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Matthew Gagne graduated from Bridgewater with a degree in Mathematics in January 2013. His research, made possible by an Adrian Tinsley Summer Research Grant, was patiently and skillfully mentored by Dr. Shannon Lockard of the BSU mathematics department. She has his deep gratitude. He would also like to thank his wife, Eunice, for her love, understanding, and support during his time at BSU. Matthew extends his thanks to the dedicated staff at the BSU Undergraduate Research Office. They provided an environment in which research can flourish. Lastly, he wants to thank the entire BSU community which made his time there so enjoyable. He looks forward to helping others to develop an understanding and appreciation of the beauty and utility of mathematics.

For centuries, people have been interested in patterns. Even in that which appears random, humans have been trying to understand the underlying order of things. Mathematicians throughout time have studied many phenomena, including infinite sequences of numbers and have been able, at times, to see structure. Many have found the satisfaction, even joy, of discovering patterns in sequences. A typical way to describe this is by a recursive formula. A recursive definition defines a term in the sequence using the previous terms in the sequence. Even more satisfying than a recursive formula is a closed formula. With this, one can find the number at any position in the sequence. A closed formula is like a locksmith cutting a master key for every lock in a building.

For this project, we focused on patterns formed in the Calkin-Wilf tree. The Calkin-Wilf tree was used by Neil Calkin and Herb Wilf to give a new enumeration of the rational numbers in their 1999 paper, "Recounting the Rationals." The paper spurred much research, to which we add this paper.

The Calkin-Wilf tree is a binary tree, where each vertex has a left and right child. These children are vertices on the tree labeled by fractions. At the top, or root, of the tree is the fraction 1/1. At each level in the tree, the labels of the left and right children of an arbitrary vertex *a/b* are formed as shown in Figure 1.

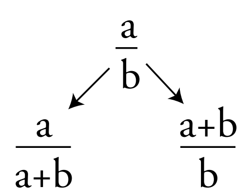

Figure 1. Left and Right Children

This diagram is used to produce labels for every vertex on the tree. The left child coming from a vertex retains the numerator of its parent and adds the numerator and denominator of its parent to form its denominator. The right child retains the denominator of its parent and forms its numerator from the sum of its parent's parts. Each of these children has two children whose labels are found using the same criteria. The infinite repetition of this process labels every node on the tree with a fraction. The first five levels of the Calkin-Wilf tree are as seen in Figure 2.

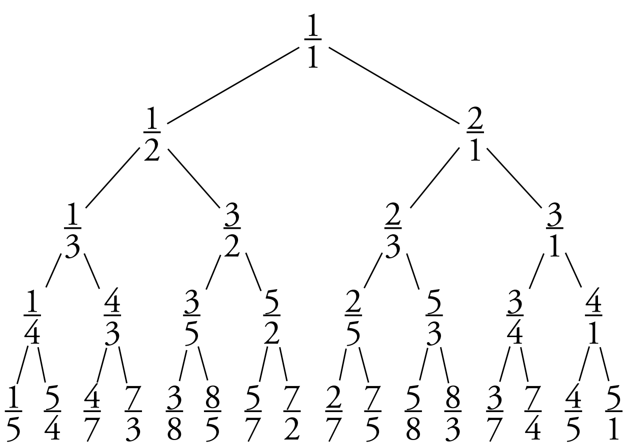

Figure 2. The Calkin-Wilf tree (1st 5 levels)

The sequence of fractions in the tree possesses many interesting properties as shown in (Calkin & Wilf, 1999) and (Aigner & Ziegler, 2010). Several of these properties follow. On each level of the tree, the corresponding left and right fractions are the inverses of the other. For instance, on the 3rd level, the 3rd fraction from the left is 3/5 and the 3rd fraction from the right is 5/3. This pattern is found throughout the tree. Additionally, every fraction is in fully reduced form. There will not appear a numerator and denominator which have a common factor. Calkin and Wilf first showed that this sequence of fractions gives an enumeration of the rationals, that is, every positive rational number will appear once and only once on the tree (Calkin & Wilf, 1999).

The sequences of numerators and denominators considered separately prove interesting as well. The sequence of numerators in the tree begins as:

1,1,2,1,3,2,3,1,4,3,5,2,5,4,1,5,4,7,3,8,5,7,2,7,5,8,3,7,4,5, . . .

The sequence of denominators in the tree begins as:

1,2,1,3,2,3,1,4,3,5,2,5,4,1,5,4,7,3,8,5,7,2,7,5,8,3,7,4,5, . . .

We notice that if we remove the first term of the sequence of numerators, we get the sequence of denominators. So these are essentially the same sequence. We name the sequence of numerators $b(n)$, for all $n \geq 0$, with n being the position in the sequence. Then for all $n \geq 1$, the sequence of fractions is given by.

$$\frac{b(n-1)}{b(n)}$$

In (Calkin & Wilf, 1999), the authors showed that the sequence $b(n)$ satisfies the following equalities:

$$b(2n+1) + b(n) \qquad (1)$$
$$b(2n) + b(n) + b(n-1) \qquad (2)$$

We will use these results later.

The paper, "Linking the Calkin-Wilf and Stern-Brocot Trees," discusses the concept of diagonals in the Calkin-Wilf tree (Bates et. al., 2010). Diagonals are sequences of fractions which share the same relative position on consecutive levels of the tree. The left diagonal L_i contains every fraction which lies i positions from the left edge of the tree. The right diagonal R_i contains every fraction which is i positions from the right. Fractions in the left diagonals are the inverse of their corresponding fraction of the opposing right diagonal. This is due to the property we discussed earlier concerning inverses in the tree. Through our research we found that the fractions in each diagonal followed certain patterns. For example, consider L_2 the diagonal containing the 2nd fraction from the left on each level of the tree. Let us consider the 1st several fractions in L_2. The first several fractions in the left diagonal L_2 are:

$$\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5} \ldots$$

We can see a pattern from these first terms. Starting from 2/1 we add 1 to the numerator and 1 to the denominator. This provides us a way of determining a fraction based on the preceding fraction. We would like to find a closed formula. A closed formula would enable us to find the fraction at any specific position in the diagonal. For L_2 it was straightforward. As we can see in the above sequence, the numerator is 1 more than the denominator. If we let the denominator be j, its position in the sequence, then the numerator would simply be $j+1$. The formula for the j^{th} fraction in L_2 is $\frac{j+1}{j}$.

We recall that we can express each fraction on the tree by $\frac{b(n-1)}{b(n)}$. Therefore,

$$\frac{b(n-1)}{b(n)} = \frac{j+1}{j}$$

for all fractions in L_2 . We will show that this is true after proving the following lemmas.

Lemma 1: $b(2^j - 1) = 1$ for all $j \geq 1$

Proof: Base case: If we let $j = 1$, then
$b(2^j - 1) = b(2^1 - 1) = b(1) = 1$. Thus this is true for the base case.

We will assume that $b(2^j - 1) = 1$.

We want to show that $b(2^{j+1} - 1) = 1$.

$$b(2^{j+1} - 1) = b(2(2^j - 1) + 1)$$
$$= b(2^j - 1) \quad \text{by equation 1}$$
$$= 1 \quad \text{by induction hypothesis}$$

Therefore, $b(2^j - 1) = 1$ by the Principle of Mathematical Induction. Q.E.D.

Lemma 2: $b(2^j) = j + 1$ for all $j \geq 1$ for all

Proof: Base case: If we let $j = 1$ then $(2^j) = b(2^1) = b(2) = 2$. Thus this is true for the base case.

We will assume that $b(2^j) = j + 1$.
We will show that $b(2^{j+1}) = j + 2$.

$$b(2^{j+1}) = b(2(2^j))$$
$$= b(2^j) + b(2^j - 1) \quad \text{by equation 2}$$
$$= j + 1 + 1 \quad \text{by induction hypothesis and Lemma 1}$$
$$= j + 2.$$

By showing that $b(2^{j+1}) = j + 2$, we show that $b(2^j) = j + 1$. Q.E.D.

Lemma 3: $b(2^j + 1) = j$ for all $j \geq 1$

Proof Base case: If we let $j = 1$ then
$b(2^j+1) = b(2^1 + 1) = b(3) = 1$. Thus this is true for the base case.

We will assume that $b(2^j + 1) = j$. We want to show that
$b(^{2j+1} + 1) = j + 1$.

$$b(2^{j+1} + 1) = b(2(2^j) + 1)$$
$$= b(2^j) \quad \text{by equation 1}$$
$$= j + 1 \quad \text{by Lemma 2}$$

Therefore, we here see that $b(2^{j+1} + 1) = j + 1$. From this, we show that, $b(2^j + 1) = j$. Q.E.D.

From the above propositions, we can find a formula for all fractions in L_2. This is shown below.

Proposition: The j^{th} fraction in the left diagonal L_2 of the Calkin-Wilf tree is given by $\frac{j+1}{j}$.

Proof: Recall, we know that the n^{th} fraction in the Calkin-Wilf tree is given by $\frac{b(n-1)}{b(n)}$.

Since the j^{th} level of the tree contains 2^j entries for $j \geq 0$, The fractions in L_2 are in the $2^j + 1$ position on the tree and so, in this case, $n = 2^j + 1$. Moreover, since one fraction from each level is selected to form the L_2 sequence, we note that j also represents the index of the fraction in L_2. Therefore,

$$\frac{b(n-1)}{b(n)} = \frac{b(2^j)}{b(2^j + 1)}$$

In Lemma 2 we showed that $b(2^j) = j + 1$. In Lemma 3, we showed that $b(2^j + 1) = j$. Therefore,

$$\frac{b(2^j)}{b(2^j + 1)} = \frac{j+1}{j}$$

Thus every fraction in L_2 can be expressed as $\frac{j+1}{j}$. Q.E.D.

We now see that the j^{th} fraction in L_2 is $\frac{j+1}{j}$.

We were able to conjecture formulas for the first 38 diagonals on the Calkin-Wilf Tree. The chart below shows the conjectured formulas for the first 10 left diagonals.

L_1	$\frac{1}{j}$	L_6	$\frac{3j+2}{2j+1}$
L_2	$\frac{j+1}{j}$	L_7	$\frac{2j+1}{3j+1}$
L_3	$\frac{j+1}{2j+1}$	L_8	$\frac{3j+1}{j}$
L_4	$\frac{2j+1}{j}$	L_9	$\frac{j+1}{4j+3}$
L_5	$\frac{j+1}{3j+2}$	L_{10}	$\frac{4j+3}{3j+2}$

Examinations of the formulas in the chart above lead us to ask questions such as, "Could we quickly find the formula for the 101st diagonal?" In essence, "Is there a formula for the formulas?" As is often the case in mathematics, we looked for patterns in the formulas. We first notice that the denominator of one diagonal is often the same as the numerator of the following diagonal. However, this didn't happen for the diagonals of a power of 2 and the following one. The pattern was re-established on the next diagonal and continued until the next diagonal that was one after a diagonal of a power of 2. We soon discovered that each fraction in every diagonal of a power of 2, $L_2^{\,i}$ was predictable. Below, we have the formulas for the first few diagonals which are a power of 2.

L_1	L_2	L_4	L_8	L_{16}	L_{32}
$\frac{1}{j}$	$\frac{j+1}{j}$	$\frac{2j+1}{j}$	$\frac{3j+1}{j}$	$\frac{4j+1}{j}$	$\frac{5j+1}{j}$

When we isolate these formulas, we see the pattern emerge. To find any numerator in L_{2^i}, we had only to multiply i by j, its position in the diagonal, and then add 1. So the numerator of L_{2^i} is given by $ij + 1$. It appears from the chart that the denominator is always j. For example, L_{2^z} is the 4^{th} diagonal, so $i = 2$. Then, the j^{th} fraction in L_{2^z} is $\frac{2j+1}{j}$. Based on these examples, we offer these conjectures:

Conjecture 1: The j^{th} fraction in the diagonal L_{2^i} is $\frac{ij+1}{j}$.

We surmised a formula for L_{2^i+1} the diagonals immediately following L_{2^i}. If we examine the formulas for L_{2^i+1} another pattern emerges.

L_2	L_3	L_5	L_9	L_{17}	L_{33}
$\frac{j+1}{j}$	$\frac{j+1}{2j+1}$	$\frac{j+1}{3j+2}$	$\frac{j+1}{4j+3}$	$\frac{j+1}{5j+4}$	$\frac{j+1}{6j+5}$

As we can see above, the numerator is $j + 1$. The denominators of the fractions in L_{2^i+1} are formed by multiplying j by $i + 1$and then adding i.

Conjecture 2: The j^{th} fraction in the diagonal L_{2^i+1} is

$$\frac{j+1}{j(i+1)+i} \quad .$$

The Calkin-Wilf tree is an interesting mathematical structure which provides intriguing patterns to investigate. Whether in puzzles, art, or mathematics, patterns stir our curiosity and wonder. When we see an order develop, we want to explore more. When we can accurately predict what comes next, it brings tremendous satisfaction. It can also produce a feeling of over-arching order. Because the Calkin-Wilf tree provides such order, it is a joy to explore.

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