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Danica Baker

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# Who Wants to Play Sadisticube?

DANICA BAKER



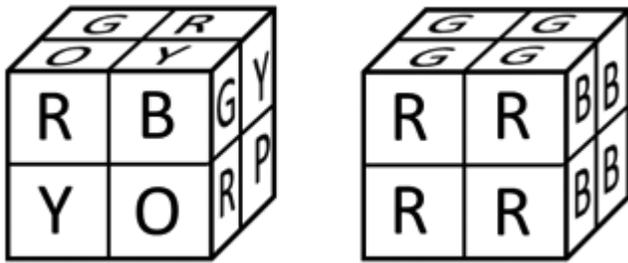
Danica Baker graduated in May 2014 with a Bachelor of Science in Mathematics and

minors in Accounting and Finance and Music. She was mentored by Dr. Shannon Lockard (Mathematics). This project was funded by a 2013 Adrian Tinsley Summer Research Grant and was presented at 2013 BSU Summer Research Symposium as well as the 2014 National Conference on Undergraduate Research (NCUR) in Lexington, KY.

**L**ogic puzzles and games are popular amongst many people for the purpose of entertainment. They also provide intriguing questions for mathematical research. One popular game that has inspired interesting research is Rubik's Cube. Researchers at MIT have investigated the Rubik's Cube to find the maximum number of moves, from any starting position, needed to win the game [6]. Another logic puzzle that has recently become very popular is Sudoku. Sudoku is a Japanese number game where a 9x9 grid is set up with a few numbers scattered on the grid. Mathematicians have been investigating Sudoku, exploring questions such as the number of possible Sudoku grids there are [7].

Sadisticube is a newer logic puzzle, created by a mathematician. A Sadisticube set is made up of eight separate blocks that form a 2x2x2 cube when placed together. The individual blocks can be rotated and swapped with each other to any position in the cube. The goal of the game is the same as in Rubik's Cube where each face of the cube needs to be one color. However, because there are trillions of ways to arrange the blocks and we do not know what our solution will look like, Sadisticube is far more difficult than Rubik's Cube to solve by hand. Fortunately, we can use mathematics to find solutions. Graphs can be used to model the cube so that a solution can be determined for any particular set of blocks. The methods used to create the matrices were adapted from a paper by Jean- Marie Magnier [5]. We will describe how to generate the matrices and their corresponding graphs and will then focus on the graphs in the second half of the paper. After describing how to generate graphs, we will discuss the analysis done on several graphs and the results we found while searching for characteristics common to all graphs.

Even though the final goal of Sadisticube is the same as the Rubik's Cube, the game is played differently. Each of the eight separate blocks is painted with one of six different colors: red (R), orange (O), yellow (Y), green (G), blue (B), or purple (P). To play the game, the individual blocks can be swapped and rotated to any position in the cube to get each face of the cube to be one color, as show below in Figure 1.



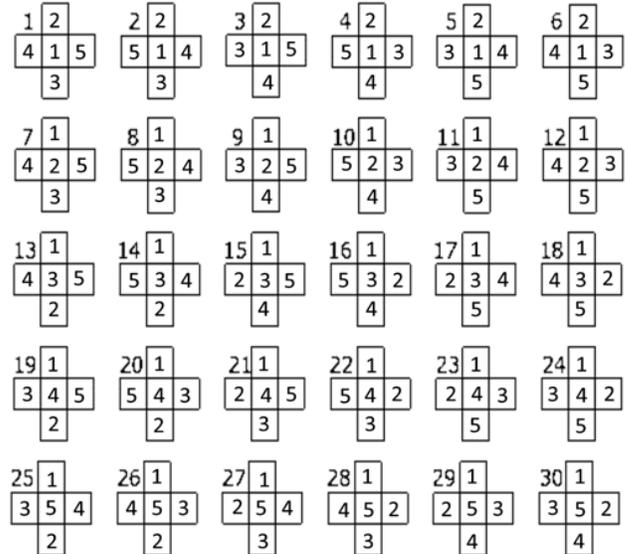
**Figure 1.** The cube on the left is unsolved while the cube on the right is solved.

Trying to win the game by trial and error is impractical. Since there are 185 trillion ways to arrange eight blocks in the form of a cube and the solution to the set of blocks is unknown, it is unimaginable luck if a player wins by trial and error. For example, say it takes ten seconds to put the blocks together in one formation. If there is one solution and it is the very last configuration put together, the 185 trillionth configuration, it would take approximately 58,000,000 years with no sleeping, eating, or doing anything other than configuring the blocks to reach the solution. Since people do not live for millions of years, it seems reasonable to represent the puzzle using math in order to solve the puzzle more efficiently.

As mentioned earlier, the blocks are painted with one of six different colors. There are 30 unique ways to color a block with six different colors, where each face of the block is a different color. Consider a single block: to count the number of ways to color a block, we will fix purple to the bottom face of that block. There are five colors used for the top and side faces of the block. Coloring the top face first, there are five choices for the color of the top face. Once the top face has a color, there are only four colors left to place on one of the faces on the side, then three colors to place on another side, two colors on the third side, and one color on the last side. In this way, we find there  $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ , or 120, ways to color the block with no restrictions on rotation. However, each rotation of an individual block does not change the way it is colored. With purple fixed to the bottom face, there are only four rotations for a block. Taking the number of colorings and dividing by the number of rotations, we find there are  $(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) / 4$ , or 30, ways to color any one block.

Thus, there are 30 different blocks from which we choose eight blocks to make a game set (Figure 2). To show the different ways to color the blocks, we represent a block two-dimensionally as a flattened box showing the top and sides of the block. In the figure below, purple is fixed to be the bottom color while the middle square gives the color of the top face and the other four colors shown are the side colors. The blocks are displayed

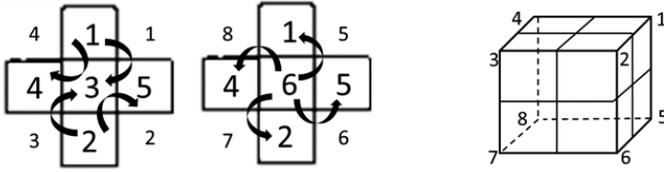
with numbers instead of colors, as seen in Figure 2. We use the following colors and numbers interchangeably: Blue = 1, Red = 2, Green = 3, Yellow = 4, Orange = 5, and Purple = 6.



**Figure 2.** Thirty different ways to color a block (picture from [5]).

Since there are thirty ways to color a block, and the solution could also look like any of these blocks, this also means that there are thirty possible solutions for any set of blocks. Previous research has indicated that a set of blocks can have anywhere from zero to five different solutions [5]. Since trial and error is an impractical approach to winning the game, other methods are useful in analyzing and solving the game. We will use the method described in [5] to generate the matrices that will be used to draw graphs. The graphs represent the relationship between a set of blocks and its possible solution cube.

We will use the diagram from Figure 2 to create sets of numbers that represent the 30 different individual blocks. To generate these sets, we look at the corners of the blocks to form three digit numbers, or triples, which represent the colors of the faces that are adjacent to each corner. Each block will have eight triples associated with it, one for each of the eight corners of the block. Figure 3 shows the diagram of block 13. On the left side of the figure is the top of the block in two-dimensional form while the middle is a similar image with the center replaced by 6, or purple, giving a representation of the bottom of the block. These diagrams are used to generate triples.



**Figure 3.** The block on the left is the top of block 13. The image in the middle shows the bottom of the block as it would be viewed through the top. The image on the right shows the cube, all eight blocks, three-dimensionally with the corners numbered.

The corners of the blocks were arbitrarily labeled from 1 to 8, but kept in the same order for all of the blocks. The image on the right of Figure 3 shows the block in 3-dimensional form. The corners are numbered to correspond to the 2-dimensional display on the left. Triples for the top of the cube are generated first. Starting with Corner 1, we list the numbers in clockwise order beginning with the smallest number in that corner. So, the first triple we find for the block in Figure 3 is 153. This process is repeated for corners 2, 3, and 4 around the top of the block, shown on the left in Figure 3, giving the triples {153, 235, 243, 134}. To find the bottom four corners, imagine the top color replaced by purple, as shown in the middle diagram in Figure 3. Starting from the center color in corner 5, we now read counterclockwise, giving the first triple for the bottom of the block as 651. Continuing in this manner, we find triples representing corners 6, 7, and 8 of the block, giving the last four triples of the set {651, 625, 642, 614}. So the complete set of triples that represents the block in Figure 3 is {153, 235, 243, 134, 651, 625, 642, 614}.

We can now use these sets of triples to generate matrices by comparing each of the blocks in the set to each of the possible solutions. Since each Sadisticube set contains eight blocks, a game set will have eight sets of triples associated with it, resulting in a matrix with eight rows, one row for each block. As an example, we consider a game set that contains blocks 3, 6, 8, 12, 13, 20, 22, and 25 from Figure 2, and compare each block to cube 8 as a possible solution. Figure 4 below gives the sets of triples for each block in the set and the possible solution cube. The row labeled B3 gives the triples for block 3, B6 gives the triples for block 6, and so on. The row labeled C8 gives the triples for cube 8, the possible solution cube to this set of blocks. The numbers 1 through 8 above the columns represent the eight corners of the cube.

	1	2	3	4	5	6	7	8
C8	{142	243	235	125	641	634	653	615}
B3	{ <u>125</u>	154	143	132	652	645	<u>634</u>	623}
B6	{123	135	154	<u>142</u>	632	<u>653</u>	643	614}
B8	{ <u>142</u>	<u>243</u>	<u>235</u>	<u>125</u>	<u>641</u>	<u>634</u>	<u>653</u>	<u>615</u> }
B12	{132	<u>235</u>	254	124	631	<u>653</u>	645	614}
B13	{153	<u>235</u>	<u>243</u>	134	651	625	642	614}
B20	{134	<u>243</u>	254	145	631	623	652	<u>615</u> }
B22	{124	234	354	145	621	632	<u>653</u>	<u>615</u> }
B25	{145	254	<u>235</u>	153	<u>641</u>	624	632	613}

**Figure 4.** This chart represents the corners of both the cube and the eight blocks. The underlined triples are the corners that the blocks share with the cube, C8.

We also use the chart in Figure 4 to see if there are any similarities between the corners of the possible solution cube and the blocks. Similarities between the blocks and the cube show that the blocks can be placed in a particular spot to create the solution. The underlined triples in the chart represent the corners that the cube and blocks have in common. If there is a block that does not have any triples in common with the cube, then that block that cannot be placed in any corner of the solution cube, so the cube is not a solution to that set of blocks. Each block has either 0, 2, or 8 similarities to the cube [5]. In Figure 4, each of the blocks has something in common with the cube, so there is a possibility that cube 8 is a solution to the set.

After finding the similarities between the blocks and the solution cube, we generate a matrix,  $A$ . When a corner between a block and the solution cube are the same, a 1 is placed in the column and row that corresponds to that corner of the cube, while corners that do not match have a 0 placed in the column corresponding to that corner. Rows in the matrix will contain 0, 2, or 8 ones based on the number of similarities [5]. Consider the solution cube and block 3 from Figure 4. This block has two triples in common with the cube, 125 and 634, so the block matches corners four and six of the cube. Since this triple represents corner 4 of the cube, a 1 is placed in the fourth column of the matrix in the row corresponding to block 3. Similarly, we will place a 1 in the sixth entry of the row since the sixth triple, or corner, in cube 8 also appears in block 3. The resulting initial row of matrix  $A$  is [ 0 0 0 1 0 1 0 0 ]. We repeat this process for each of the blocks, resulting in the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

The matrices that we find in this way will be used to draw graphs that represent the relationship between the blocks in the set and the possible solution cube.

Each row of the matrix represents a block of the game set and an edge on the graph. Columns represent a corner of the 2x2x2 cube and correspond to vertices on the graph. Remember that each of the rows will only have zero, two, or eight “1”s in the row. If the rows can be swapped so that the main diagonal of the matrix has a one in each entry, then the cube is a solution to the set of blocks. If there is a “1” in the main diagonal, the block represented by the row will fit in the corner of the cube represented by that column. So if there are all “1”s in the main diagonal, a block will fit in every corner of the cube, meaning there is a solution. If the rows cannot be swapped to have a one in each entry of the main diagonal, the cube is not a solution to the set of blocks because some corners of the cube will not have any blocks that match the coloring of the solution. In our example, it is possible to swap the rows of matrix  $A$  so that the main diagonal is all ones, indicating that cube 8 is a solution.

Graphs can be used to represent the cubes instead of using matrices. Since there are thirty possible solutions, each set of blocks has thirty graphs we call Sadisticube graphs. We find each of the Sadisticube graphs for one set of blocks by comparing the set to all 30 possible solutions and then use these graphs to determine if a particular cube is a solution to the set of blocks. If the graph indicates the cube is a solution, we call that graph a Sadisticube solution graph, — or simply, a solution graph.

The matrices we find using the procedure outlined above are used as adjacency matrices and tell us where to place edges on the graph. To draw the graph for our example set of blocks, consider the first row of the example matrix  $A$ . In this row,

there are “1”s in the fourth and sixth column of the matrix. This means the block represented by this row of the matrix and edge of the graph has Corners 4 and 6 in common with the solution. So on the graph, an edge is drawn between vertices 4 and 6.

We draw an edge in this way for all the rows that have two “1”s. When a row of all “1”s appears, as we see in the third row of the matrix  $A$ , we must do something a little different. A row of all ones indicates that each corner of that particular block is identical to the solution being considered. So this block could be placed in any of the corners of the cube. This means that we are allowed to draw an edge between any two of the eight vertices. If there is a row of all zeros, implying the block shares nothing in common with the possible solution cube, we do not draw an edge and the cube is not a solution. The graph for our example is shown below.

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

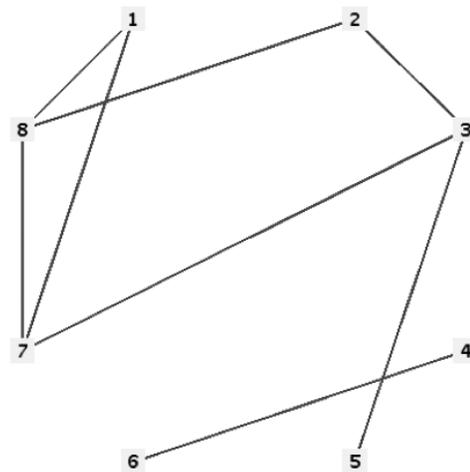


Figure 5. The graph of matrix  $A$ .

For this graph, the edge between vertices 1 and 8 represents the row of all ones.

All Sadisticube graphs have eight vertices, representing the cor-

ners of the  $2 \times 2 \times 2$  cube. The number of edges on the graph varies depending on the set of blocks and the cube to which we compare it. So, if there was block with few similarities to the cube, there would be fewer edges on the graph. Even though the number of edges varies for each set of blocks, we can find an upper bound on the number of edges in a Sadisticube graph.

**Theorem:** There are at most 8 edges on a Sadisticube graph.

**Proof:** The edges on the graph are determined by the rows of the corresponding matrix. If a row has all “0”s, there is no edge. If a row has a “1” in column  $i$  and another one in column  $j$ , one edge is drawn between vertex  $i$  and vertex  $j$ . If a row has all ones, one edge is drawn between any two vertices on the graph. Since there are eight blocks, there are eight rows in the matrix. Since there are eight rows, and each row corresponds to zero or one edges on the graph, there can be at most eight edges.

Recall that if a matrix has a row of all “1”s, we can place an edge between any two vertices on the graph. In the remainder of this paper, if there is a row of all ones, we will draw an edge between vertex 1 and a vertex that remains isolated after drawing the other edges of the graph. If there is no isolated vertex after we have drawn the other edges, we will simply place an edge between vertex 1 and vertex 8. The graph in Figure 5 is drawn using this convention.

As stated before, every set of eight blocks has thirty possible solutions and thirty corresponding graphs. Since not all of these graphs are solution graphs, we looked closely at the graphs to determine the characteristics that are present in solution graphs but are not present in non-solution graphs. This gives us a set of identifying graph characteristics that can be used to determine if the corresponding cube is a solution cube for that set of graphs. In order to generate a large number of matrices and corresponding graphs, we wrote two programs using Maple. The first program generates matrices given a set of blocks. The second program uses these matrices to draw the corresponding graphs. Figure 5 above is one graph that was generated by these programs.

After the graphs were drawn, we considered several different characteristics that are typically observed in graph theory. One characteristic we observed was planarity. A graph is planar if the vertices can be rearranged so the edges of the graph do not cross. All of the Sadisticube graphs we looked at were planar whether or not they were solution graphs. Since this characteristic was seen in both solution and non-solution graphs, we did not consider this to be a defining characteristic for solution graphs. Another interesting characteristic of graphs is the diameter. To find the diameter of a graph, we first determine the length of the shortest path between each pair of points on the graph. The

diameter is then the longest of these lengths. Diameters for the Sadisticube graphs ranged from 1 to 7 while the diameters of solution graphs varied between 2 and 7. Since the range of possible values for the diameters was similar for solution graphs and non-solution graphs, we did not consider diameters to be a defining characteristic for solution graphs either.

However, after analyzing nearly 1,000 graphs for over 20 different characteristics, we did find three characteristics that are always present in solution graphs. All Sadisticube graphs have eight vertices because the vertices represent the eight corners of the  $2 \times 2 \times 2$  cube, so solution graphs also have eight vertices. By the theorem above, Sadisticube graphs have at most 8 edges. Each edge of a graph represents a block that can be placed in the solution cube. Since we must be able to place all 8 blocks in the cube to have a solution, solution graphs must have exactly 8 edges. Solution graphs also have at least one cycle as a result of the following lemma.

**Lemma:** Suppose  $G$  is a connected graph with  $n$  vertices and  $n$  edges. Then  $G$  contains a cycle.

**Proof:** Assume  $G$  is a connected graph with  $n$  edges and  $n$  vertices. Suppose by way of contradiction that there are no cycles. Since  $G$  is connected and contains no cycles, it is a tree. Since it is a tree with  $n$  vertices, there must be  $n-1$  edges. This is a contradiction to the hypothesis that there are  $n$  edges on the graph. Therefore the graph must contain a cycle.

By this lemma, since all solution graphs have 8 vertices and 8 edges, they must also contain at least one cycle. However, we found that these are not the only characteristics needed to define a graph class for solution graphs. We have found Sadisticube graphs that have eight vertices, eight edges, and at least one cycle that are not solution graphs, implying there is at least one more identifying characteristic. In fact, we have found that isolated vertices are also very important. An isolated vertex on a Sadisticube graph indicates that a corner of the possible solution cube does not appear in any of the blocks in the set. It is important for each vertex to have an incident edge because this means a block can be placed in each corner of the cube. If there is no block that can fill a corner of the solution cube, it is impossible for the cube to be a solution to a particular set of blocks. This is summarized in the following theorem. Recall that when a row of the adjacency matrix contains all ones, we draw an edge between vertex 1 and an isolated vertex if there is one present and between vertices 1 and 8 if there are no isolated vertices.

**Theorem:** Suppose a graph  $G$  is a Sadisticube graph. If  $G$  has an isolated vertex, then the corresponding cube is not a solution to that set.

**Proof:** Suppose  $G$  is a Sadisticube graph. Then it has eight vertices. Assume  $G$  has an isolated vertex. We will show the cube is not a solution to the set. Since there is an isolated vertex, there is a column of zeros in the matrix. Since the matrix has a column of zeros, there must be one zero on the main diagonal after swapping rows, implying the cube is not a solution. Thus, if there is an isolated vertex, the cube is not a solution to the set of blocks.

Sadisticube graphs represent the relationship between the blocks in a game set and its possible solutions. There are characteristics that help determine if a Sadisticube graph is a solution graph or not. We have seen that solution graphs always have eight vertices, eight edges, and at least one cycle. However, these characteristics are not enough to define a graph class since some graphs have these three characteristics but are not solution graphs. Since an isolated vertex in a graph implies the corresponding cube is not a solution, solution graphs cannot have isolated vertices. Thus every Sadisticube solution graph has eight edges, eight vertices, at least one cycle, and no isolated vertices.

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