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Extremal Graph Theory: Turán's Theorem

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1 History of Turán's Theorem

Extremal graph theory is a branch of graph theory that involves finding the largest or smallest graph with certain properties. Extremal graph theory began with this question [3]: if we have a graph of order n, what is the maximum size allowed so that it does not contain a triangle? A *graph* $G = (V, E)$ is a set of vertices V together with a set of edges E. A triangle is simply a graph with 3 vertices and 3 edges. We call the number of vertices of a graph G the *order* of G, and the number of edges in G the *size* of G. Consider a graph G of order $n = 4$ and size $m = 4$ as in Figure 1.

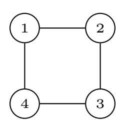

Figure 1: Graph G of order $n = 4$ and size $m = 4$.

This graph is a cycle of order 4, denoted C_4. Clearly G does not contain a triangle. However, if we were to add one more edge to G, then no matter where we attempt to put the edge we are guaranteed to create a triangle (see Figure 2).

Figure 2: Graphs of order $n = 4$ and size $m = 5$ that contain a triangle.

In fact, in any graph with 4 vertices and 5 edges we are forced to have a triangle. This is easier to see for a graph of relatively small order. In this case there are only six different labeled graphs with 4 vertices and 5 edges, as seen in Figure 3.

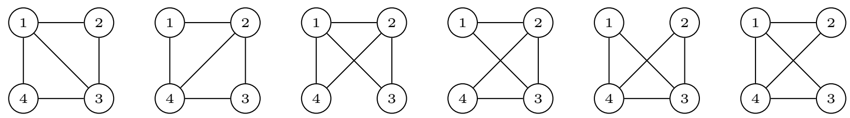

Figure 3: All labeled graphs of order $n = 4$ and size $m = 5$.

Therefore, we see that for a graph of order $n = 4$, the maximum number of edges so that the graph does not have to contain a triangle is $m = 4$.

This question was answered in general by Mantel, a Dutch mathematician, in 1907 [3]. Mantel gave a lower bound on the number of edges in a graph so that it must contain a triangle.

Theorem 1.1. (Mantel, [3]) If G is a graph of order $n \geq 3$ and size $m > \left\lfloor \frac{n^2}{4} \right\rfloor$, then G contains a triangle.

Notice that for the graph G in Figure 1, the number of edges $m = 4$ is not greater than $\left\lfloor \frac{4^2}{4} \right\rfloor = 4$. However, if we add one more edge to have 5 edges, then 5 is indeed greater than 4, so by Theorem 1.1 we are guaranteed to have a triangle as we saw in Figure 2.

Similarly, consider a graph H of order $n = 5$. First, note that $\left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{5^2}{4} \right\rfloor = 6$. We see that it is possible to draw a graph H without a triangle using 6 edges, like the graph on the left in Figure 4, but impossible with $m = 7$ edges, which is indeed greater than $\left\lfloor \frac{5^2}{4} \right\rfloor$, as seen by the graph on the right in Figure 4.

Figure 4: Graph H on the left that does not contain a triangle, and graph H' on the right that does contain a triangle.

Of course, mathematicians have built upon Mantel's work and more results have been given. Some of these results guarantee whether or a not a graph of a specific size contains a triangle as a subgraph. For example, it has been shown that for a graph of even order $n = 2\ell$ and size $m = \ell^2$ for some positive integer ℓ, the only graph with no triangle as a subgraph is $K_{\ell,\ell}$ [3]. Additionally, a graph of odd order $n = 2\ell + 1$ and size $m \geq \ell^2 + \ell + 1$ must contain a triangle [3].

It is important to note that a triangle is a *complete graph* on $p = 3$ vertices.

Definition 1.1. A graph G is *complete* if every two distinct vertices of G are adjacent, or if there is an edge between every two distinct vertices. The complete graph with p vertices is denoted K_p.

Indeed, a triangle is K_3. Some examples of complete graphs can be seen in Figure 5.

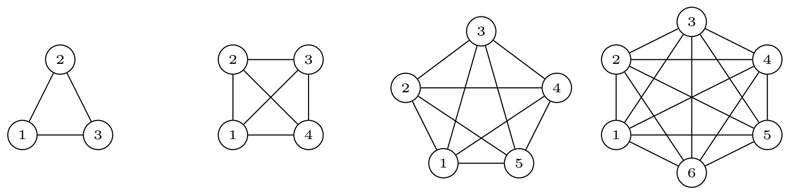

Figure 5: Complete graphs K_3, K_4, K_5, and K_6 respectively.

Definition 1.2. A graph $G = (V, E)$ is *regular* if the vertices of G all have the same degree. If $\deg(v) = r$ for every vertex $v \in V(G)$, then G is called an r-regular graph.

Notice that the vertices in each of the graphs in Figure 5 all have the same degree. Respectively, these graphs are 2-regular, 3-regular, 4-regular, and 5-regular.

Theorem 1.1 provides us with parameters regarding graphs containing K_3 as a *subgraph*.

Definition 1.3. A graph $H = (V_H, E_H)$ is a *subgraph* of $G = (V_G, E_G)$, denoted $H \subseteq G$, if $V_H \subseteq V_G$ and $E_H \subseteq E_G$.

Naturally, the next question to ask is: what is the maximum number of edges in a graph with n vertices that does not contain a complete subgraph with p vertices? That is, what is the maximum number of edges in a graph of order n that does not contain K_p as a subgraph? This generalization is precisely the question Pál Turán was interested in answering.

Pál Turán was a Hungarian mathematician born on August 28, 1910 in Budapest. While he was primarily a probabilistic and analytical number theorist, his renowned work in graph theory is considered to be the birth of extremal graph theory [3]. Turán's Theorem is a major result in extremal graph theory that gives an upper bound on the number of edges in a graph with no K_p as a subgraph. First we have a related definition.

Definition 1.4. A *p-clique* is a subset of p vertices of a graph $G = (V, E)$ such that each of the p vertices are adjacent. That is, a p-clique is a subgraph of G that is isomorphic to K_p.

Now we state Turán's Theorem.

Turán's Theorem - Version 1. [1] If a graph $G = (V, E)$ on n vertices has no p-clique, $p \geq 2$, then

$$|E| \leq \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}. \tag{1}$$

Being such an important theorem in extremal graph theory, there are many different methods to go about proving Turán's Theorem. We will continue by presenting a few of these proofs, explaining the logic and tools used in them, and bringing attention to notable differences between them.

2 A Probabilistic Proof

This first proof, as adapted from a proof in Chapter 40 of [1], uses tools from probability theory. The proof outline is as follows. We first prove a lemma about the order of a complete subgraph of a graph G, which will later be useful in proving Turán's Theorem. To prove the lemma we take a random *permutation* of vertices of G and extract a subset of vertices from the permutation in such a way that this subset forms a clique.

Definition 2.1. A *permutation* of a list, or set, is a reordering of the elements of the list.

Then we find the *expected value* of the *random variable* assigned to keep track of the order of this set and compare it to the order of a largest clique, which completes the proof of the lemma.

Definition 2.2. A *random variable* X is measurable function on the sample space of an experiment which attaches numbers to outcomes.

Definition 2.3. The *expected value* of a discrete random variable X is the sum of all values of X times the probability that each value occurs. That is,

$$E(X) = \sum_{k=1}^{n} k \cdot P(X = k).$$

We are then able to deduce the result in Turán's Theorem using the Cauchy-Schwarz inequality to give a needed bound, Theorem 2.1, and the lemma that follows.

The Cauchy-Schwarz inequality states that for any sequences $\{a_i\}$ and $\{b_i\}$ we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right). \tag{2}$$

The following theorem is often called the First Theorem of Graph Theory, as it easily relates the degrees of the vertices of a graph to the size of the graph [3].

Theorem 2.1. If $G = (V, E)$ is a graph of order n and d_i is the degree of vertex v_i for $1 \leq i \leq n$, then

$$\sum_{i=1}^{n} d_i = 2|E|$$

Proof. Each edge in G is counted twice when adding the degrees of each vertex. Thus, the sum of the degrees is twice the number of edges as claimed. □

We will now prove the lemma.

Lemma 2.2. Let $\omega(G)$ be the order of a largest clique in a graph $G = (V, E)$ and let d_i be the degree of a vertex v_i in G. Then

$$\omega(G) \geq \sum_{i=1}^{n} \frac{1}{n - d_i}.$$

Proof. Let $G = (V, E)$ be a graph on n vertices. Let π be any permutation of the set $\{1, 2, ..., n\}$. Then π induces an ordering $v_{\pi(1)}, v_{\pi(2)}, ..., v_{\pi(n)}$ on the vertex set $V(G)$. Using this notation, a vertex $v_{\pi(k)}$ is listed in the kth position in π, and the position a vertex v_j in π is given by $\pi^{-1}(j)$. Consider the subset C_π of vertices of G defined by $C_\pi = \{v_k \in V : v_k \sim v_j \text{ for all } j \text{ with } \pi^{-1}(j) < \pi^{-1}(k)\}$. Note that $v_{\pi(1)}$ is always in C_π by construction.

Let Π be chosen uniformly at random from all $n!$ permutations. Then C_Π is a subset of vertices of G such that all vertices in C_Π are adjacent. Hence these vertices form a clique.

Now let

$$X_i = \begin{cases} 1, & v_i \in C_\Pi \\ 0, & v_i \notin C_\Pi \end{cases}$$

be the indicator of the event that $v_i \in C_\Pi$.

Let

$$X = |C_\Pi| = \sum_{i=1}^{n} X_i,$$

be a random variable corresponding to C_Π that tracks the order of C_Π.

We are interested in the probability that a vertex v_i is in C_Π. A vertex v_i has d_i neighbors. There are $\binom{n}{d_i}$ ways to locate the d_i neighbors amongst all n vertices. Then, there are $(d_i)!$ ways to arrange the d_i neighbors amongst themselves. Notice that if any of the $n - d_i$ non-neighbors are listed before v_i in Π, then v_i would not be adjacent to all previous vertices hence v_i would not be in C_Π. Finally, there are $(n - d_i - 1)!$ ways to order the non-neighbors with v_i listed first. Thus the number of permutations π for which v_i is in C_π is

$$\binom{n}{d_i} \cdot (d_i)! \cdot (n - d_i - 1)! = \frac{n!(d_i)!(n - d_i - 1)!}{(n - d_i)!(d_i)!} = \frac{n!}{n - d_i}.$$

As each of the $n!$ permutations is equally likely, we have

$$P(v_i \in C_\Pi) = \frac{1}{n!} \cdot \frac{n!}{n-d_i} = \frac{1}{n-d_i}.$$

It follows that

$$P(v_i \notin C_\Pi) = 1 - \frac{1}{n-d_i}.$$

Thus by linearity of expectation we have

$$\begin{aligned}
E[X] &= E\left[\sum_{i=1}^{n} X_i\right] \\
&= \sum_{i=1}^{n} E[X_i] \\
&= \sum_{i=1}^{n} 1 \cdot P(v_i \in C_\Pi) + 0 \cdot P(v_i \notin C_\Pi) \\
&= \sum_{i=1}^{n} 1 \cdot \left(\frac{1}{n-d_i}\right) + 0 \cdot \left(1 - \frac{1}{n-d_i}\right) \\
&= \sum_{i=1}^{n} \frac{1}{n-d_i}
\end{aligned}$$

As the average cannot exceed the maximum, we see that a largest clique has order

$$\omega(G) \geq \sum_{i=1}^{n} \frac{1}{n-d_i}. \tag{3}$$

Thus the lemma is proved. □

Now to prove Turán's Theorem - Version 1.

Proof. To prove Turán's Theorem we will make use of the Cauchy-Schwarz inequality in (2). Let $a_i = \sqrt{n-d_i}$ and $b_i = \frac{1}{\sqrt{n-d_i}}$. Substituting into (2), we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 = \left(\sum_{i=1}^{n} \sqrt{n-d_i} \cdot \frac{1}{\sqrt{n-d_i}}\right)^2 = \left(\sum_{i=1}^{n} 1\right)^2 = n^2$$

and

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) = \left(\sum_{i=1}^{n} n - d_i\right)\left(\sum_{i=1}^{n} \frac{1}{n-d_i}\right).$$

Thus

$$\begin{aligned}
n^2 &\le \left(\sum_{i=1}^{n} n - d_i\right)\left(\sum_{i=1}^{n} \frac{1}{n-d_i}\right) \\
&\le \left(\sum_{i=1}^{n} n - d_i\right)\omega(G), \text{ by Lemma 2.2} \\
&\le \left(\sum_{i=1}^{n} n - d_i\right)(p-1), \text{ since } G \text{ has no } p\text{-clique}, \omega(G) \le p-1 \\
&= \Big(n^2 - 2|E|\Big)(p-1), \text{ by linearity and Theorem 2.1.}
\end{aligned}$$

Hence

$$n^2 \le \Big(n^2 - 2|E|\Big)(p-1).$$

Solving for $|E|$, we see

$$|E| \le \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}.$$

Thus Turán's Theorem is proved. □

3 A Combinatorial Proof

This next proof involves taking a more combinatorial approach, and is also adapted from a proof in Chapter 40 of [1]. The proof outline is as follows. We first assume a graph G has the most possible edges. We use the assumption in Turán's Theorem - Version 1 to obtain a $(p-1)$-clique as a subgraph and call those vertices the set A. We then count the number of edges in A, the number of edges between the remaining vertices of G, and finally the number of edges between the vertices of A and the remaining vertices of G. We use these three quantities to bound the total number of edges in G above by the bound given in Turán's Theorem.

In the following proof, we will also use the fact that the number of edges in a complete graph of order n is $\binom{n}{2} = \frac{n(n-1)}{2}$.

Now let us prove Turán's Theorem.

Proof. We will prove Turán's Theorem - Version 1 by strong induction on n.

For our base case, we will show that the theorem holds for all $n < p$. To begin, suppose a graph G on n vertices has no p-clique. Note that since $n < p$, we have $n \le p - 1$, hence $\frac{-1}{n} \le \frac{-1}{p-1}$. Since G has no p-clique and $n < p$, we see that G has at most $\binom{n}{2}$ edges. Thus the number of edges in G is at most

$$\begin{aligned} |E| &\leq \binom{n}{2} \\ &= \frac{n(n-1)}{2} \\ &= \left(1 - \frac{1}{n}\right)\frac{n^2}{2} \\ &\leq \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}. \end{aligned}$$

Thus the base case holds.

For our induction hypothesis, suppose that for some $k \geq p$, a graph $G = (V, E)$ of order ℓ, where $1 \leq \ell \leq k$, with no p-clique has at most

$$\left(1 - \frac{1}{p-1}\right)\frac{\ell^2}{2}$$

edges.

We want to show that for a graph $G = (V, E)$ on $k+1$ vertices with no p-clique, we have

$$|E| \leq \left(1 - \frac{1}{p-1}\right)\frac{(k+1)^2}{2}.$$

Let G be a graph on $k+1$ vertices with no p-clique, $p \geq 2$. Since we are interested in finding an upper bound on the number of edges in G, suppose G has maximal edges. Since G has maximal edges and no p-clique, G must have a $(p-1)$-clique, otherwise we could add more edges. Let A be the subset of vertices that form a $(p-1)$-clique. Since A is complete, each of the $p-1$ vertices are adjacent to the remaining $p-2$ vertices. Thus the number of edges in A, denoted e_A, is

$$e_A = \binom{p-1}{2} = \frac{(p-1)(p-2)}{2}. \tag{4}$$

Let $B = V \setminus A$ be the set containing the remaining vertices in G. Then B has $(k+1) - (p-1) = k - p + 2$ vertices. Since B has no p-clique, by our induction assumption we have

$$e_B \leq \left(1 - \frac{1}{p-1}\right)\frac{(k-p+2)^2}{2}. \tag{5}$$

The number of edges in G is now given by $e_G = e_A + e_B + e_{A,B}$, where $e_{A,B}$ is the number of edges between A and B. To find $e_{A,B}$, consider a vertex v_j in B. If v_j is adjacent to $p-1$ vertices in A, then G would have a p-clique which is a contradiction. So v_j is adjacent to at most $p-2$ vertices in A, for any $v_j \in B$. Hence

$$e_{A,B} \leq (k-p+2)(p-2). \tag{6}$$

Combining the quantities in (4), (5), and (6) gives the number of edges in G as

$$\begin{aligned}
e_G &= e_A + e_B + e_{A,B} \\
&\le \frac{(p-1)(p-2)}{2} + \left(1 - \frac{1}{p-1}\right)\frac{(k-p+2)^2}{2} + (k-p+2)(p-2) \\
&= \frac{p-2}{2(p-1)}\left((p-1)^2 + (k-p+2)^2 + 2(p-1)(k-p+2)\right) \\
&= \frac{p-2}{2(p-1)}\left((p-1) + (k-p+2)\right)^2 \\
&= \frac{p-2}{2(p-1)}(k+1)^2 \\
&= \left(1 - \frac{1}{p-1}\right)\frac{(k+1)^2}{2}.
\end{aligned}$$

Thus

$$|E| \le \left(1 - \frac{1}{p-1}\right)\frac{(k+1)^2}{2},$$

which completes the proof of Turán's Theorem. □

4 Graphs of Maximum Size

Upon initial inspection, the bound given in Version 1 of Turán's Theorem seems unforeseen. To better understand bounds in graph theory, it often helps to look at different types and families of graphs. Of course, there are many different types and families of graphs, each of which have their own interesting properties and usefulness. We will proceed by discussing a particular family of graphs and discover how the upper bound in Version 1 was obtained.

First we will turn our attention to *k-partite* graphs.

Definition 4.1. A graph G is *k-partite* if $V(G)$ can be partitioned into k subsets $V_1, V_2, ...V_k$, called *partite sets*, such that if $uv \in E(G)$, then u and v belong to different partite sets.

We also have *complete k-partite graphs.*

Definition 4.2. A k-partite graph G is a *complete k-partite* graph if every two vertices in different partite sets are joined by an edge. For each partite set V_i, if $|V_i| = n_i$ for $1 \le i \le k$, then the complete k-partite graph is denoted $K_{n_1,n_2,..,n_k}$.

Some complete k-partite graphs can be seen in Figure 6. The first and second graphs in Figure 6 are 2-partite, which are typically called bipartite graphs, and the third graph is 4-partite.

While working to find certain graphs of maximum size, Turán focused on a particular family of finite graphs, namely *Turán graphs.*

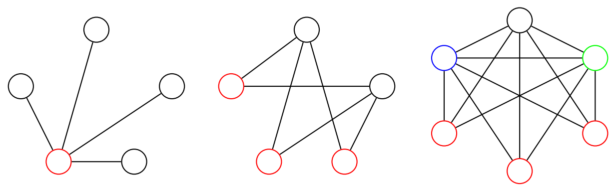

Figure 6: Complete partite graphs $K_{1,4}$, $K_{2,3}$, and $K_{1,1,1,3}$ respectively.

Definition 4.3. The *Turán graph* $T_{n,k}$ is the complete k-partite graph of order n, the cardinalities of whose partite sets differ by at most one.

The Turán graph $T_{n,k}$ has n vertices and k partite sets. Since the cardinalities of each partite set differ by at most one, each partite set will have either $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$ vertices. If k divides n, then each partite set will have the same cardinality and $T_{n,k}$ will be regular. If k does not divide n and r is the remainder when n is divided by k, then exactly r of the partite sets of $T_{n,k}$ will have cardinality $\lceil \frac{n}{k} \rceil$ and $k - r$ partite sets will have cardinality $\lfloor \frac{n}{k} \rfloor$ [3]. Since $T_{n,k}$ is complete k-partite, every vertex x will be adjacent to every vertex y unless x and y are in the same partite set. Thus if $U_1, U_2, ..., U_k$ are the partite sets of $T_{n,k}$, then every vertex in U_i is adjacent to every vertex in U_j, for $i \neq j$.

As an example, consider the Turán graph $T_{10,3}$. To construct $T_{10,3}$, we note that each of the three partite sets will have either $\lfloor \frac{10}{3} \rfloor = 3$ or $\lceil \frac{10}{3} \rceil = 4$ vertices. Since 3 does not divide 10, exactly 1 of the partite sets will have $\lceil \frac{10}{3} \rceil = 4$ vertices, and 2 partite sets will have $\lfloor \frac{10}{3} \rfloor = 3$ vertices. We then draw an edge from each vertex to every other vertex not in the same partite set. Figure 7 shows two isomorphic representations of $T_{10,3}$.

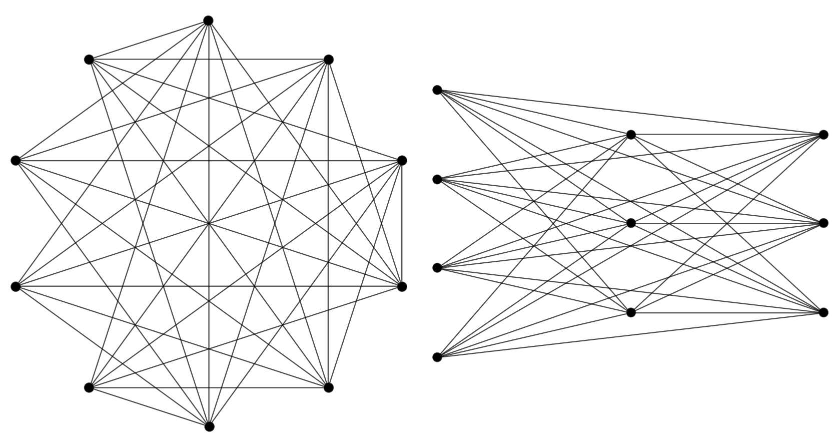

Figure 7: The Turán graph $T_{10,3}$.

Specifically, the graph on the right in Figure 7 illustrates the three partite sets vertically, each consisting of 4 vertices, 3 vertices, and 3 vertices respectively. Additionally, Figure 8 shows some other Turán graphs.

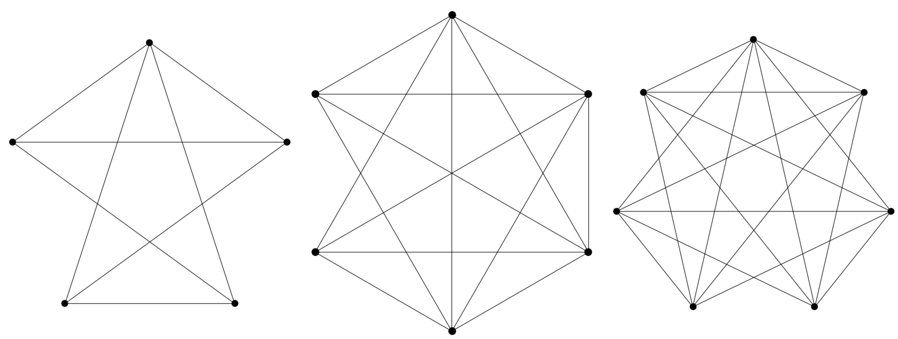

Figure 8: Turán graphs $T_{5,3}$, $T_{6,5}$, and $T_{7,4}$ respectively.

Let us compute the size of a Turán graph $T_{n,p-1}$ with maximal edges. When $p-1$ divides n, there are $\left(\frac{n}{p-1}\right)^2$ edges between any two partite sets. Also, there are $\binom{p-1}{2}$ ways to choose any two different partite sets from all $p-1$ partite sets. Thus the total number of edges in $T_{n,p-1}$ when $p-1$ divides n is

$$\binom{p-1}{2}\left(\frac{n}{p-1}\right)^2. \tag{7}$$

Notice that since $T_{n,p-1}$ has $p-1$ partite sets, it cannot contain a p-clique. Recalling Turán's Theorem - Version 1, the maximum number of edges in this graph is bounded above by

$$|E| \leq \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}.$$

In fact, from (7) we have that for $T_{n,p-1}$,

$$|E| \leq \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2} = \binom{p-1}{2}\left(\frac{n}{p-1}\right)^2.$$

Therefore, we see now that Turán graphs actually attain the maximum number of edges without some complete subgraph. We observe that when $p-1$ divides n, the number of edges in $T_{n,p-1}$ is equal to the upper bound given in Version 1. Correspondingly when $p-1$ does not divide n, the number of edges in $T_{n,p-1}$ is strictly less than this upper bound.

As an example, consider a graph F of order $n = 5$ such that F does not contain a 4-clique. There are many ways to construct F so that it does not contain K_4 as a subgraph, such as the graphs in Figure 9.

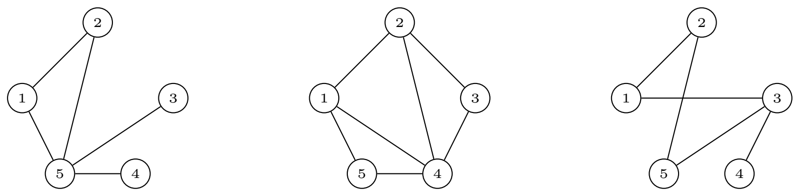

Figure 9: Graphs of order $n = 5$ that do not contain a K_4 as a subgraph.

We see that these graphs have size $m = 5$, $m = 7$, and $m = 5$ respectively. Turán's Theorem - Version 1 says that for any graph on $n = 5$ vertices with no 4-clique, the number of edges is bounded above by

$$\left(1 - \frac{1}{4-1}\right)\frac{5^2}{2} = \frac{25}{3} = 8.33.$$

Hence the maximum number of edges possible in such a graph is $m = 8$. Notice that the graphs in Figure 9 all have less than 8 edges, so these graphs do satisfy the bound given in Version 1.

The question now becomes: can we construct a graph so that it actually attains the maximum of $m = 8$ edges? Indeed as we saw above, the Turán graph $T_{5,3}$ is a graph with no 4-clique that attains the maximum number of edges. Figure 8 shows $T_{5,3}$ as the leftmost graph. Notice that $T_{5,3}$ has $m = 8$ edges, which is more than the graphs in Figure 9 and, in fact, the maximum number of possible edges.

5 A Graph Theoretic Proof

Turán graphs are graphs whose sizes meet the upper bound given by Version 1 of Turán's Theorem. These graphs lead us to a second version that we will prove in this section.

Turán's Theorem - Version 2. [4] The largest graph with n vertices that contains no subgraph isomorphic to K_{k+1} is a complete k-partite graph $K_{n_1,n_2,...,n_k}$ with $n = n_1 + n_2 + ... + n_k$ and $|n_i - n_j| \leq 1$ for all $i, j \in \{1, 2, ..., k\}$ with $i \neq j$.

The final proof is adapted from a proof in Chapter 4 of [4]. We will need the following lemmas and terminology.

Definition 5.1. A graph H is an *induced subgraph* of a graph G provided if $u, v \in V(H)$ and $uv \in E(G)$, then $uv \in E(H)$.

Lemma 5.1. [4] The graph of maximum size with n vertices and k partite sets is a complete k-partite graph $K_{n_1,n_2,...,n_k}$ with $n = n_1 + n_2 + ... + n_k$ and $|n_i - n_j| \leq 1$.

Proof. Let G be a k-partite graph of maximum size with partite sets V_i for $1 \leq i \leq k$. Let n_i be the number of vertices in the partite set V_i. Adding the vertices in each partite set, we have $n = n_1 + n_2 + ... + n_k$. By assumption, G has maximal edges, so every pair of vertices in different partite sets must be adjacent. Thus by definition, G is the complete k-partite graph $K_{n_1,n_2,...,n_k}$.

Denote the number of edges in a k-partite graph $K_{m_1,m_2,...,m_k}$ by

$$e(m_1, m_2, ..., m_k).$$

Consider the graph H obtained by removing the edges between the vertices in V_1 and V_2. Notice that H is a $(k-1)$-partite graph with partite sets $U_1, U_2, ..., U_{k-1}$ where $U_1 = V_1 \cup V_2$ and $U_i = V_{i+1}$ for $2 \leq i \leq k-1$. Therefore, $H = K_{n_1+n_2,n_3,...,n_k}$. Then the number of edges in H is

$$e_H = e(n_1 + n_2, n_3, ..., n_k).$$

When comparing H to G, note that H does not have the edges between V_1 and V_2. Thus the number of edges in G is

$$e_G = n_1 n_2 + e_H = n_1 n_2 + e(n_1 + n_2, n_3, ..., n_k). \tag{8}$$

Suppose by way of contradiction that the number of vertices in two of the partite sets of G differs by more than 1. Without loss of generality, suppose $n_1 \geq n_2 + 2$. Consider a new graph $\hat{G}$ with partite sets $W_1, W_2, ..., W_k$ such that $\hat{G} = K_{n_1-1,n_2+1,n_3,...,n_k}$. Notice that W_1 in $\hat{G}$ has 1 less vertex than V_1 in G, and W_2 in $\hat{G}$ has 1 more vertex than V_2 in G. Since $n_1 \geq n_2 + 2$, the number of vertices in W_1 and W_2 in $\hat{G}$ differ by at least 2. From (8), the number of edges in $\hat{G}$ is

$$\begin{aligned} e_{\hat{G}} &= e(n_1 - 1, n_2 + 1, n_3, ..., n_k) \\ &= (n_1 - 1)(n_2 + 1) + e((n_1 - 1 + n_2 + 1), n_3, ..., n_k). \end{aligned}$$

Then the number of edges in $\hat{G}$ minus the number of edges in G is

$$\begin{aligned} e_{\hat{G}} - e_G &= (n_1 - 1)(n_2 + 1) + e((n_1 - 1 + n_2 + 1), n_3, ..., n_k) \\ &\quad - (n_1 n_2 + e(n_1 + n_2, n_3, ..., n_k)) \\ &= (n_1 - 1)(n_2 + 1) - n_1 n_2 \\ &= n_1 - n_2 - 1. \end{aligned}$$

Since $n_1 \geq n_2 + 2$, we have that $n_1 - n_2 - 1 \geq 1$. Thus

$$e_{\hat{G}} - e_G = n_1 - n_2 - 1 \geq 1.$$

Hence when G has two partite sets such that $n_1 \geq n_2 + 2$, there exists a graph, namely $\hat{G}$, with at least one more edge than G. Therefore G cannot have the maximum number of edges when the number of vertices in any two partite sets differs by more than 1, a contradiction. Thus for any two partite sets V_i, V_j of the graph $G = K_{n_1,n_2,...,n_k}$, we have $|n_i - n_j| \leq 1$. This completes the proof. □

Lemma 5.2. [4] If G is a graph on n vertices that contains no K_{k+1}, then there is a k-partite graph H with the same vertex set as G such that

$$\deg_G(z) \leq \deg_H(z)$$

for every vertex z of G.

Proof. This proof is by strong induction on k.

For our base case, let $k = 2$. Let $G = (V, E_G)$ be a graph on n vertices that does not contain a K_3. We want to show there is a bipartite graph $H = (V, E_H)$ with the same vertex set as G such that

$$\deg_G(z) \leq \deg_H(z)$$

for every vertex z in G.

Let $\Delta(G) = \max\{\deg(v) : v \in V(G)\}$ and choose a vertex x in G such that $\deg_G(x) = \Delta(G)$. Let W be the set of vertices adjacent to x. Note that any two vertices in W cannot be adjacent, otherwise G would contain a K_3. Also, W only contains vertices adjacent to x, so x is in the set of vertices $V \setminus W$.

Define a new graph H on the vertices of G such that H is a complete bipartite graph with partite sets W and $V \setminus W$. Note that H only has the same vertex set as G, and not necessarily the same edge set as G. We want to show that $\deg_G(z) \leq \deg_H(z)$ for every vertex z in G.

Consider a vertex z in $V(G) = V(H)$. First, suppose $z \in W$. Note that no two vertices in W are adjacent in G or H. Then since z is adjacent to all vertices in $V \setminus W$ in H we have

$$\deg_H(z) = n - |W| \geq \deg_G(z).$$

Now suppose $z \in V \setminus W$. Then

$$\begin{aligned} \deg_H(z) &= \deg_H(x), \text{ since } W \text{ consists of all vertices adjacent to } x \\ &= \deg_G(x), \text{ since } x \text{ is adjacent to all vertices in } W \text{ in the graph } G \\ &\geq \deg_G(z), \text{ since we chose } x \text{ to have maximal degree in } G. \end{aligned}$$

Thus there is a bipartite graph H with the same vertex set as G such that

$$\deg_G(z) \leq \deg_H(z)$$

for every vertex z in G. Therefore the base case holds.

For our induction hypothesis, suppose that for a graph G of order n with no $K_{\ell+1}$ as a subgraph, for all $1 \leq \ell < k$, there exists an ℓ-partite graph H with the same vertex set as G such that $\deg_G(z) \leq \deg_H(z)$ for every vertex z in G.

We want to show that for a graph G of order n with no K_{k+1} as a subgraph that there exists a k-partite graph H such that $\deg_G(z) \leq \deg_H(z)$ for every vertex z in G.

Let $G = (V, E)$ be a graph on n vertices with no K_{k+1} as a subgraph. Let x be a vertex in V with maximal degree. Let W be the set of vertices adjacent to x.

Let G_0 be the subgraph induced by the vertices in W. Then G_0 cannot contain K_k as a subgraph, otherwise G would contain a K_{k+1} as a subgraph since x is adjacent to all vertices in W. By our induction hypothesis, there exists a $(k-1)$-partite graph H_0 such that

$$\deg_{G_0}(z) \leq \deg_{H_0}(z) \tag{9}$$

for every vertex z in W.

Define a new graph H on the vertices of G where the E_H includes all the edges in H_0 and the edges between every vertex in W and in $V \setminus W$. First, consider a vertex z in $V \setminus W$. Recall that x is also in $V \setminus W$. Then we have

$$\deg_G(z) \leq \deg_G(x)$$

since the vertex x was chosen with maximal degree in G. Then

$$\deg_G(x) = \deg_H(x) = \deg_H(z)$$

since all vertices in W in H are adjacent to all vertices in $V \setminus W$ in H, and $deg_G(x) = |W|$. Thus for every vertex z in $V \setminus W$, we have

$$\deg_G(z) \leq \deg_H(z) \tag{10}$$

since all vertices in W are adjacent to all vertices in $V \setminus W$ in H.

Now consider a vertex z in W. Then

$$\deg_G(z) \leq \deg_{G_0}(z) + |V \setminus W|$$

since G_0 does not contain the vertices in $V \setminus W$. Then, adding $|V \setminus W|$ to (9) we have

$$\deg_{G_0}(z) + |V \setminus W| \leq \deg_{H_0}(z) + |V \setminus W|.$$

Notice

$$\deg_{H_0}(z) + |V \setminus W| = \deg_H(z),$$

because H was constructed by connecting every vertex in W to every vertex in $V \setminus W$, in addition to keep all the adjacencies in H_0. Thus

$$\deg_G(z) \leq \deg_H(z) \tag{11}$$

for every vertex z in W. Now by (10) and (11), we have that

$$\deg_G(z) \leq \deg_H(z)$$

for every vertex z in V. Thus we have shown that there is a k-partite graph H such that $\deg_G(z) \leq \deg_H(z)$ for every vertex z in G, hence the lemma is proved. □

Now to prove Turán's Theorem - Version 2.

Turán's Theorem - Version 2. [4] The largest graph with n vertices that contains no subgraph isomorphic to K_{k+1} is a complete k-partite graph $K_{n_1,n_2,...,n_k}$ with $n = n_1 + n_2 + ... + n_k$ and $|n_i - n_j| \leq 1$ for all $i, j \in \{1, 2, ..., k\}$ with $i \neq j$.

Proof. Let G be a graph on n vertices that does not contain a subgraph isomorphic to K_{k+1}. By Lemma 5.2, we have that there exists a k-partite graph H with the same vertex set as G such that $\deg_G(z) \leq \deg_H(z)$ for every vertex z in G. Therefore the size of H is greater than or equal than the size of G. By Lemma 5.1, we have that the largest k-partite graph with n vertices is the complete k-partite graph $K_{n_1,n_2,...,n_k}$ with $|n_i - n_j| \leq 1$. Thus the largest graph with n vertices with no K_{k+1} as a subgraph is a complete k-partite graph $K_{n_1,...,n_k}$ with $n_1 + n_2 + ... + n_k = n$ and $|n_i - n_j| \leq 1$. Hence Turán's Theorem is proved. □

Indeed, these two versions of Turán's Theorem are equivalent. Version 1 gives an upper bound on the number of edges in a graph without some p-clique. In section 4 we found the size of Turán graphs $T_{n,p-1}$ to meet this bound when $p - 1$ divides n, or to fall just below this bound when $p - 1$ does not divide n. Version 2 says that Turán graphs are the largest graphs that satisfy these conditions. It has been shown in section 11.2 in [3] that Turán graphs are in fact the unique graphs of maximum size that satisfy Turán's Theorem.

Finally we have seen three very different proof of Turán's Theorem. In the probabilistic proof we relied on the expected value of the order of a largest clique in a graph to deduce the upper bound for the number of edges. In the combinatorial proof we cleverly counted the number of edges in any graph without some p-clique to arrive at that same upper bound. In the graph theoretic proof we utilized the structure of Turán graphs to show that these graphs attain the maximum number of edges. Of course, there are more than just three ways to prove Turán's Theorem, and more than just two ways to state it. The proofs presented here used various techniques from distinct areas of mathematics. Each of them has their own practicality and elegance in extremal graph theory. It remains the reader's decision to choose which proof of Turán's Theorem is most favorable.

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