositions 1.2 and 1.3 gives us lower and upper bounds on the toughness of a non-complete graph. That is, $\frac{\kappa(G)}{\beta(G)} \leq \tau(G) \leq \frac{\kappa(G)}{2}$ for $G \neq K_n$.

2 Known Toughness Results

Since the introduction of this measure of connectivity, several results have been proven for classes of graphs that provide insight into toughness and assist us in proving toughness results for new classes of graphs. The toughness of a path on n vertices is equal to $\frac{1}{2}$ for $n > 2$. The toughness of a cycle on n vertices is equal to 1 for $n > 3$. Let W_n denote a wheel, a graph that is a cycle of length $n-1$ with an additional vertex that is adjacent to every vertex of the cycle. Then $\tau(W_n) = \frac{\lceil \frac{n}{2} \rceil}{\lceil \frac{n}{2} \rceil - 1}$ for $n > 4$. The toughness of a tree T, which is an acyclic graph, is $\frac{1}{\Delta T}$. The toughness of a hypercube is equal to 1. Results have also been found for the cross products of some of these classes of graphs. The *Cartesian Product* of two graphs, G and H, denoted $G \times H$, is the graph whose vertex set, $V(G \times H)$, is $V(G) \times V(H)$ and $(u, v) \epsilon V(G \times H)$ is adjacent to $(w, z) \epsilon V(G \times H)$ if either $u = w$ and $(v, z) \epsilon E(H)$, or $v = z$ and $(u, w) \epsilon E(G)$.

Theorem 2.5 Goddard and Swart [1990]

(i) $\tau(P_m \times P_n) = 1$, for mn even, $m, n \geq 2$;

(ii) $\tau(P_m \times P_n) = \frac{mn-1}{mn+1}$, for m, n odd, $m, n \geq 3$;

(iii) $\tau(P_m \times C_n) = 1$, for n even (m even or odd);

(iv) $\tau(C_m \times C_n) = 1$, for m, n even.

In 1999, W. Heilman proved the conjectures of Goddard and Swart that,

(i) $\tau(P_m \times C_n) = \frac{n}{n-1}$, for n odd, m even, $m \geq 2, n \geq 3$;

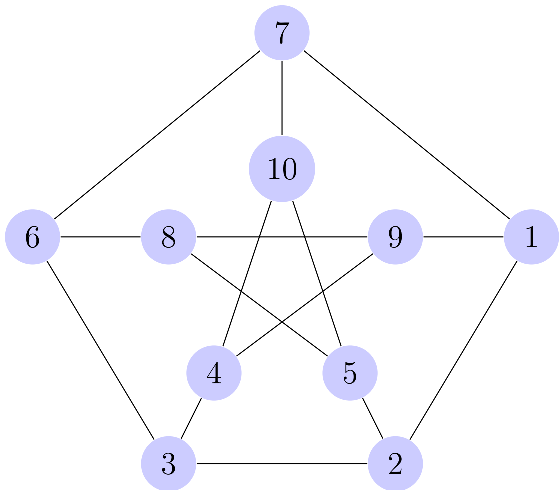
(ii) $\tau(P_m \times C_n) = \frac{mn-1}{mn-m}$, for m, n odd, $m, n \geq 3$;

(iii) $\tau(C_m \times C_n) = \frac{n}{n-1}$, for m even, n odd, $m \geq 2, n \geq 3$;

(iv) $\tau(C_m \times C_n) = \frac{mn-1}{(m-1)(n-1)}$, for m, n odd, $m, n \geq 3$.

Some other commonly known graphs have toughness results as well. For example, the Petersen Graph shown below has a toughness of $\frac{4}{3}$ by deleting vertices 1, 3, 8 and 10 to get three components {2,5}, {4,9}, and {6,7}.

Petersen Graph



Another known toughness result is on the Platonic solids. The toughness of the tetrahedron is infinite since it is the complete graph K_4. The toughness of the hexahedron is 1 since it is a bipartite graph with equal parts in the partitions. The toughness of the octohedron is 2, the toughness of the icosahedron is $\frac{5}{2}$, and the toughness of the dodecahedron is $\frac{3}{2}$.

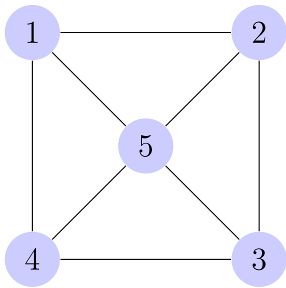
3 Johnson Solids

In Geometry, we have known of the Platonic and Archimedean solids for centuries. Also, it is quite obvious that there are several other polyhedra that can be formed with regular polygons, however it was not always known just how many were possible. In 1966, Norman Johnson proposed that there are exactly 92 non-uniform, regular-faced polyhedra, and in 1969 Victor Zalgaller proved Johnson's conjecture [2],[3]. Johnson classified all 92 of these objects in his conjecture with particular symbols, nomenclature, even by edges and dihedral angles. A table can be found in the appendix of [2]. For our purposes it will suffice to refer to them by their symbol, and provide their name and its corresponding graph.

4 Toughness of the Johnson Solids

All of the Johnson solids can be viewed as 2-dimensional graphs since we know all of their vertices and edges. Because of this, the Johnson solids are another class of graphs for which we can attempt to find toughness results. Here we will prove several toughness results on these solids.

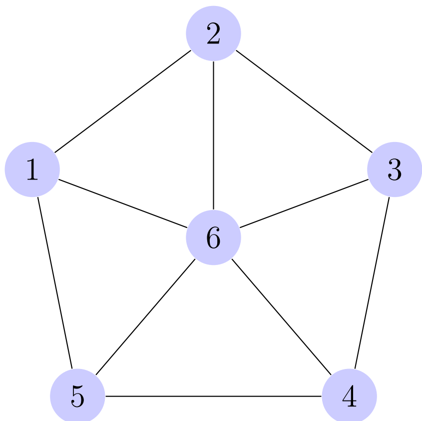
Square Pyramid: Y_4



Theorem 1: $\tau(Y_4) = \frac{3}{2}$

Proof. $\kappa(Y_4) = 3$ and $\beta(Y_4) = 2$, so by Proposition 1.2 we have $\tau(Y_4) \geq \frac{3}{2}$. If we begin by removing v_5, we are then left with a cycle of length 4. We know cycles are 2-connected so we must remove at least two more vertices to disconnect the graph. Thus we remove three vertices from Y_4 to get $\tau(Y_4) \leq \frac{3}{2}$. Since $\tau(Y_4) \geq \frac{3}{2}$ and $\tau(Y_4) \leq \frac{3}{2}$, we conclude $\tau(Y_4) = \frac{3}{2}$. □

Pentagonal Pyramid: Y_5*

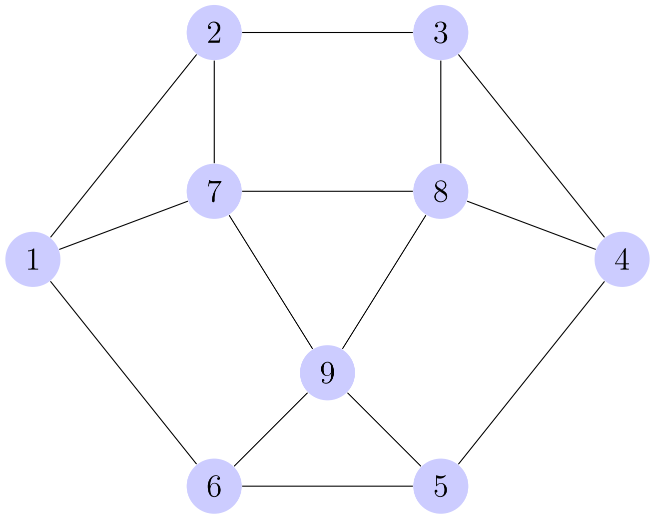


Theorem 2: $\tau(Y_5) = \frac{3}{2}$

Proof. $\kappa(Y_5) = 3$ and $\beta_0(Y_5) = 2$, so by Proposition 1.2 we have $\tau(Y_5) \geq \frac{3}{2}$. Next remove v_6. We are then left with a cycle which we know requires us to remove two more vertices to disconnect the graph. Thus we remove three vertices to get two components which yields $\tau(Y_5) \leq \frac{3}{2}$. Since $\tau(Y_5) \geq \frac{3}{2}$ and $\tau(Y_5) \leq \frac{3}{2}$, we conclude $\tau(Y_5) = \frac{3}{2}$. □

*Alternatively, both Y_4 and Y_5 are wheels, so we can also use the rule mentioned earlier to get the same results or $\frac{3}{2} = \frac{\kappa}{\beta} \leq \tau \leq \frac{3}{2} = \frac{\kappa}{2}$.

Triangular Cupola: Q_3



Theorem 3: $\tau(Q_3) = \frac{3}{2}$.

Proof. We know $1 \leq \tau(Q_3) \leq \frac{3}{2}$ since the graph is hamiltonian along with proposition 1.3. Furthermore we know it is pointless to remove either of the two cycles contained in this graph completely, since the resulting graph would still be connected and needing another two vertices removed from the other cycle to disconnect the graph. The resulting ratio would be much larger than $\frac{3}{2}$. So we know we will be removing at least one vertex from each cycle and never removing all of the vertices from either cycle. We begin by removing one vertex from the outer cycle to create our vertex-cut.

Case 1: Without loss of generality delete v_1. From here delete v_7. Next, if we delete v_3, we have disconnected the graph, and get a ratio value that is equal to that of proposition 1.3. From there we see if the removal of additional vertices yields a lower toughness bound. Obviously we won't bother deleting v_2 as it is already its own component and its removal would just make the toughness bound we get for this case larger. The remaining five vertices we are concerned with then are v_4, v_5, v_6, v_8, and v_9 which form a C_5 with one additional edge between two of the vertices. Hence we must remove at least two of them to create more components. By exhausting these cases we can see that at best we can remove two vertices to get two more components. This results in $\tau(Q_3) \leq \frac{5}{3}$, but $\frac{5}{3} > \frac{3}{2}$, so we already have a better bound.

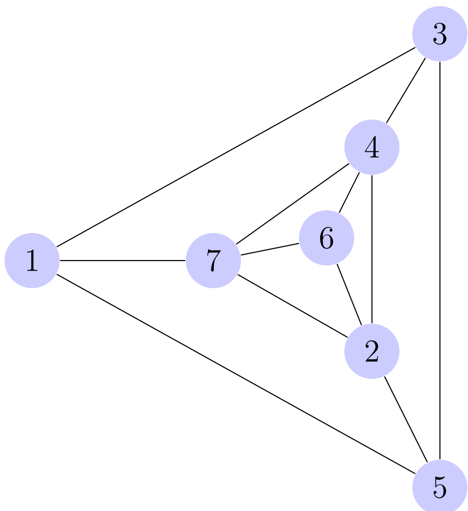
Case 2: Delete v_1 and v_7. The other way to proceed from Case 1 is to remove a vertex other than v_3. Removing v_2 is of no use, since it is only adjacent to v_3, so we have five vertices to concern ourselves with here. The vertices v_4, v_5, v_6, v_8, and v_9 form a C_5 with one additional edge between two of the vertices. So we know we must remove at least two more vertices to disconnect the graph. Removing v_8 and v_4 results in $\tau(Q_3) \leq 2$ which is no better bound than we have already. Therefore we know at least one of v_4 or v_8 must remain as we proceed in this case. If we remove v_4 and keep v_8, the best we can do is remove v_9 which yields $\tau(Q_3) \leq 2$. If we remove v_8 and keep v_4, the only way to disconnect the graph in this case is to remove v_5, which yields $\tau(Q_3) \leq 2$. The only other option in this case is now keeping the four vertices v_2, v_3, v_4, and v_8. But from here the only option is removing v_5 and v_9 which

again yields $\tau(Q_3) \leq 2$.
Case 3: Delete v_1 and v_8. Suppose we delete v_3. We can't delete v_9 otherwise we get cases isomorphic to Case 1 and Case 2. Also removing v_2 is also pointless due to the same reasons in the first cases. If we delete v_4, we only add to the toughness bound since the degree of this vertex is already one at this point, so it is of no use to delete it. Removing v_7 does nothing to disconnect the graph any better, since it is the last vertex adjacent to v_2 and we can leave it as the component {2,7} and remove other vertices. Removing v_5 yields $\tau(Q_3) \leq 2$, and removing v_9 results in the same bound. If we remove v_5 and v_9 we get $\tau(Q_3) \leq \frac{5}{3}$.
Case 4: Delete v_1 and v_8. We now know v_2, v_3, and v_7 must all remain for the last scenarios. Thus we can proceed by deleting either v_4, v_5, v_6, or v_9. If we remove v_4, we arrive at a graph identical to Case 3, so v_4 will not be in our cut-set either. If we remove only one of v_5, v_6, or v_9, the graph is still connected. So we must remove some combination of them, but not all three to disconnect the graph. If we remove v_5 and v_6 we are left with a P_5, which is clearly still connected. Removing v_6 and v_9 does the same. Lastly, removing v_5 and v_9 results in $\tau(Q_3) \leq 2$.

In no case is there a toughness ratio that is less than $\frac{3}{2}$. Hence we have shown that for all of these cases $\tau(Q_3) \geq \frac{3}{2}$, and that the graphs of all other cases will be isomorphic to these cases. Also by proposition 1.3 we know $\tau(Q_3) \leq \frac{3}{2}$. Thus $\tau(Q_3) = \frac{3}{2}$.

□

Elongated Triangular Pyramid: Y_3P_3

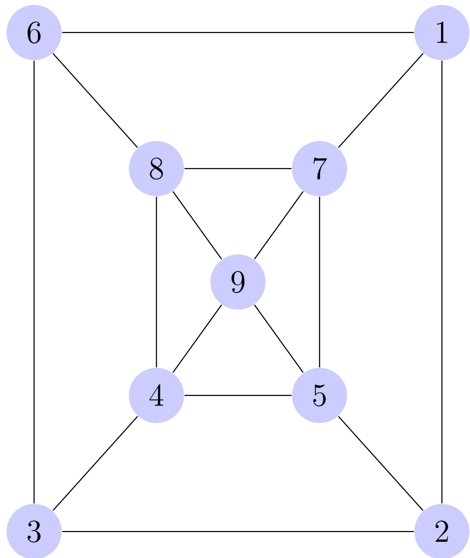


Theorem 4: $\tau(Y_3P_3) = \frac{3}{2}$

Proof. Since $\kappa(Y_3P_3) = 3$ and $\beta(Y_3P_3) = 2$, proposition 1.2 gives us $\tau(Y_3P_3) \geq \frac{3}{2}$ and proposition 1.3 tells us that $\tau(Y_3P_3) \leq \frac{3}{2}$. Therefore $\tau(Y_3P_3) = \frac{3}{2}$.

□

Elongated Square Pyramid: Y_4P_4



Theorem 5: $\tau(Y_4P_4) = \frac{5}{4}$.

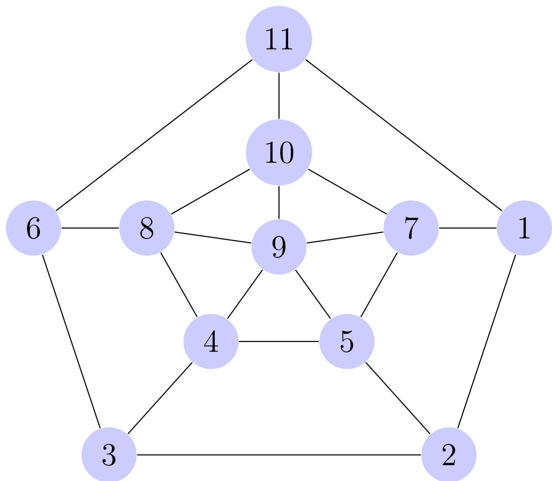
Proof. The graph is made up of two cycles both on four vertices where each distinct vertex of one cycle is adjacent to one distinct vertex of the other, which forms a cube. It also has an additional vertex v_9 which is adjacent to each vertex of one of the cycles. The graph is hamiltonian so by Proposition 2.1 it is at least 1-tough. So $1 \leq \tau(Y_4P_4)$. Also, we can see the case where if we remove v_2, v_4, v_6, v_7, and v_9 we are then left with four components. Thus $\tau(Y_4P_4) \leq \frac{5}{4}$. It is impossible to remove five vertices and get five components (which would imply the toughness is 1), since the graph has only nine vertices. Since we know the toughness is bounded between 1 and $5/4$, it will suffice to show that there are no three vertices we can remove which will split the graph into three components, and no four vertices we can remove which would split the graph into four components. If we remove the center vertex we are left with a cube, and we get the bound $\tau(Y_4P_4) \leq \frac{5}{4}$ which we already have. So we know we are only concerned with cases in which we remove vertices from the two cycles.

Case 1: Since $\delta(Y_4P_4) = 3$, in order to achieve a disconnected graph our cut-set must include three vertices that are all adjacent to the same vertex. Moreover, this same vertex must be one from the outer cycle since the outer cycle is where the vertices of degree 3 are located. Due to the symmetry of the graph we can see that whichever three vertices we remove to disconnect one of the vertices of the outer cycle from the rest of the graph, the rest of the graph is still connected which yields $\tau(Y_4P_4) \leq \frac{3}{2}$.

Case 2: If we remove all four vertices from the outer cycle, the graph is still connected. If we remove all four vertices from the inner cycle we get an upper bound of 2. So we know we must proceed by removing vertices from both of them. Suppose we remove three vertices from the outer cycle and one from the inner cycle. Without loss of generality remove v_1, v_2, v_3. If we remove v_8 we get $\tau(Y_4P_4) \leq 2$. If we remove any of the other vertices from the inner cycle instead of v_8 the graph is still connected. Next, suppose instead that we remove three vertices from the inner cycle. Without loss of generality remove v_4, v_5, and v_7. If we remove v_6, we again

get $\tau(Y_4P_4) \leq 2$. If we remove any of the other vertices from the outer cycle instead of v_6 the graph is still connected.
Case 3: The only other way to proceed in finding a lower upper bound is to remove two vertices from each of the cycles. Due to the symmetry of the graph the only unique possibilities are including two adjacent vertices form the outer cycle or two non-adjacent vertices from the outside cycle in our cut-set. Without loss of generality remove v_1 and v_2. We could then remove v_4 and v_8 to get $\tau(Y_4P_4) \leq 2$. Other than that no matter which two vertices from the inner cycle we remove the graph is still connected. Next suppose instead we remove two non-adjacent vertices from the outer cycle, v_1 and v_3. If we then remove v_4 and v_5, we get $\tau(Y_4P_4) \leq 2$. If we remove v_4 and v_7, the graph is still connected. If we remove v_4 and v_8, we again get $\tau(Y_4P_4) \leq 2$. If we remove v_5 and v_7 we still get $\tau(Y_4P_4) \leq 2$. If we remove v_5 and v_8, we get $\tau(Y_4P_4) \leq \frac{4}{3}$ and $\frac{4}{3} \geq \frac{5}{4}$. Lastly if we remove v_7 and v_8 we get $\tau(Y_4P_4) \leq 2$.
For all of these cases the resulting ratios were all larger than $\frac{5}{4}$ or equal to $\frac{5}{4}$. Hence $\tau(Y_4P_4) \geq \frac{5}{4}$, and the graphs of all other cases will be isomorphic to these cases. Also from our cut-set mentioned earlier we know $\tau(Y_4P_4) \leq \frac{5}{4}$. Thus $\tau(Y_4P_4) = \frac{5}{4}$. □

Elongated Pentagonal Pyramid: Y_5P_5

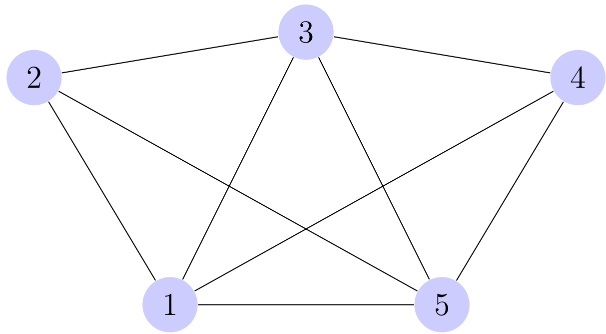


Theorem 6: $1 \leq \tau(Y_5P_5) \leq \frac{3}{2}$.

Proof. The graph is hamiltonian and $\kappa(Y_5P_5) = 3$, so by propositions 1.3 and 2.1 we get that $1 \leq \tau(Y_5P_5) \leq \frac{3}{2}$.

□

Triangular Dipyramid: Y_{3^2}

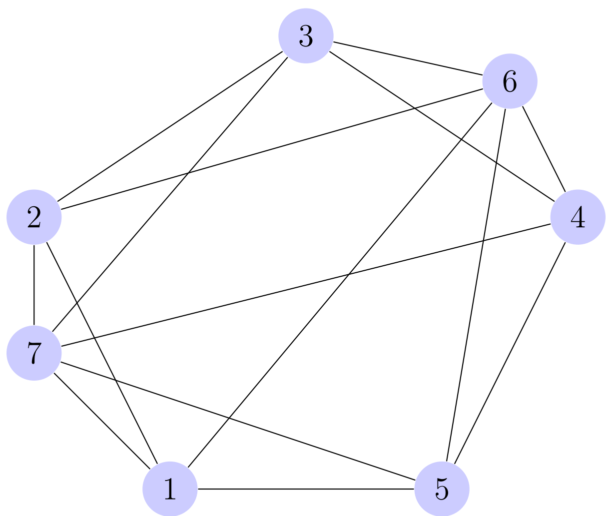


Theorem 7: $\tau(Y_{3^2}) = \frac{3}{2}$.

Proof. The graph is a K_3 with two additional vertices v_2 and v_4 that are each adjacent to every vertex in the K_3, but are not adjacent to each other. Thus if we remove either one of these two additional vertices we are left with a K_4. If we remove both, a K_3. So we must remove at least one vertex from the K_3 to disconnect the graph. But since each vertex of the K_3 is adjacent to every other vertex, we must remove all three of these vertices to split the graph into two components and arrive at the result $\tau(Y_{3^2}) = \frac{3}{2}$.

□

Pentagonal Dipyramid: Y_{5^2}

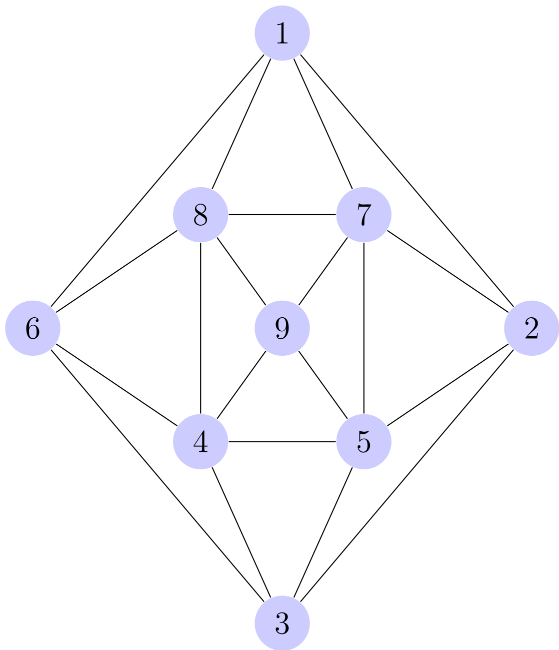


Theorem 8: $\tau(Y_{5^2}) = 2$.

Proof. Similar to Y_{3^2}, the graph is a C_5 with two additional vertices v_6 and v_7 that are each adjacent to every vertex of the cycle. That is, for these two additional vertices v_6 andv_7, $(v_6, v_i)\epsilon E(Y_{5^2})$ and $(v_7, v_i)\epsilon E(Y_{5^2})$ for every $v_i \epsilon C_5$. If we remove either one of the two additional vertices v_6 or v_7 then we are left with a wheel on six vertices, so $\tau(Y_{5^2}) \leq 2$. If we remove both of the additional two vertices we get a C_5 and again arrive at $\tau(Y_{5^2}) \leq 2$. Finally, if we keep both and therefore proceed to remove vertices from the C_5, since each of the additional two vertices are adjacent to every vertex of the cycle, we must remove the entire cycle resulting in $\tau(Y_{5^2}) \leq \frac{5}{2}$. Since there is no case in which we get a ratio of 2, we also know that $\tau(Y_{5^2}) \geq 2$ Thus the toughness is equal to 2.

□

Gyroelongated Square Pyramid: Y_4S_4



Theorem 9: $\tau(Y_4S_4) = 2$.

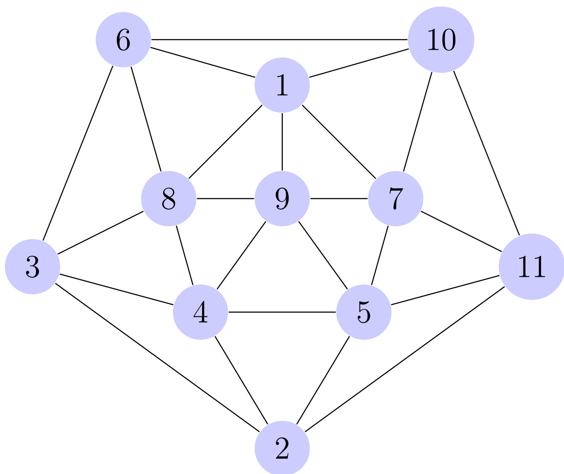
Proof. The independence number is 3 and the graph is 4-connected so by proposition 1.2 and 1.3 we can say $\frac{4}{3} \leq \tau(Y_4S_4) \leq \frac{4}{2} = 2$. To attain equality on this upper bound it suffices to show we can not make a cut-set of five vertices that results in three components. If we can't achieve three components by removing five vertices then we certainly can not with less vertices, and the smallest ratio that can be achieved by removing six vertices is 2. Also, while a toughness upper bound of $\frac{5}{4}$ is certainly less than 2, if we can not even get three components by removing five vertices, we won't be able to get four components. If we remove all four vertices from the inner cycle, then the last of the five vertices removed is either the center vertex or any of the outer vertices. Removing the inner one just takes away one of our already made components, and the outer cycle is still connected if we only remove one vertex from it. Hence we can not include the entire inner cycle when creating our cut-set of five vertices. Similarly, we wouldn't include the entire outer cycle in our cut-set. Then the only cases are where we remove the center vertex along with four other vertices, where the vetices come form both cycles, or we leave the center vertex and remove five vertices from the cycles. Also, notice that we can only remove two from one cycle and three from the other and vice versa since removing more would take away a whole cycle.
Case 1: Suppose we remove the center vertex, we are left with a 4-regular graph with an independence number of 2. By proposition 1.2 and 1.3 we get that the toughness of this subgraph would be 2, and thus if we are removing the center vertex the minimum toughness ratio of our original solid would be $\frac{2a+1}{a} > 2$ for some natural

number a. Since we know by proposition 1.3 that $\tau(Y_4S_4) \leq 2$, it is clear we must not include the center vertex v_9 in our cut-set.
Case 2: Suppose we remove two vertices from the inner cycle that are not adjacent. Due to the symmetry of the graph, without loss of generality remove v_4 and v_7. Then we must include three vertices from the outer cycle in our cut-set. By symmetry, removing any three vertices from the outer cycle leaves in a connected graph.
Case 3: Suppose we remove two adjacent vertices form the inner cycle. Without loss of generality remove v_4 and v_5. Then we must remove three vertices from the outer cycle again. If we remove v_1, v_2, and v_3, the graph is still connected. If we remove v_1, v_2, and v_6, we get $\tau(Y_4S_4) \leq \frac{5}{2}$. If we remove v_1, v_3, and v_6, the graph is still connected. If we remove v_2, v_3, and v_6, the graph is again still connected.
Case 4: Suppose we remove three vertices from the inner cycle. Without loss of generality remove v_4, v_5, and v_7. Then we must remove two vertices from the outer cycle. If we remove v_1 and v_2, the graph is still connected. If we remove v_1 and v_3, we get $\tau(Y_4S_4) \leq \frac{5}{2}$. If we remove v_1 and v_6, we again get $\tau(Y_4S_4) \leq \frac{5}{2}$. Removing v_2 and v_3 does not disconnect the graph. Removing v_2 and v_6 results in $\tau(Y_4S_4) \leq \frac{5}{2}$. If we remove v_2 and v_3, the graph is still connected. Finally, removing v_3 and v_6 again leaves a connected graph.

We have shown that for all possible cases, $\tau(Y_4S_4) \geq 2$, and we know $\tau(Y_4S_4) \leq 2$. Therefore $\tau(Y_4S_4) = 2$.

□

Gyroelongated Pentagonal Pyramid: Y_5S_5

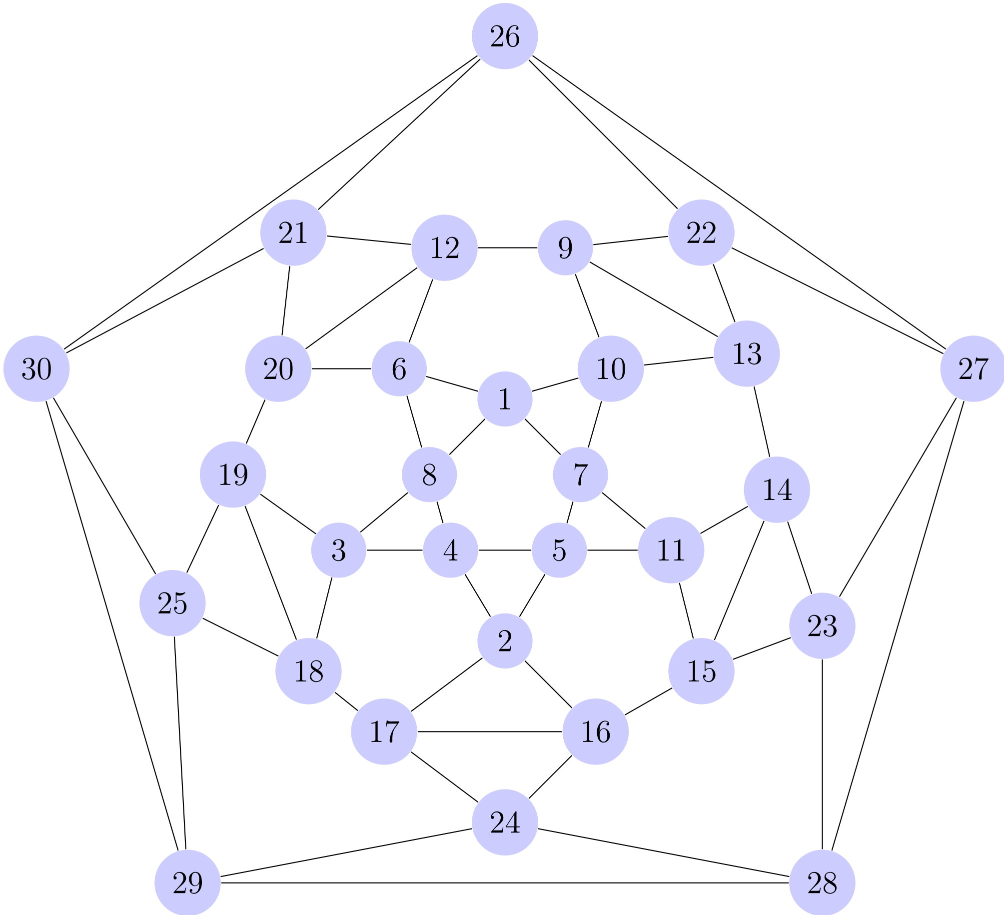


Theorem 10: $1 \leq \tau(Y_5S_5) \leq 2$.

Proof. The graph is hamiltonian and is 4-connected. So by propositions 1.3 and 2.1 we see that $1 \leq \tau(Y_5S_5) \leq 2$.

□

Pentagonal Orthobirotunda: R_{5^2}



Theorem 11: $1 \leq \tau(R_{5^2}) \leq 2$.

Proof. The graph is hamiltonian so by proposition 2.1 we know it is at least 1-tough. $S = \{v_1, v_3, v_4, v_5, v_{10}, v_{11}, v_{12}, v_{13}, v_{16}, v_{17}, v_{20}, v_{21}, v_{22}, v_{23}, v_{25}, v_{26}, v_{28}, v_{29}\}$ is a cut-set of this graph which yields $\tau(R_{5^2}) \leq 2$. Thus $1 \leq \tau(R_{5^2}) \leq 2$.

□

5 References

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