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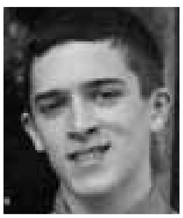

Alex Roche is a graduating senior of the class of 2013 with a major in physics and mathematics. He intends to continue his physics education in graduate school.

The purpose of this research was to create a computer code that would numerically test a Poisson equation relating the mass distribution of a lens galaxy cluster to weak gravitational shear. Einstein's theory of general relativity predicts that space-time is bent by massive objects, and in weak gravitational lensing, galaxy clusters act as lenses. The observable result is that galaxies far behind the gravitational lens will appear slightly more elliptical than they actually are. The ellipticity of the background galaxies is quantifiable and is directly related to the weak gravitational shear, and the shear is used to determine the mass distribution of the lensing cluster. Within the gravitational lensing community, there are two other well known methods of determining the mass distribution of a lensing galaxy cluster from the weak gravitational shear. This new method is unique in that it uses a Poisson equation, potentially simplifying the approach to the numerical integrations required. Part of the overall goal of this project was to clarify the effectiveness of this Poisson equation method in relation to the existing methods, with the hope that the Poisson method is more accurate. The Poisson equation method described here refers to the method of numerical relaxation to determine the mass distribution of a lensing galaxy cluster by using the Poisson equation. Over 2000 lines of original C++ code were written for this project, and the code simulates a typical lensing galaxy cluster mass distribution, calculates the weak gravitational shear from the simulated cluster, adds realistic random noise to the shear data, then finally applies a second order Taylor series expansion of the Poisson equation to the noisy shear data and checks how closely the computed mass distribution matches the original simulated galaxy cluster's distribution. This project has provided the tools needed to perform more rigorous testing of the Poisson equation.

Initial testing suggests that the Poisson method for determining the mass distribution of a galaxy cluster is about as good as the existing methods. Exactly where it lies in relation to them is yet to be determined. The Poisson method can detect the total mass of a galaxy cluster $\pm 2.4 \cdot 10^{14} M_{\odot}$. Substructure was detected 80% of the time when the total mass of the system was above $1.6 \cdot 10^{15} M_{\odot}$ and the mass of the substructure was $4 \cdot 10^{14} M_{\odot}$. It was detected 50% of the time when the total mass of the system was $6.3 \cdot 10^{14} M_{\odot}$ and the substructure's total mass was $1.6 \cdot 10^{14} M_{\odot}$. Numerical error inherent in the method was less than 3% in total.

Introduction

General Relativity predicts that the path of light is bent by the gravity of massive objects. A galaxy cluster will behave approximately as a thin lens.[3] Any light coming from far behind the galaxy cluster will be bent when it touches the lens plane. The amount that it is bent is related to the strength of gravity at that point. This phenomenon is called gravitational lensing. If the light from a galaxy sized circle, originating from far behind the lens, passes through the lens plane, it will appear as an ellipse to an observer on earth. The amount of elliptical stretching is the observable quantity in weak gravitational lensing.

Weak gravitational lensing can be used to determine the mass distribution of a galaxy cluster by analyzing the observed shapes of the galaxies far behind the galaxy cluster. Thomas Kling and Bryan Campbell derived, from fundamental principles of General Relativity, a new relationship between the mass distribution of a galaxy cluster and the observed shapes of the galaxies far behind the galaxy cluster.[2] The purpose of this research was to test new relationship, a Poisson equation relating weak gravitational shear to mass distribution of the lensing cluster.

Several outcomes were meant to be determined the C++ implementation of this new relationship between the mass distribution and the weak gravitational shear. The first goal was to determine if the method could accurately determine the mass distribution, without noise, to a low, acceptable amount of error. The second goal was to see if the method could accurately determine the mass distribution, with noise, of galaxy clusters ranging from 10^{14} to $10^{15} M_{\odot}$ (solar masses), including substructure of about 25% of the total mass along with location and shape of the cluster.

The Poisson method is designed to eventually be used with real data, which would come from a large field telescope. The galaxies that are far behind the galaxy cluster can be selected from the image by analyzing the color and intensity of their light. Since a single galaxy on its own could be oriented in any direction, its lensed image tells nothing about the mass distribution of the lensing galaxy cluster. To make a connection between image shape and mass distribution, a large number of galaxies in a given area have to be averaged. For this reason, the image taken by the telescope would be broken into equal bins, like boxes on grid paper. The shapes of the selected images in a given bin would be averaged and that bin would be interpreted as a single object, located in the middle of the bin area.

The mass distribution computation requires three grids of data. Each grid location corresponds to a physical location on the lens plane. Two of the grids hold the values of weak gravitational shear. From real data, the shear would be computed from the observed shapes of the galaxies, but with our simulation we are able to compute it from the mass distribution. The third grid will hold the mass density that will be computed using the Poisson method. Before the Poisson equation is applied, the values on the edges of the mass density grid are estimated. If the data are taken with a wide enough field of view, the mass density should be about zero at the edges. Information gained using a wide field of view data set can be used to interpolate values for the edges of a narrow field of view data set, where the mass density would not be close to zero. If the Poisson method was applied to real data, a ground based telescope would be used for the wide field of view data and a more accurate space telescope would be used for the narrow field of view data.

Simulation of Data

The simulated mass distribution (mass model) used for the lensing galaxy cluster was a truncated isothermal sphere with a core radius. The 3-dimensional mass density is described by,

$$\rho = \frac{v^2}{2\pi G}\,\frac{1}{r_c^2 + r^2}\,\frac{r_t^2}{r_t^2 + r^2} \qquad 1$$

Where r_c is the core radius, r_t is the truncation constant, v is the velocity dispersion. The truncation constant makes the mass finite at the origin. The core radius makes the total mass finite. This mass distribution is highest, but finite, at the origin and it decays gradually outward. In gravitational lensing theory, the thin lens approximation is used to describe a lensing cluster. For this reason, the 3-dimensional mass density is projected onto the lens plane by integrating along the line of sight.

$$\Sigma = \int_{-\infty}^{\infty} \rho dz \;\; \frac{v^2}{2G}\,\frac{r_t^2}{r_t^2 - r_c^2}\left(\frac{1}{\sqrt{s^2 + r_c^2}} - \frac{1}{\sqrt{s^2 + r_t^2}}\right) \qquad 2$$

$$s \equiv \sqrt{x^2 + y^2} \qquad 3$$

The total mass of the truncated isothermal sphere model is,

$$M = \int \kappa^s \rho dV = \frac{\pi r_t^2 v2}{G(r_c + r_t)} \qquad 4$$

The weak gravitational shear quantities, γ_1 and γ_2, can be computed from this mass model.[2]

$$\gamma_1 = \frac{\pi v^2 r_t^2}{r_t^2 - r_c^2}(x^2 - y^2)Q \quad 5$$

$$\gamma_2 = \frac{\pi v^2 r_t^2}{r_t^2 - r_c^2}(2xy)Q \quad 6$$

$$Q \equiv \frac{1}{\sqrt{s^2 + r_t^2}\,(r_t + \sqrt{s^2 + r_t^2})^2} - \frac{1}{\sqrt{s^2 + r_c^2}\,(r_c + \sqrt{s^2 + r_c^2})^2} \quad 7$$

The C++ program calculated the mass distribution that a typical lens galaxy cluster would have. From the simulated mass distribution, it then calculated the two weak gravitational shear quantities, γ_1 and γ_2, that would normally be obtained from the observed shapes of the background galaxies in a telescope image. Randomly generated noise was added to our observable quantities to simulate real data. The added noise is described by a normal distribution, where the average is taken to be the exact value for the shear and the standard deviation is given by,[1]

$$\sigma = \frac{0.3}{\sqrt{(\textit{number of objects per bin})}} \quad 8$$

This standard deviation comes from the probability that galaxies are aligned due to gravitational lensing or due to some other cause. We assume a constant number of objects per bin. This form of noise is the generally accepted noise assumed in all weak lensing measurements.[1] Finally, the Poisson equation was applied to the noisy observable data.

The Theory

The point of this project was to test the effectiveness of this new Poisson equation at finding the total mass of a galaxy cluster.[2]

$$\frac{\partial^2 \kappa}{\partial x^2} + \frac{\partial^2 \kappa}{\partial y^2} = \frac{\partial^2 \gamma_1}{\partial x^2} - \frac{\partial^2 \gamma_1}{\partial y^2} + 2\frac{\partial^2 \gamma_2}{\partial xy} \quad 9$$

This Poisson equation was solved for κ by using second order Taylor expansions of κ, γ_1 and γ_2.

$$\kappa(x,y) = \tfrac{1}{4}[\kappa(x + h,y) + \kappa(x - h,y) + \kappa(x,y + h) + \kappa(x,y - h) - F] \quad 9$$

$$F \equiv \gamma_1(x + h,y) + \gamma_1(x - h,y) - \gamma 1(x,y + h) - \gamma_1(x,y - h + \tfrac{1}{2}[\gamma_2(x + h, y + h) + \gamma_2(x - h, y - h) - \gamma_2(x + h, y - h) - \gamma_2(x - h, y + h)] \quad 10$$

Where h was the width and height of a bin. With this equation, if the values for γ_1 and γ_2 are known, then the values for at each bin can be determined, so long as the boundary conditions for are known.

The weak gravitational shear that the C++ program computes from the simulated mass distribution can also be computed directly from the telescope image. The C++ code does not simulate a telescope image, it simply computes γ_1 and γ_2 from the mass distribution. The moments of intensity are first calculated from the image data.[1]

$$q_{xx} = \frac{\Sigma(x-\bar{x})^2 w(\bar{x}-x, y-\bar{y}) I(x,y)}{\Sigma w(x-\bar{x}, y-\bar{y}) I(x,y)} \quad 11$$

$$q_{yy} = \frac{\Sigma(y-\bar{y})^2 w(\bar{x}-x, y-\bar{y}) I(x,y)}{\Sigma w(x-\bar{x}, y-\bar{y}) I(x,y)} \quad 12$$

$$q_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y}) w(x-\bar{x}, y-\bar{y}) I(x,y)}{\Sigma w(x-\bar{x}, y-\bar{y}) I(x,y)} \quad 13$$

Where $(\bar{x}, \bar{y})$ is the center of the image, $w(x, y)$ is a weight function that goes to zero outside the object, and $I(x, y)$ is the intensity of the light. The γ_1 and γ_2 from the Poisson equation are related to the observable quantities as follows.[1,6,7]

$$\Upsilon_1 = \frac{1}{2}\frac{q_{xx} - q_{yy}}{q_{xx} + q_{yy}} \quad 14$$

$$\Upsilon_2 = \frac{1}{2}\frac{q_{xy}}{q_{xx} + q_{yy}} \quad 15$$

Data & Discussion

As mentioned in the introduction, data with a wide field of view (the large grid) were used to get boundary conditions for a narrow field of view data set (the small grid). For each data point on the graphs below, the following steps are done in the program, after picking a large and small grid size:

1. Create new matrices for γ_1,γ_2 and the exact values of κ
2. Add random noise to γ_1 and γ_2
3. Apply the relaxation method
4. Compute the error in the relaxed κ matrix
5. Repeat steps two through four to gain statistical significance
6. Compute averages and standard deviations of the errors

The error in an average is given by,[5]

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}} \quad (16)$$

where σ is the standard deviation of the errors computed in step 6 and N is the number of noisy simulations we used for a given data point, as in step 5.
The error in a standard deviation is given by,[5]

$$\sigma_{\sigma} = \frac{\sigma}{\sqrt{2(N-1)}} \quad (17)$$

The advantage of using a large number of bins is that it is possible to have a more detailed description of a galaxy cluster's substructure. The disadvantage is that there will be more noise, as there will be fewer objects per bin. The disadvantages of having too few bins are that the numerical method will be less accurate and we will have less resolution. A practical upper limit to our number of bins is that there cannot be less than one object in a bin. This limit is at around 150x150 bins for the large grid and 40x40 for the small grid. This is assuming that the large grid data is from a ground based telescope with a field of view of 12 mpc and the small grid data is from a space based telescope with a field of view of 1.14 mpc.

For noisy data, average error for all grid sizes is about zero. However, the standard deviation in the errors increases with bin number.

For computing the total mass of a cluster with the Poisson method, the result will be more accurate with fewer bins for noisy data but not that much less accurate for higher bin sizes. Changing bin size has a small effect. We see 8% more accuracy with an 11x11 small grid than a 39x39 small grid. For the 11x11 grid, one standard deviation is within 11% of the true total mass.

Without noise, we see that the accuracy increases, as expected, with more bins.

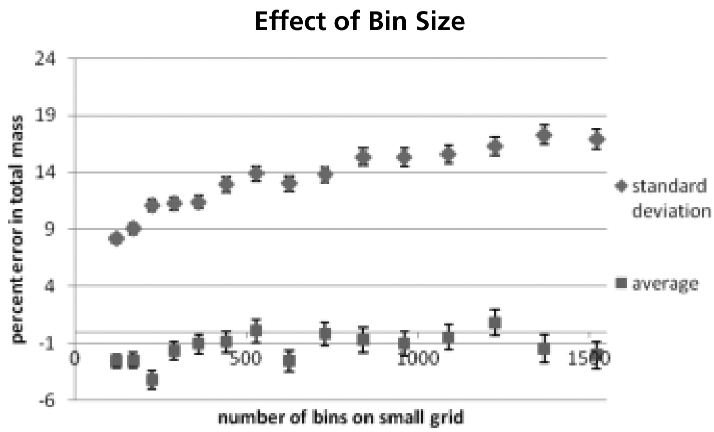

Figure 1. Small grid. 200 simulations per point. v = 0.005c, r_c = 0.1 mpc, r_t = 1.5 mpc.

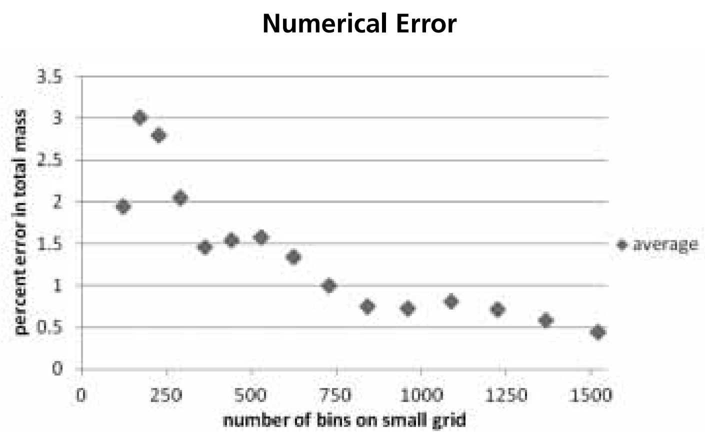

Figure 2. Small grid, 11x11. v = 0.005c, r_c = 0.1 mpc, r_t = 1.5 mpc.

The accuracy is acceptable in all cases with noiseless data. Numerical error is at worst 3% and at best 0.5% from true the total mass.

Naturally occurring galaxy clusters come in different widths. We tested the method to see how sensitive it was to a cluster getting wider and flatter but maintaining constant mass.

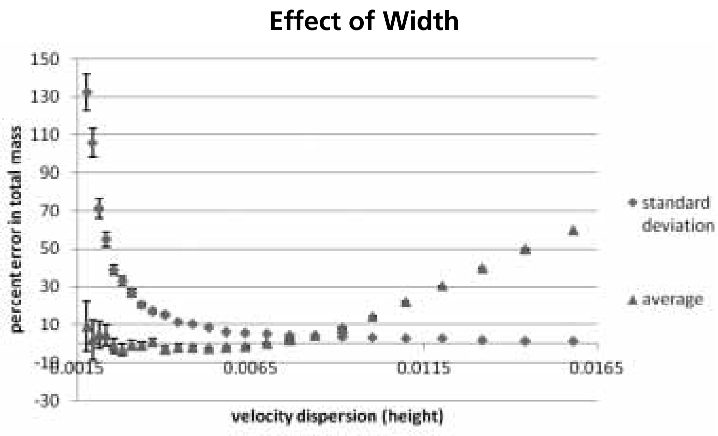

Figure 3. Small grid 11x11. 100 simulations per point.

Figure 3 shows that the data have a very wide tolerance for varying width of a galaxy cluster. So long as the cluster is not very dense or very disperse, reasonable accuracy is maintained. One cause of the inaccurate detection of highly disperse galaxy clusters is that as the mass is flattened, the mass density at the edges should increase. As a result, the choice of zero mass density as the boundary condition becomes less reasonable. When width and height are varied, the data have a minimum when v = 0.007c and r_c = 0.05mpc. One standard deviation of the error can go beyond 20% error in the total mass when v is above 0.01c or below 0.033c.

A hope for the Poisson method was for it to be able to detect lensing galaxy clusters in the range of 10^{14} to $10^{15}M_{\odot}$. Figure 4 is a plot of total error versus total mass of the galaxy cluster.

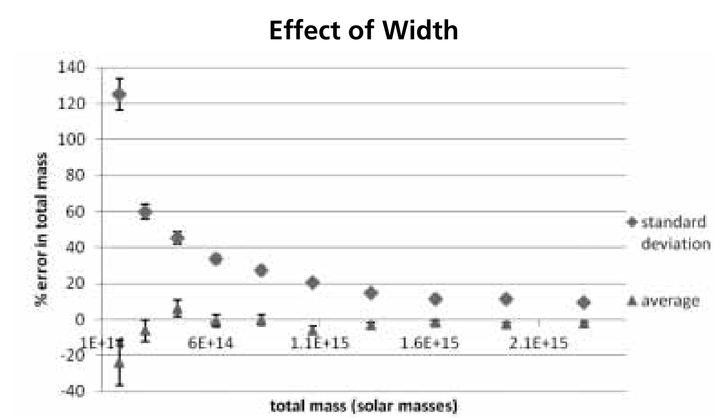

Figure 4. Small grid, 11x11. 100 simulations per point. r_c = 0.1 mpc, r_t = 1.5 mpc.

Clusters that were as low in mass as ~$5 \cdot 10^{14}$ $M_{\odot}$ were detected. This was about as good as other existing methods of determining the mass distribution of a lensing galaxy cluster, in common use. When the total mass goes below $1.3 \cdot 10^{15}M_{\odot}$, one standard deviation from the average was beyond 20% error in the total mass. At about $4.5 \cdot 10^{14}M_{\odot}$, one standard deviation from the average is beyond 50% error in the total mass. By converting percent error to error in mass, it was shown that one standard deviation of the data is within an average of $2.4 \cdot 10^{14}M_{\odot} \pm 4\%$ of the correct mass. Essentially, the resolution of the Poisson method, for the total mass, has a standard deviation of $2.4 \cdot 10^{14}M_{\odot}$.

The ability to detect substructure of about 25% of the total mass of a cluster was also one of the project goals. The program was used to generate two peaks that were separated enough that they could be resolved with the grid spacing, as shown in figure 6. Below is a plot of peaks detected versus total mass of the galaxy cluster.

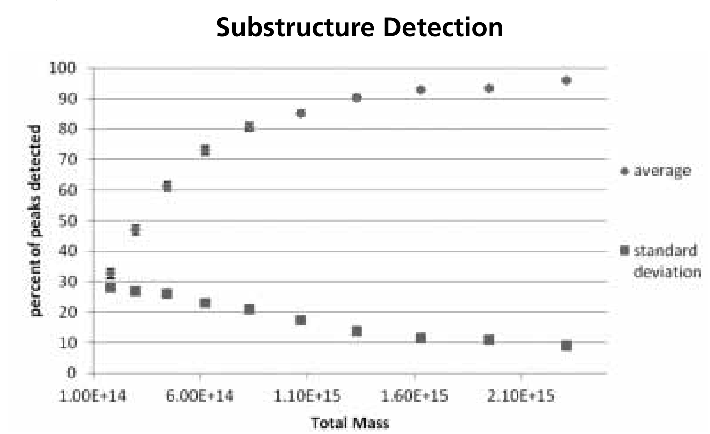

Figure 5. Small grid, 11x11. 300 simulations per point. r_c = 0.1 mpc and r_t = 1.5 mpc for both peaks.

The substructure was consistently detected as long as the total mass of the cluster was sufficiently large. Only peaks that were above 3 standard deviations were considered.

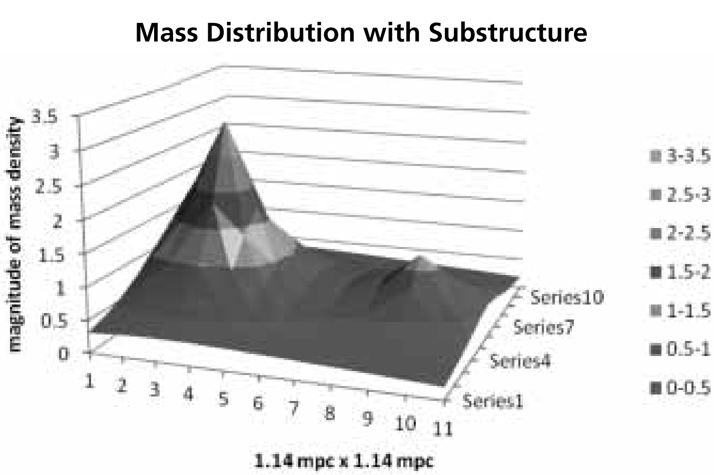

Figure 6. Small grid, 11x11. $2.3 \cdot 10^{15}M_{\odot}$. Exact values. This is the noiseless shape of the distribution used to detect substructure.

The substructure was detected 80% of the time when the total mass of the system was above $1.6 \cdot 10^{15} M_\odot$ and the mass of the substructure was $4 \cdot 10^{14} M_\odot$. It was detected 50% of the time when the total mass of the system was $6.3 \cdot 10^{14} M_\odot$ and the substructure's total mass was $1.6 \cdot 10^{14} M_\odot$.

There was less than 1% difference in total mass error between our interpolation and a perfect interpolation.

Conclusion

This project provided the tools needed to perform more rigorous testing of the Poisson equation. Initial testing suggests that our method for determining the mass distribution of a galaxy cluster is about as good as the existing methods. Exactly where it lies in relation to them is yet to be determined. The Poisson method can detect the total mass of a galaxy cluster $\pm 2.4 \cdot 10^{14} M_\odot$. Substructure was detected 80% of the time when the total mass of the system was above $1.6 \cdot 10^{15} M_\odot$ and the mass of the substructure was $4 \cdot 10^{14} M_\odot$. It was detected 50% of the time when the total mass of the system was $6.3 \cdot 10^{14} M_\odot$ and the substructure's total mass was $1.6 \cdot 10^{14} M_\odot$. Numerical error inherent in the method was less than 3% in total. There was less than 1% difference in total mass error between our interpolation and a perfect interpolation.

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