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Symmetric Full Spark Frames

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Abstract

A full-spark frame of an n-dimensional vector space is a finite collection of m vectors ($m \geq$ n) with the following property: every subset of cardinality n of the given collection is a basis for the vector space. In this thesis, we realize the symmetric group S_n as a matrix group of invertible matrices with n^2 entries for $n > 2$. This representation induces a natural linear action on the vector space $\mathbb{C}^n$ and we prove that S_n admits an orbit which is a full-spark frame if and only if $n \leq 3$.

Contents

1 Introduction

The main objective of the present thesis is to investigate the existence and explicit construction of full-spark frames generated by subgroups of the symmetric group. Full-spark frames for a finite-dimensional vector space are defined as spanning sets enjoying the additional property that every subset whose cardinality is equal to the dimension of the given vector space is a basis for the ambient space. Full-spark frames are generally difficult to construct and are appealing for a variety of reasons [9]. In addition to the fact that full-spark frames are interesting in their own right, they are also widely applicable. They provide effective tools that can be exploited to design signal transmission methods that are resilient to noise and are maximally robust to erasures. As far as we can tell, the classification problem of representations of finite groups giving rise to full-spark frames is a difficult problem which has yet to be settled. In fact, only a few examples of full-spark frames are known in the literature. Most of these examples are obtained by considering representations of the following groups: finite cyclic groups, the finite Heisenberg groups, and the Dihedral groups [9, 6, 8, 7]. For the Dihedral groups, the problem of existence of full-spark frames arising from the action of a class of representations was recently completely settled in a series of two papers in [8, 7].

In this thesis, we investigate the existence of maximal subgroups of the symmetric groups having the Haar property. To be precise, let $\pi : S_n \longrightarrow GL(n, \mathbb{C})$ be the canonical representation of the symmetric group. This representation induces an action of the symmetric group on the n-dimensional complex vector space $\mathbb{C}^n$ and we look for maximal subgroups of $\pi(S_n)$ with the Haar property. Moreover, we shall prove that the group $\pi(S_n)$ has the Haar property if and only if $n \leq 3$.

Our work is organized as follows: in the second section, we formalize the concepts of Haar property and full-spark frames. In the third section, we prove that every cyclic subgroup of order n of the symmetric group has the Haar property. The fourth section reviews known results of Dihedral representations having the Haar property and a proof of our main result, that S_3 is the only symmetric group with the Haar property, is given in the fifth section. Furthermore, examples are given throughout the work to help the reader follow the stream of ideas presented throughout the work.

2 Full spark frames and the symmetric group

2.1 A toy example

Let us start our discussion by presenting a toy example which illustrates the main concepts developed in this thesis. As a starting point, we fix

$$v = v^{(0)} = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{C}^4.$$

Next, we are interested in the construction of a finite subset of $\mathbb{C}^4$ obtained by permuting the coordinates of v. A new vector may be obtained by rotating, each entry of v down by one row. Following this scheme, we generate

$$v^{(1)} = \begin{pmatrix} v_3 \\ v_0 \\ v_1 \\ v_2 \end{pmatrix}.$$

Next, we define inductively $v^{(k+1)}$ as the vector obtained by applying the procedure described above to $v^{(k)}$. That is,

$$v^{(2)} = \begin{pmatrix} v_2 \\ v_3 \\ v_0 \\ v_1 \end{pmatrix}, v^{(3)} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_0 \end{pmatrix}$$

and finally,

$$v^{(4)} = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} = v^{(0)}.$$

Note that the fourth element in our iteration coincides with the initial one. Next, we ask for which complex numbers v_0, v_1, v_2, v_3, is the set

$$\left\{v^{(k)} : 0 \le k \le 3\right\} \subset \mathbb{C}^4$$

linearly independent. In order to answer this question, we consider the matrix

$$\begin{pmatrix} v_0 & v_3 & v_2 & v_1 \\ v_1 & v_0 & v_3 & v_2 \\ v_2 & v_1 & v_0 & v_3 \\ v_3 & v_2 & v_1 & v_0 \end{pmatrix}$$

and we compute its determinant

$$\det \begin{pmatrix} v_0 & v_3 & v_2 & v_1 \\ v_1 & v_0 & v_3 & v_2 \\ v_2 & v_1 & v_0 & v_3 \\ v_3 & v_2 & v_1 & v_0 \end{pmatrix} = v_0^4 - 4v_0^2 v_1 v_3 - 2v_0^2 v_2^2 + 4v_0 v_1^2 v_2 + 4v_0 v_2 v_3^2 - v_1^4 + 2v_1^2 v_3^2 - 4v_1 v_2^2 v_3 + v_2^4 - v_3^4.$$

Thus, as long as

$$v_0^4 - 4v_0^2 v_1 v_3 - 2v_0^2 v_2^2 + 4v_0 v_1^2 v_2 + 4v_0 v_2 v_3^2 - v_1^4 + 2v_1^2 v_3^2 - 4v_1 v_2^2 v_3 + v_2^4 - v_3^4 \neq 0$$

the collection

$$\beta(v) = \left\{ \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}, \begin{pmatrix} v_3 \\ v_0 \\ v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} v_2 \\ v_3 \\ v_0 \\ v_1 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_0 \end{pmatrix} \right\}$$

is a basis for $\mathbb{C}^4$. Note that, it is also possible to generate the basis $\beta(v)$ by exploiting symmetry. Indeed, let Γ be the subgroup generated by

$$T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then clearly, Γ is a cyclic subgroup of order four consisting of matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Moreover,

$$\beta(v) = \{\gamma v : \gamma \in \Gamma\}$$

is the Γ-orbit of the vector v and as long as

$$v_0^4 - 4v_0^2 v_1 v_3 - 2v_0^2 v_2^2 + 4v_0 v_1^2 v_2 + 4v_0 v_2 v_3^2 - v_1^4 + 2v_1^2 v_3^2 - 4v_1 v_2^2 v_3 + v_2^4 - v_3^4 \neq 0$$

$\beta(v)$ is a basis for $\mathbb{C}^4$ generated by the action of the cyclic group Γ. There are questions that naturally arise from this exploration.

- Which property of Γ gives us the desired result? In order to obtain similar construction, how important is it to require that Γ is cyclic or commutative?

- Noting that

$$k \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^k$$

  is a matrix representation of the cyclic group $\mathbb{Z}_4$, is it true that every matrix representation of $\mathbb{Z}_4$ has an orbit which is a basis for $\mathbb{C}^4$?

- Are there other matrix groups enjoying similar properties?

- What is the largest matrix group obtained by permuting the identity matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

  such that every subset of order four of one of its orbit is a basis for $\mathbb{C}^4$?

In order to investigate answers to the questions above, we need to introduce the concepts of Haar property and full-spark frames.

2.2 Haar property of matrix groups and full-spark frames

Let $\mathbb{Z}_n = \{0, 1, 2, ..., n-1\}$ be a cyclic group endowed with modular arithmetic. The set of all bijective maps from $\mathbb{Z}_n$ into itself, endowed with the operation of function composition forms a group known as the permutation group of $\mathbb{Z}_n$. This group, which is commonly denoted S_n, has order $n!$ and is generally non-abelian.

Example 1 *If $n = 3$, then $\mathbb{Z}_3 = \{0, 1, 2\}$, and the symmetric group of $\mathbb{Z}_3$ consists of functions of the type $f : \mathbb{Z}_3 \mapsto \mathbb{Z}_3$ such that f is a bijection. More precisely S_3 consists of functions $f_1, f_2, \cdots, f_6$ defined as follows:*

$$f_1(0) = 0, f_1(1) = 1, f_1(2) = 2$$

$$f_2(0) = 2, f_2(1) = 0, f_2(2) = 1$$

$$f_3(0) = 1, f_3(1) = 2, f_3(2) = 0$$

$$f_4(0) = 0, f_4(1) = 2, f_4(2) = 1$$

$$f_5(0) = 1, f_5(1) = 0, f_5(2) = 2$$

$$f_6(0) = 2, f_6(1) = 1, f_6(2) = 0$$

Also, it is easy to verify that $(S_3, \circ)$ is isomorphic with the Dihedral group of order 6.

Let Γ be a subgroup of S_n such that $|\Gamma| = m \geq n$. Fixing an ordering for the elements of Γ, we write

$$\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_n, \gamma_{n+1}, ..., \gamma_m\}.$$

Next, let $v \in \mathbb{C}^n$ such that

$$v = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}.$$

Define O to be a function mapping the vector space $\mathbb{C}^n$ into the set of all $m \times n$ matrices such that

$$O(v) = \begin{pmatrix} v_{\gamma_1(0)} & v_{\gamma_1(1)} & \cdots & v_{\gamma_1(n-1)} \\ v_{\gamma_2(0)} & v_{\gamma_2(1)} & \cdots & v_{\gamma_2(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ v_{\gamma_m(0)} & v_{\gamma_m(1)} & \cdots & v_{\gamma_m(n-1)} \end{pmatrix}.$$

We say that Γ has the **Haar property** if for every submatrix $O_1(v)$ of order n of $O(v)$, the function $v \mapsto \det(O_1(v))$ is a nonzero polynomial in the coordinates of v. If there exists a vector w such that $w \mapsto \det(O(w))$ satisfies the properties above, we say that the set $\{\gamma w : \gamma \in \Gamma\}$ is a **full spark frame**. Furthermore, we are interested in the following general question.

Problem 2 *Given a natural number $n \geq 3$, which maximal subgroups of S_n have the Haar property?*

3 Full spark frames generated by cyclic groups

The objective of this subsection is to prove that all cyclic sugroups of the symmetric group have the Haar property. In order to set the stage for the generalization to come, let us first present a few examples.

Example 3 *If $n = 2$ and Γ is the group generated by $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then*

$$T^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly, Γ is the cyclic group of order 2. Moreover, $m = n = 2$. Let $v = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$ be an *arbitrary element of the vector space $\mathbb{C}^2$, and define γ_1 to be the identity matrix of order two and $\gamma_2 = T$. Then*

$$T(v) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_0 \end{pmatrix},$$

and the matrix $O(v)$ defined above takes the form

$$O(v) = \begin{pmatrix} v_0 & v_1 \\ v_1 & v_0 \end{pmatrix}.$$

The first row of $O(v)$ is the transpose of v and the second one is the transpose of $T(v)$. Next, the determinant of the matrix $(O(v))$ is the bivariate polynomial given by $v \mapsto v_0^2 - v_1^2$. Since $v \mapsto v_0^2 - v_1^2$ is a nonzero polynomial in the variables v_0, v_1 it follows that Γ has the Haar Property and if $v \in \mathbb{C}^2$ such that $v_0^2 - v_1^2 \neq 0$ then $\{v, Tv\}$ is a basis for $\mathbb{C}^2$.

Example 4 *Let $n = 3$, and let Γ be the cylcic group generated by*

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We verify that $T^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and $T^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Moreover, $|\Gamma| = 3$ and $m = n = 3$. It follows that

$$O(v) = \begin{pmatrix} v_0 & v_1 & v_2 \\ v_2 & v_0 & v_1 \\ v_1 & v_2 & v_0 \end{pmatrix}$$

Next,

$$\det O(v) = -(v_0 + v_1 + v_2)(v_0^2 - v_0 v_1 + v_1 - v_1 v_2 + v_2^2)$$

is a nonzero polynomial in the variables v_0, v_1, v_2. Therefore, the cyclic group of order three which is generated by T has the Haar Property.

We shall now generalize the examples above. To this end, let

$$T = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

be a square matrix of order n. Then clearly T is a *circulant* matrix since each column of T is a cyclic permutation of the first vector column. Next, let Γ be the matrix group generated by T.

Lemma 5 Γ *is a cyclic group of order n.*

Proof. Let

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, ..., e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Define

$$T(e_k) = \begin{cases} e_{k+1}, & \text{if} \quad 1 \leq k \leq n-1 \\ e_1, & \text{if} \quad k = n \end{cases}$$

In general, $T(e_k) = e_{(k+1) \mod n}$. Thus in general $T^j(e_k) = e_{(k+j) \mod n}$ and the smallest natural number for which $T^j(e_k) = e_k$ for all k is when $j = n$. Thus T^n is the identity matrix and the group generated by T has order n. ■

In the subsequential paragraphs, we shall prove that the group Γ has the Haar property. First, we fix

$$v = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{n-1} \end{pmatrix} \in \mathbb{C}^n.$$

Next, we consider the map

$$v \mapsto O(v) = \begin{pmatrix} v_0 & v_1 & \dots & v_{n-1} \\ v_{n-1} & v_0 & \dots & v_{n-2} \\ v_{n-2} & v_{n-1} & \dots & v_{n-3} \\ \vdots & \vdots & & \vdots \\ v_2 & v_3 & \dots & v_1 \\ v_1 & v_2 & \dots & v_0 \end{pmatrix}$$

obtained by rearranging the Γ-orbits of v in row form. To prove that Γ has the Haar property, it is sufficient to prove that the function $v \mapsto \det(O(v))$ is a non-trivial polynomial. Since the determinant of $O(v)$ is equal to the product of its eigenvalues, we need to compute all eigenvalues of the matrix $O(v)$. To this end, we shall first establish that for each vector v, the matrix is $O(v)$ is diagonalizable. Recall that a matrix is called a diagonal matrix if all of its non-diagonal entries are equal to zero. Moreover, a matrix A is called **diagonalizable** if there exists an invertible matrix P such that PAP^{-1} is a diagonal matrix.

Example 6 *Let*

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Then

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and it follows that A is a diagonalizable matrix. Note that it is not the case that every matrix is diagonalizable. For example, the matrix

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is not diagonalizable. Otherwise, suppose that there exists an invertible matrix $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$PXP^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{(ac - ad + bc)}{ad - bc} & \frac{a^2}{ad - bc} \\ -\frac{c^2}{ad - bc} & \frac{(ac + ad - bc)}{ad - bc} \end{pmatrix}$$

is a diagonal matrix. This would imply the existence of a, b, c, d such that $ad - bc \neq 0$ and $a = c = 0$ which is absurd.

Lemma 7 *Let*

$$T = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Then T is a square matrix of order n. Then T is diagonalizable and its eigenvalues are the roots of unity obtained by solving the complex equation $z^n = 1$.

Proof. To prove this result, it suffices to show that the inverse of T is diagonalizable. To this end, we shall prove that T^{-1} has n distinct eigenvalues which are precisely the complex roots of the polynomial equation $z^n = 1$. Indeed,

$$T^{-1} - \lambda \cdot I = \begin{pmatrix} -\lambda & 1 & 0 & ... & 0 \\ 0 & -\lambda & 1 & ... & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \\ 1 & 0 & 0 & ... & -\lambda \end{pmatrix}$$

and

$$\det(T^{-1} - \lambda I) = -\lambda \begin{pmatrix} -\lambda & 1 & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & 1 \\ 0 & 0 & ... & -\lambda \end{pmatrix} + (-1) \begin{pmatrix} 0 & 1 & ... & 0 \\ 0 & 0 & ... & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & ... & -\lambda \end{pmatrix}$$

$$= -\lambda(-\lambda)^{n-1} + (-1)^{1+n}$$

$$= (-1)^n \lambda^n + (-1)^{1+n}.$$

Note that if n is even then $\det(T^{-1}-\lambda I)=\lambda^n-1$. Otherwise, if n is odd then

$$\det(T^{-1}-\lambda I)=-\lambda^n+1.$$

For $k\in\{0,1,...,n-1\}$, let us define $\lambda=\exp\left(\frac{2\pi ki}{n}\right)$. Clearly, the complex numbers $\exp(\frac{2\pi ki}{n})$ are eigenvalues for the inverse of T. Next, we would like to compute the eigenvectors corresponding to each eigenvalue $\exp(\frac{2\pi ki}{n})$. To this end, we compute the following matrix.

$$(T^{-1}-\exp(\frac{2\pi ki}{n}))=\begin{pmatrix}-\exp(\frac{2\pi ki}{n}) & 1 & 0 & ... & 0\\ 0 & -\exp(\frac{2\pi ki}{n}) & 1 & ... & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & 0 & 0 & ... & -\exp(\frac{2\pi ki}{n})\end{pmatrix}.$$

Next,

$$(T^{-1}-\exp(\frac{2\pi ki}{n}I))\begin{pmatrix}1\\ \exp(\frac{2\pi ki}{n})\\ \vdots\\ \exp(\frac{2\pi k(n-2)i}{n})\\ \exp(\frac{2\pi k(n-1)i}{n})\end{pmatrix}$$

$$=\begin{pmatrix}-\exp(\frac{2\pi ki}{n}) & 1 & 0 & ... & 0\\ 0 & -\exp(\frac{2\pi ki}{n}) & 1 & ... & 0\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & 0 & 0 & ... & -\exp(\frac{2\pi ki}{n})\end{pmatrix}\begin{pmatrix}1\\ \exp(\frac{2\pi ki}{n})\\ \vdots\\ \exp(\frac{2\pi k(n-2)i}{n})\\ \exp(\frac{2\pi k(n-1)i}{n})\end{pmatrix}$$

$$
= \begin{pmatrix} -\exp(\frac{2\pi ki}{n} + \exp(\frac{2\pi ki}{n}) \\ -\exp(\frac{2\pi 2ki}{n}) + \exp(\frac{2\pi 2ki}{n}) \\ \vdots \\ -\exp(\frac{2\pi k(n-1)i}{n})) + \exp(\frac{2\pi k(n-1)i}{n}) \\ 1 - \exp(\frac{2\pi kni}{n}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}.
$$

Letting

$$
E_k = \begin{pmatrix} 1 \\ \exp(\frac{2\pi ki}{n}) \\ \vdots \\ \exp(\frac{2\pi k(n-1)i}{n}) \end{pmatrix}
$$

it follows that the element E_k spans the null space of $T - \exp(\frac{2\pi ki}{n})I$ making it an eigenbasis for the eigenvalue $\exp(\frac{2\pi ki}{n})$. As a result, $\{E_k : k \in \mathbb{Z}_n\}$ forms a basis for $\mathbb{C}^n$ consisting of eigenvectors for the inverse of T. ■

Definition 8 *For a fixed natural number n, we define the **Fourier matrix** as the following square matrix of order n as*

$$
F = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & \dots & \dots & 1 \\ 1 & w & w^2 & \dots & w^{n-1} \\ \vdots & w^2 & w^{2(2)} & \dots & w^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{n-1} & w^{2(n-1)} & \dots & w^{(n-1)(n-1)} \end{pmatrix}
$$

where

$$
w = \exp\left(\frac{2\pi i}{n}\right) = \cos\left(\frac{2\pi}{n}\right) + i \cdot \sin\left(\frac{2\pi}{n}\right).
$$

Note that

$$E_0 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, E_1 = \begin{pmatrix} 1 \\ \exp\left(\frac{2\pi i}{n}\right) \\ \vdots \\ \exp\left(\frac{2\pi i(n-1)}{n}\right) \end{pmatrix}, \cdots, E_{n-1} = \begin{pmatrix} 1 \\ \exp\left(\frac{2\pi i(n-1)}{n}\right) \\ \vdots \\ \exp\left(\frac{2\pi i(n-1)(n-1)}{n}\right) \end{pmatrix}$$

and

$$\frac{1}{\sqrt{n}} E_k = \mathrm{col}_k\,(T)\,.$$

Thus, conjugating the inverse of T by the Fourier matrix gives the diagonal matrix

$$FT^{-1}F^{-1} = \begin{pmatrix} 1 & & & \\ & w & & \\ & & \ddots & \\ & & & w^{n-1} \end{pmatrix}.$$

The following statements will be useful for the subsequent discussion.

Lemma 9

1. *The group obtained by conjugating the group generated by FTF^{-1} is isomorphic to the cyclic group of order n.*
2. *The Haar property is a conjugation-invariant property of a matrix group.*

Proof. To prove the first part, let $M = FT^{-1}F^{-1}$ and let G be the group generated by M. Then clearly, the map $M^k \mapsto T^k$ defines a group isomorphic between G and $< T >$. It follows that G is isomorphic with the cyclic group of order n.

To prove the second part, let us suppose that Γ is a group acting on $\mathbb{C}^n$. Moreover, assume that Γ has the Haar property. Next, let P be an invertible square matrix of order n and $\Lambda = \{P\gamma P^{-1} : \gamma \in \Gamma\}$.

By assumption, there exists some vector $v \in \mathbb{C}^n$ such that every subset of $\{\gamma v : \gamma \in \Gamma\}$ of cardinality n is a basis for $\mathbb{C}^n$. Now, let $z \in \mathbb{C}^n$ such that $P^{-1}z = v$. Then $\gamma P^{-1} z = \gamma v$ and every subset of cardinality n of the collection $\{\gamma P^{-1} z : \gamma \in \Gamma\} = \{\gamma v : \gamma \in \Gamma\}$ is a basis for $\mathbb{C}^n$. Finally, we note that every subset of cardinality n of $\{P\gamma P^{-1} z : \gamma \in \Gamma\} = \{\lambda z : \lambda \in \Lambda\}$ is the image of a basis under a linear map P. Consequently, Λ has the Haar property and that Haar property is preserved under conjugation. ■

To establish that $\langle T \rangle$ has the Haar property, we consider the matrix obtained by arranging the $F\Gamma F^{-1}$-orbit of a generic element

$$f = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{pmatrix}$$

of $\mathbb{C}^n$ as follows

$$O_f = \begin{pmatrix} f_0 & f_1 & \dots & f_{n-1} \\ f_0 & wf_1 & \dots & w^{n-1} f_{n-1} \\ f_0 & w^2 f_1 & \dots & w^{n-1} f_{n-1} \\ \vdots & \vdots & & \vdots \\ f_0 & w^{n-1} & \dots & w^{(n-1)(n-1)} f_{n-1}. \end{pmatrix}.$$

Next,

$$\det(O_f) = \det \begin{pmatrix} f_0 & f_1 & \cdots & f_{n-1} \\ f_0 & wf_1 & \cdots & w^{n-1} f_{n-1} \\ f_0 & w^2 f_1 & \cdots & w^{2(n-1)} f_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ f_0 & w^{n-1} f_1 & \cdots & w^{(n-1)(n-1)} f_{n-1} \end{pmatrix}.$$

Appealing to the multilinearity of the determiant function, we obtain

$$\det\begin{pmatrix} f_0 & f_1 & \cdots & f_{n-1} \\ f_0 & wf_1 & \cdots & w^{n-1}f_{n-1} \\ f_0 & w^2f_1 & \cdots & w^{2(n-1)}f_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ f_0 & w^{n-1}f_1 & \cdots & w^{(n-1)(n-1)}f_{n-1} \end{pmatrix} = f_0f_1\cdots f_{n-1}\cdot\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & w & \cdots & w^{n-1} \\ 1 & w^2 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n-1} & \cdots & w^{(n-1)(n-1)} \end{pmatrix}.$$

The above is reformulated as

$$\left(\prod_{k=0}^{n-1} f_k\right)\cdot\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & w & \cdots & w^{n-1} \\ 1 & w^2 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n-1} & \cdots & w^{(n-1)(n-1)} \end{pmatrix} = (*)\,.$$

Having previously proved that the collection

$$\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ w \\ w^2 \\ \vdots \\ w^{n-1} \end{pmatrix}, \cdots, \begin{pmatrix} 1 \\ w^{n-1} \\ w^{2(n-1)} \\ \vdots \\ w^{(n-1)(n-1)} \end{pmatrix}\right\}$$

forms a basis for $\mathbb{C}^n$, it follows that

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & w & \cdots & w^{n-1} \\ 1 & w^2 & \cdots & w^{2(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & w^{n-1} & \cdots & w^{(n-1)(n-1)} \end{pmatrix}$$

is nonzero. Referring back to $(*)$, the mapping

$$(f_0, f_1, \cdots, f_{n-1}) \mapsto \det(O_f)$$

is a nonzero polynomial in the variables $f_0, f_1, \cdots, f_{n-1}$. As a result, the matrix groups $F\Gamma F^{-1}$ and Γ both have the Haar property.

4 Full spark frames generated by the Dihedral groups

In this section, we will explore known results of full-spark frames generated by the Dihedral group. Let

$$T = \begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & & 1 \\ \vdots & \vdots & \iddots & \vdots \\ 0 & 1 & & 0 \end{pmatrix}$$

be matrices of order $n \geq 3$. Note that $T^n = I_n$ and $D^2 = I_n$. Moreover, it is not hard to verify that $DT = T^{n-1}D$. As a result, the group generated by T and D is a non-commutative matrix group. In fact, letting Γ be the group generated by T, D we may write

$$\Gamma = \left\langle T, D : T^n = I_n, D^2 = I_n, DTD = T^{-1} \right\rangle$$

and it is not hard to verify that Γ is isomorphic to the dihedral group of order $2n$. In these settings, it is known that the Haar property depends on the parity of the order of T. To be more precise, the following results are proved in [7, 8]

Theorem 10 Γ *has the Haar property if and only if n is odd.*

Furthermore, in the case where n is odd, we have obtained in [7] an algorithm for constructing vectors generating full-spark frames under the action of the Dihedral group. Our procedure is described as follows. Let n be an odd natural number greater than or equal to 3.

1. Let
$$F_n = \left\{ w \in \mathbb{C}^n : w = \begin{pmatrix} \sum_{k=0}^{n-1} \lambda^k \\ \sum_{k=0}^{n-1} \lambda^k e^{-\dfrac{2i\pi k}{n}} \\ \vdots \\ \sum_{k=0}^{n-1} \lambda^k e^{-\dfrac{2i\pi(n-1)k}{n}} \end{pmatrix} \right\}$$
where λ is **transcendental** or is an algebraic number over $\mathbb{Q}$ whose degree is at least $n^2 - n + 1$.

2. Given $v \in F_n$ the set $\{\gamma f : \gamma \in \Gamma\}$ is a full spark frame of $2n$ elements generated by the Dihedral group. As an application of the algorithm presented above, let us consider the following example.

Example 11 *Let $n = 3$ and define*

$$v = \begin{pmatrix} \sum_{k=0}^{2} \pi^k \\ \sum_{k=0}^{2} \left(\pi^k e^{-\frac{2\pi i k}{3}}\right) \\ \sum_{k=0}^{2} \left(\pi^k e^{-\frac{4\pi i k}{3}}\right) \end{pmatrix} = \begin{pmatrix} \pi + \pi^2 + 1 \\ -\frac{1}{2}(\pi - 1)\left(\pi - i\sqrt{3}\pi + 2\right) \\ -\frac{1}{2}(\pi - 1)\left(\pi + i\sqrt{3}\pi + 2\right) \end{pmatrix} \in \mathbb{C}^3.$$

Next, the orbit of the vector above under the action of the Dihedral is given by

$$\left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{j} \begin{pmatrix} \pi + \pi^2 + 1 \\ -\frac{1}{2}(\pi - 1)\left(\pi - i\sqrt{3}\pi + 2\right) \\ -\frac{1}{2}(\pi - 1)\left(\pi + i\sqrt{3}\pi + 2\right) \end{pmatrix} : (k, j) \in \mathbb{Z}_3 \times \mathbb{Z}_2 \right\}.$$

The above is an example of a full-spark frame in $\mathbb{C}^3$. Listing the elements of the set described above, it follows that every subset of cardinality three of this collection

$$\begin{pmatrix} -\frac{1}{2}(\pi - 1)\left(\pi + i\sqrt{3}\pi + 2\right) \\ -\frac{1}{2}(\pi - 1)\left(\pi - i\sqrt{3}\pi + 2\right) \\ \pi + \pi^2 + 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2}(\pi - 1)\left(\pi - i\sqrt{3}\pi + 2\right) \\ \pi + \pi^2 + 1 \\ -\frac{1}{2}(\pi - 1)\left(\pi + i\sqrt{3}\pi + 2\right) \end{pmatrix}, \begin{pmatrix} \pi + \pi^2 + 1 \\ -\frac{1}{2}(\pi - 1)\left(\pi + i\sqrt{3}\pi + 2\right) \\ -\frac{1}{2}(\pi - 1)\left(\pi - i\sqrt{3}\pi + 2\right) \end{pmatrix}$$
$$\begin{pmatrix} -\frac{1}{2}(\pi - 1)\left(\pi - i\sqrt{3}\pi + 2\right) \\ -\frac{1}{2}(\pi - 1)\left(\pi + i\sqrt{3}\pi + 2\right) \\ \pi + \pi^2 + 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2}(\pi - 1)\left(\pi + i\sqrt{3}\pi + 2\right) \\ \pi + \pi^2 + 1 \\ -\frac{1}{2}(\pi - 1)\left(\pi - i\sqrt{3}\pi + 2\right) \end{pmatrix}, \begin{pmatrix} \pi + \pi^2 + 1 \\ -\frac{1}{2}(\pi - 1)\left(\pi - i\sqrt{3}\pi + 2\right) \\ -\frac{1}{2}(\pi - 1)\left(\pi + i\sqrt{3}\pi + 2\right) \end{pmatrix}$$

forms a complex basis for $\mathbb{C}^3$. In other words, every minor of order three of

$$\begin{pmatrix} -\frac{1}{2}(\pi - 1)\left(\pi + i\sqrt{3}\pi + 2\right) & -\frac{1}{2}(\pi - 1)\left(\pi - i\sqrt{3}\pi + 2\right) & \pi + \pi^2 + 1 \\ -\frac{1}{2}(\pi - 1)\left(\pi - i\sqrt{3}\pi + 2\right) & \pi + \pi^2 + 1 & -\frac{1}{2}(\pi - 1)\left(\pi + i\sqrt{3}\pi + 2\right) \\ \pi + \pi^2 + 1 & -\frac{1}{2}(\pi - 1)\left(\pi + i\sqrt{3}\pi + 2\right) & -\frac{1}{2}(\pi - 1)\left(\pi - i\sqrt{3}\pi + 2\right) \\ -\frac{1}{2}(\pi - 1)\left(\pi - i\sqrt{3}\pi + 2\right) & -\frac{1}{2}(\pi - 1)\left(\pi + i\sqrt{3}\pi + 2\right) & \pi + \pi^2 + 1 \\ -\frac{1}{2}(\pi - 1)\left(\pi + i\sqrt{3}\pi + 2\right) & \pi + \pi^2 + 1 & -\frac{1}{2}(\pi - 1)\left(\pi - i\sqrt{3}\pi + 2\right) \\ \pi + \pi^2 + 1 & -\frac{1}{2}(\pi - 1)\left(\pi - i\sqrt{3}\pi + 2\right) & -\frac{1}{2}(\pi - 1)\left(\pi + i\sqrt{3}\pi + 2\right) \end{pmatrix}$$

is nonzero

5 Maximal subgroups with the Haar property

Let $GL_n(\mathbb{C})$ be the group of invertible complex matrices of order n. Next, define

$$\pi : S_n \to GL_n(\mathbb{C})$$

as the canonical representation of S_n. This representation is uniquely determined by the equations

$$\pi(\sigma)\begin{pmatrix} v_0 \\ \vdots \\ v_{n-1} \end{pmatrix} = \begin{pmatrix} v_{\sigma(0)} \\ \vdots \\ v_{\sigma(n-1)} \end{pmatrix} \text{ where } \begin{pmatrix} v_0 \\ \vdots \\ v_{n-1} \end{pmatrix} \in \mathbb{C}^n.$$

Clearly, the mapping $\sigma \mapsto \pi(\sigma)$ is a group isomorphism.

In this subsection, we are mainly interested in describing maximal subgroups of $\pi(S_n)$ having the Haar property.

Proposition 12 *$\pi(S_3)$ has the Haar property.*

Proof. Note that

$$\pi(S_3) = \left\langle \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle = D_{2(3)}$$

is isomoprhic with the Dihedral group of order six. Since the element

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

has order three then it follows from Theorem 2 of [8] that $\pi(S_3)$ has the Haar property. ■

Proposition 13 *Let n be a natural number which is even and greater than 2. Then $\pi(S_n)$ does not have the Haar property.*

Proof. Since the group generated by the matrices

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ \vdots & \vdots & & & 0 \\ 0 & 0 & 1 & & \vdots \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

is a subgroup of $\pi(S_n)$, if n is even then S_n does not have the Haar property. ∎

Definition 14 *Let Γ be a subgroup of S_n. We say that Γ is a maximal subgroup of S_n if there is no subgroup of S_n other than S_n containing Γ.*

Proposition 15 *The only subgroups of $\pi(S_4)$ having the Haar property are subgroups isomorphic to A_4 (the even permutations), and subgroups isomorphic to the cyclic group $\mathbb{Z}_4$.*

Proof. First, according to its lattice of subgroups, (see the Figure 1), S_4 admits nine families of non-isomorphic subgroups. The subgroups isomorphic to the trivial group and the cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ do not have the Haar property. This is due to the fact that for $k = 2, 3$ the order of $\mathbb{Z}_k$ is less than four. Next, let $\Gamma_1, \Gamma_2, \Gamma_3$ be the subgroups of $\pi(S_4)$ isomorphic to D_8, A_4 and S_3 respectively.

Up to conjugation Γ_1 is generated by

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{and} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

and by [8], Theorem 2, we know that since $n = 4$ that Γ_1 does not have the Haar property. Next, Γ_2 consists of the following matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The matrix corresponding to the orbit of $v : \{\gamma v : \gamma \in \Gamma_2\}$ is given by

$$O(v) = \begin{pmatrix} v_0 & v_1 & v_2 & v_3 \\ v_0 & v_2 & v_3 & v_1 \\ v_0 & v_3 & v_1 & v_2 \\ v_1 & v_0 & v_3 & v_2 \\ v_1 & v_2 & v_0 & v_3 \\ v_1 & v_3 & v_2 & v_0 \\ v_2 & v_0 & v_1 & v_3 \\ v_2 & v_1 & v_3 & v_0 \\ v_2 & v_3 & v_0 & v_1 \\ v_3 & v_0 & v_2 & v_1 \\ v_3 & v_1 & v_0 & v_2 \\ v_3 & v_2 & v_1 & v_0 \end{pmatrix}.$$

With straightforward computations, it is not hard to verify that every minor of $O(v)$ of order

4 is not equal to zero. Therefore, Γ_2 has the Haar property.

Finally, up to conjugation Γ_3 is generated by

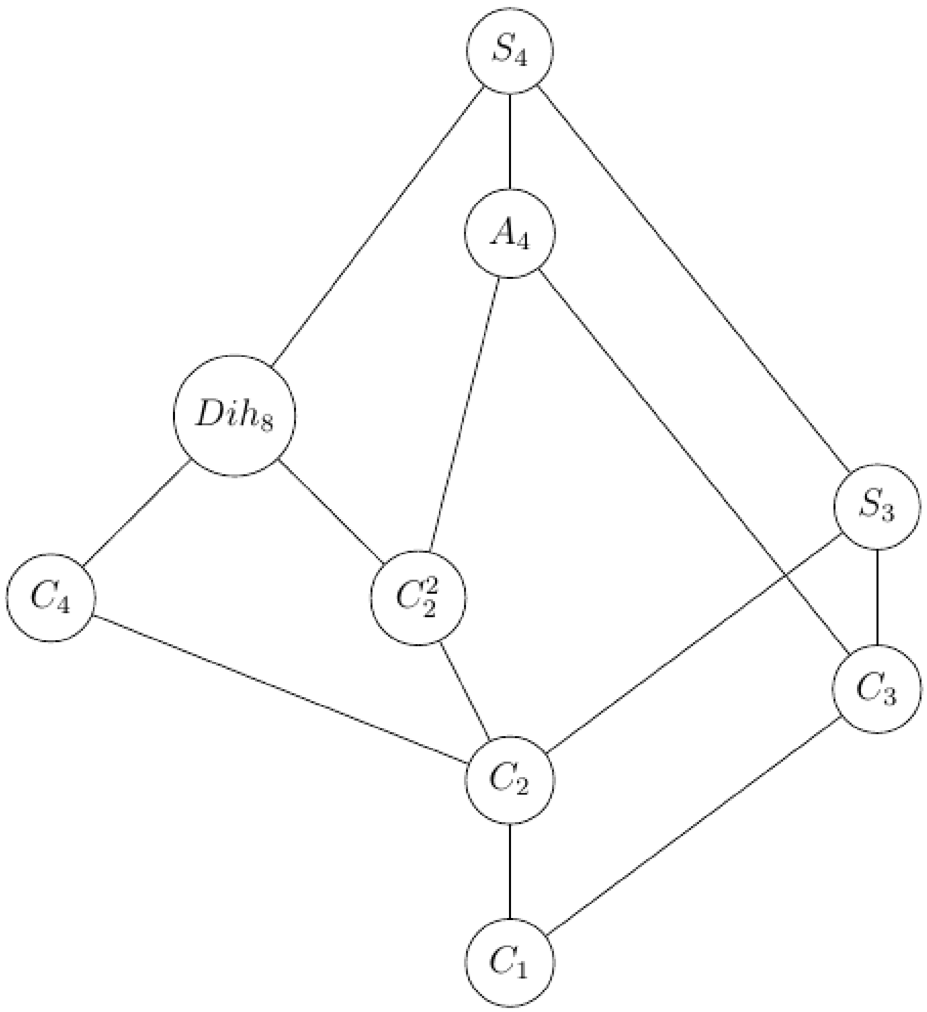
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and given $v \in \mathbb{C}^3$, there exists an ordering for Γ_3 such that

$$O(v) = \begin{pmatrix} v_0 & v_1 & v_2 & v_3 \\ v_0 & v_1 & v_3 & v_2 \\ v_0 & v_3 & v_1 & v_2 \\ v_0 & v_2 & v_1 & v_3 \\ v_0 & v_2 & v_3 & v_1 \\ v_0 & v_3 & v_2 & v_1 \end{pmatrix}.$$

Every minor of $O(v)$ of order 4 is equal to zero. Therefore $\Gamma_3 \cong S_3$ does not have the Haar property. $\blacksquare$

Figure 1: Lattice Diagram of S_4



Proposition 16 *If $n \geq 4$, then $\pi(S_n)$ does not have the Haar Property.*

Proof. To establish this result, it suffices to show that there exists a subset of $\pi(S_n)$ which is linearly dependent. Indeed, let R, S be two invertible matrices of order $n \geq 4$ such that

$$R = \begin{pmatrix} 0 & 0 & 0 & 1 & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & 0 & & & & \\ 0 & 0 & 1 & 0 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}, S = \begin{pmatrix} 1 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 1 & 0 & & & & \\ 0 & 1 & 0 & 0 & & & & \\ & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \end{pmatrix}$$

It can be shown that $R^4 = I_n$, $S^2 = I_n$, $SRS = R^{n-1} = R^{-1}$ and the group Γ generated by R, S is isomorphic to D_8. Next, we claim that

$$I + R^2 - S - R^2 S = \begin{pmatrix} 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix}.$$

To see this, it suffices to verify that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^2 - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} + \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} - \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} - \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}.$$

Next, given any vector $v \in \mathbb{C}^n$, it is clear that $Iv + R^2v - Sv - R^2Sv = 0$. As a result, $\{v, R^2v, Sv, R^2Sv\}$ is never linearly independent. Consequently, $\pi(S_n)$ does not have the Haar property if $n \geq 4$. ■

Appealing to Proposition 12 and Proposition 16, the following is immediate.

Theorem 17 *Let n be a natural number larger than two. Then $\pi(S_n)$ has the Haar property if and only if $n \leq 3$.*

6 Concluding remarks

Let n be a natural number larger than two. We have shown that any cyclic subgroup of $\pi(S_n)$ of order n has the Haar property. We have also proved that $\pi(S_n)$ has the Haar property if and only if $n \leq 3$. Furthermore, we have shown that if $n = 4$ then $\pi(A_n)$ has the Haar property. However, we have not been able to determine if $\pi(A_n)$ (the group of even permutations) has the Haar property if $n \geq 5$. Note that $\pi(A_5)$ is a group of order 60. Since the set of all subsets of cardinality 5 of a set of cardinality 60 is equal 5461512, the brute force approach of verifying if $\pi(A_5)$ has the Haar property is computationally too expensive and thus not feasible. We conclude this thesis with the following open question which will be the focus of a future study.

Problem 18 *Let n be a natural number larger than five. Does $\pi(A_n)$ have the Haar property?*

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