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$e^{\pi i} + 1 = 0$: The History & Development

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$$e^{\pi i} + 1 = 0$$

The History & Development

DAWNE CHARTERS-NELSON



Dawne is a senior and mathematics major with two prior degrees in art and engineering. As a returning student

Dawne brings many life experiences to her education including teaching high school mathematics, running a computer consulting business with her husband Jeffrey, and working as an engineer for Wang Laboratories and Digital Equipment Corporation. She wrote this paper for a Math History course under the encouragement and mentorship of Dr. Philip Scalisi without whom she would have never known of this opportunity. After graduation Dawne plans to return to teaching high school to spread her enthusiasm for learning!

I have on occasion run across the equation $e^{\pi i} + 1 = 0$ in books, articles and in conversation with other mathematicians. In each of these encounters the person alluded to a fascination with this equation which links the five most important constants in the whole of analysis:

- 0 = The additive identity
- 1 = The multiplicative identity
- π = The circular constant
- e = The base of the natural logarithms
- i = The imaginary unit

Being a novice mathematician, I wondered how all these fundamental constants could end up in one equation and what it meant. Along with this thought came the realization that there was some fun investigating to be done. In this paper I will trace the growth of trigonometric ratios and sequences of exponential growth that lead to the equation: $e^{\pi i} + 1 = 0$. My objective is to appreciate and understand the math that lead up to this equation and help the reader understand how it came to be over two thousand years.

There are basically three “tributaries” of development that lead to this equation. They are, not necessarily in this order, the development of trigonometric ratios and their extension to periodic functions, the development of the theory of infinite series, and the development and understanding of exponential growth and logarithms. These areas don’t always develop independently of each other or in consecutive order as listed, so I will let the natural evolution of the equation dictate the flow of the paper.

Initially I will decompose the equation to show how I determined its historical roots. In other words I will start with $e^{\pi i} + 1 = 0$ and work backwards to see how it untangles mathematically. I will follow that with the historical story of what I consider the important mathematical contributions.

Decomposition

To understand how this equation came to be, I naturally began to decompose the equation to see what I could get. At each step of the way I researched the transformed equation and found that many famous mathematicians in history had played around with this equation or the mathematics associated with it. To begin our summary of the mathematical transformations, let’s observe what

happens when we work backwards from the final equation: $e^{\pi i} = -1$. Rearranging terms we get: $e^{\pi i} = -1$. Representing -1 as a complex number $-1+0i$, and further substituting $\cos\pi = -1$ and $\sin\pi = 0$, we get $-1 = \cos\pi + i\sin\pi$. This leads to: $e^{\pi i} = (\cos\pi + i\sin\pi)$ an odd equality until you look at the power expansions of each:

$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\ e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\end{aligned}$$

The sum of $\sin x$ and $\cos x$ almost equal e^x . This is too coincidental to ignore.

Introducing complex variables into the mix yields the following where ix is imaginary:

$$\begin{aligned}e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} \dots \\ e^{ix} &= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} \dots\end{aligned}$$

This leads to Euler's formula

$$e^{ix} = \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right] + i \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right]$$

which simplifies to $e^{ix} = \cos x + i\sin x$. An interesting corollary of the above proof occurs when you let $x=\pi$:

$$e^{\pi i} = \cos \pi + i \sin \pi = -1 + i \cdot 0 = -1$$

This can be rewritten as our famous identity: FORM 1!

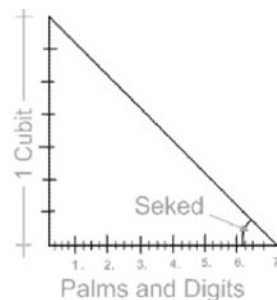
I will now pick up the threads of historical development of the three tributaries that lead to our equation: $e^{\pi i} = -1$: trigonometric ratios, infinite series and exponents.

Trigonometric Functions

The development of the sine and cosine trigonometric functions, have a long and detailed history. I will focus primarily on how and why the sine and cosine ratios came to be and how they transitioned into their well known representations as functions so that I can lead into the analytic forms of both functions.

The oldest known trigonometric artifact from antiquity is the Mesopotamian Plimpton 322 tablet (c. 1900 - 1600 BC) in the G. A. Plimpton Collection at Columbia University. Its actual interpretation is disputed, but it appears to contain the oldest known trigonometric table.

Another one of the oldest recorded references involving trigonometry was the Rhind Papyrus, transcribed by the Egyptian scribe Ahmes in 1659 B.C and discovered by Henry Rhind in a bazaar in Cairo in the late 19th century. The Rhind Papyrus was an important mathematical source used in ancient Egypt, and is a compilation of different mathematical techniques. Problems 56 through 60 specifically deal with the building of the pyramids and use practical trigonometry relating the sides of a right triangle to find the seked of the pyramid. The seked is the angle of inclination of the face of a right pyramid to the horizontal base, in other words the inverse cosine or angle whose cosine is the ratio of the length of the base to the length of the face of the pyramid. It is interesting to note that these problems don't explicitly refer to angles at all.



Here is a sample problem from the Rhind Papyrus dealing with trigonometry. It is problem 59 and is a translation of the problem with some of the original words.

A mir [pyramid] has 12 as its ukha-thebet [side] and its per-em-us [height] is 8.

Perform the operation on 8 which gives 6; [the latter value] being 1/2 of the ukha-thebet [side]. [The result of performing this operation is]:

- | | |
|--------|---|
| 1) 1 | 8 |
| 2) 1/2 | 4 |
| 3) 1/4 | 2 |

Take 1/2 + 1/4 of 7 [palms]; [this latter value] being a cubit.

[The result of performing this operation is]:

- | | |
|--------|---------------|
| 1) 1 | 7 |
| 2) 1/2 | 3 + 1/2 |
| 2) 1/4 | 1 + 1/2 + 1/4 |

The result is 5 palms, [1 finger]. Behold this is the seked of the mir [pyramid]. (Herz-Fischler 34)

This problem is an example of how to find the angle of inclination, or seked, of the pyramid face. The Egyptians used a common form of halving on the value of the height (8) to get six, half of the pyramid face (12). The reduction factors used

on the height, in this case $1/2$ and $1/4$, are then applied to the standard measurement of a cubit to get the *seked* in terms of palms.

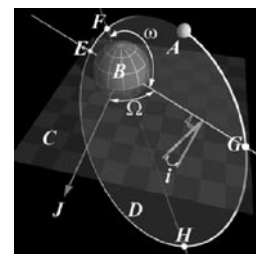
Although all of problems 56 through 60 deal with solving for the *seked* of the pyramid, it is not conclusively known whether these techniques were ever used to actually aid in building the pyramids in Egypt.

There are many other examples in recorded history of the application and study of trigonometric ratios. Babylonian astronomers in Mesopotamia from 1830 to 1531 B.C. used a device called a *gnomon* for telling the time or season from the length of a shadow cast by a vertical rod. It was essentially a device for computing the cotangent function (Maor 20). The ancient Egyptians built decorated obelisks of stone to keep time. They were located on a horizontal surface that was marked to aid in telling time. The shadow cast by the obelisk was like the hour hand on a clock and marked the time, seasons and equinoxes. The actual method used by the Egyptians is still disputed.

Moving on to the Greeks; the mathematician Hipparchus of Nicaea (ca. 190-120 B.C.) was considered the greatest astronomer of antiquity. He computed a table of trigonometric ratios for plane and sphere, using a triangle inscribed in a circle so that each side is a chord and computing various parts of the triangle as functions of the central angle. Unfortunately although all of Hipparchus' work has been lost, much of his work has been documented in the *Almagest* by Ptolemy. Claudius Ptolemaeus, commonly known as Ptolemy (ca. 85-ca.165 A.D.), lived in Alexandria and wrote the *Almagest* around 150 A.D. It is perhaps the most important summary of ancient Greek mathematical astronomy based on the earth as the stationary center of the universe and was used as an Islamic and European astronomy reference until the early 17th century. It consists of thirteen sections, or "books". Chapters ten and eleven of book one address the lengths of chords in a circle as a function of the central angle. Ptolemy goes on to use this information to solve any planar triangle in a manner similar to Hipparchus (Maor 25).

The Hindu *Surya Siddhanta* (ca. 400 A.D.) is a treatise of Indian astronomy and continues the half-chord work of Ptolemy. It uses sine, cosine and inverse sine for the first time and details how to calculate and determine planetary displacements in their elliptical orbits. Below is verse thirty-nine from section 3 of the *Surya Siddhanta*. It is an example of the type of information this book details:

39. By the corrected epicycle multiply the *bhujajya* [base-sine] and *kotijya* [perpendicular-sine] respectively, and divide by the number of degrees in a circle: then, the arc corresponding to the result from the *bhujajyaphala* [base-sine] is the equation of the *manda phalli* [apsis], in minutes, etc.



The Apsis is Line

This problem describes the final process by which the equation of the apsis is ascertained (Burgess 206). The apsis runs along the major axis of the elliptical orbit and intersects the orbit at the points of greatest and least distance from the planets center of attraction.

Around 510 A.D. the Indian mathematician Aryabhata wrote the astronomical treatise the *Aryabhatiya*. The text is written in Sanskrit and was highly influential as is evidenced by the many commentaries written about it by other influential mathematicians during Aryabhata's lifetime and subsequently. In it we find the first mention of the sine, or *jiva*, as a function of an angle. One of the most notable stanzas in the work is the concluding stanza of the Dasagitika:

10. The (twenty-four) sines reckoned in minuets of arc are 225, 224, 222, 219, 215, 210, 205, 199, 191, 183, 174, 164, 154, 143, 131, 119, 106, 93, 79, 65, 51, 37, 22, 7 (Aryabhata 19).

This stanza is actually the equivalent of our modern sine table. The numbers stand for the differences between half-chord lengths for a specific angle and circle (Gongol). This illustrates the dependence of calculating the sine as a function of the given angle. The cosine function, which is also mentioned, came about because of the need to compute the sine of the complementary angle, and thus was called the *co-sine* or *kotijya* (Maor 35).

For a thousand years use of the sine, cosine and tangent served applicable purposes as ratios of side of triangular representations. Then in the 18th century Abraham Gotthelf Kastner (1719 - 1800, Germany) changed all that. He was the first to define trigonometric functions as pure numbers, rather than ratios of sides in a triangle. He wrote: "If x denotes the angle expressed in degrees, then the expressions $\sin x$; $\cos x$; $\tan x$ etc. are numbers, which correspond to every angle" (Maor 53). This enabled mathematicians to consider the sine, cosine and all other trigonometric relationships as functions where the independent variable could be any real number rather than

an angle. The next section will show how the trigonometric functions became analytic, or represented by an infinite series. The analytic representation of the trigonometric functions helped to bring about the unification of trigonometry and exponentiation into our equation $e^{\pi i} + 1 = 0$.

Infinite Series

How do the sine and the cosine functions become analytic? In other words, how were they written as infinite series? Here we turn to the Swiss mathematician Leonard Euler (1707 - 1783) for much of the discoveries relating to $e^{\pi i} + 1 = 0$. It is impossible to over emphasize Euler's contributions to the discovery of this equation, which is also known as *Euler's Identity*. In my opinion, Euler was one of the most adventurous mathematicians in history. His intuitive and creative leaps uncovered a wealth of mathematical connections. Building on a theorem of Abraham De Moivre and using a liberal dose of imaginary numbers, Euler derived the famous sine and cosine series expansions.



Leonard

To begin, though, it is necessary to recognize the important contributions of Abraham DeMoivre (1667 - 1754). DeMoivre was born in France and move to England when he was a teenager. He made many contributions to probability theory and analytic geometry. In 1722 he restated his famous DeMoivre's Theorem as follows:



Abraham DeMoivre

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$$

This formula is a cornerstone of modern algebra and was used by Euler as a justification in his proof on finding roots of any number. More importantly, Euler used DeMoivre's theorem in his *Introductio* to derive the following trigonometric expansions:

$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \cdots\end{aligned}$$

Given the importance of this expansion to the evolution of $e^{\pi i} + 1 = 0$, I have included Euler's proof of the sine and cosine expansions in Appendix A.

Exponential Growth

To uncover the birth of the natural base e , and the discovery

of its analytic expansion, we need to trace the history of logarithms. A number of famous mathematicians played a part in this exposition, and we shouldn't be surprised to find Leonard Euler once again making inventive contributions.

First we will start with John Napier, a Scottish mathematician (1550 - 1617) who invented logarithms to be used as tools to simplify multiplication and division calculations into mere addition and subtraction. It seems that he worked primarily in isolation. He wrote *Mirifici Logarithmorum Canonis Descriptio* (A Description of the Wonderful Canon of Logarithms) in 1614. Although he did not consider it his most important work, it contained the first set of logarithmic tables, ninety pages of natural logarithms in all. In the appendix of that work Napier alludes to, but does not name, the constant e , also known as the natural base.

The next mathematician to tackle work associated with the discovery of the natural base was Jakob Bernoulli. Bernoulli was a Swiss mathematician who lived from 1654 to 1705 and was part of the famous and talented Bernoulli family, many of whom were mathematicians, scientists, architects and writers (Herman Hesse is a descendent of the Bernoulli family). For much of Bernoulli's career he was a Professor of mathematics at University of Basel, Switzerland and made numerous contributions to mathematics. He developed the Bernoulli equation in 1695, a type of separable differential equation. He may be best known for his *Ars Conjectandi* which was published posthumously, and contained known work on probability theory and enumeration as well as applications of probability theory and an introduction to his theorem on the laws of large numbers. While working on a problem involving compound interest, Bernoulli attempted to determine what evaluated to. He determined $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$ that it was equal to e , although he did not call the result such.

Enter Euler, once again. In his work to obtain an infinite series expansion for exponential and logarithmic functions, Euler used rigorous but somewhat elementary mathematics, meaning no differentiation or integration. The mathematics we are interested in occurs in chapter seven of his *Introductio*. In that chapter, Euler begins by finding the series expansion for the exponential function $y = a^x$ where $a > 1$. To begin he lets $a^\omega = 1 + \psi$ and notes that both ω and ψ are infinitely small. He then goes on to choose a constant k such that $\psi = k\omega$. He knew from experimentally substituting values into $a^\omega = 1 + k\omega$ that k was dependent upon values of the base a . Ultimately Euler was looking for the expansion of a^x and so raised both sides of the equation $a^\omega = 1 + k\omega$ to the j power yielding $(a^\omega)^j = (1 + k\omega)^j$. Euler then boldly let $j = \frac{x}{\omega}$, noting that j is infinitely large

because i is infinitely small. This will be a crucial distinction as we will see. By employing the binomial expansion for $(1+k\omega)^j$ and making use of the idea that j is infinitely large Euler wrote the infinite series expansion:

$$a^x = 1 + kx + \frac{(kx)^2}{2!} + \frac{(kx)^3}{3!} + \frac{(kx)^4}{4!} + \dots \quad \text{Equation 1}$$

At this point he chose to exercise the special case of $X=1$ yielding:

$$a = 1 + k + \frac{k^2}{2!} + \frac{k^3}{3!} + \frac{k^4}{4!} + \dots$$

Typical of Euler's insight, he wondered what would happen if he also let $k=1$. The equation becomes

$$a = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

Euler then computed the approximation of this infinite series and found that $a \approx 2.718$. He designated this, "for the sake of brevity", by the letter e . Further, by letting $a=e$ and $k=1$ in equation 1, he arrived at the expansion we are most interested in:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

The Final Unification

Now it's time to bring all the mathematical tributaries together into one equation. We followed the developments of trigonometric development to the point of analytic representation of the sine and cosine functions. We also saw how the natural base, e obtained an analytic representation as an infinite series. At this point it is obvious to ask "who discovered the final unification?" Who realized that $e^{\pi i} + 1 = 0$? I'm sure you can take a pretty good guess at who it was.

Of course it was Euler! Leonard Euler was already hot on the trail to this discovery when he determined that.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Ever the mathematically curious, Euler began to wonder what would happen if he employed imaginary numbers in this equation. Specifically, what would e^x evaluate to if he let it become complex by setting $x=xz$? We have the following theorem:

Theorem: For any real x , $e^{xi} = \cos x + i \sin x$

Proof: Replacing x by xi in the expansion of e^x , we obtain the following equations:

$$e^{xi} = 1 + xi + \frac{(xi)^2}{2!} + \frac{(xi)^3}{3!} + \frac{(xi)^4}{4!} + \dots$$

$$e^{xi} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) + i \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)$$

$$e^{xi} = \cos x + i \sin x$$

Q.E.D.

In the final corollary to this proof Euler replaces x by π to get the following

$$e^{\pi i} = \cos \pi + i \sin \pi$$

$$e^{\pi i} = -1 + i(0)$$

$$e^{\pi i} = -1$$

$$e^{\pi i} + 1 = 0$$

Being an exploration addict, mathematics is my perfect storm. This paper on $e^{\pi i} + 1 = 0$ demonstrates how seemingly disconnected areas of mathematics, can become connected with some bold intuitive leaps and fresh outlooks on old topics. Being a novice is no liability because sometimes imagination is more important than knowledge!

Appendix A

Euler's Proof of the Sine and Cosine Expansions:

Theorem:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Proof: For any $n \geq 1$ Euler knew that

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$$

and

$$(\cos \theta - i \sin \theta)^n = \cos(n\theta) - i \sin(n\theta)$$

Adding these two identities together and dividing by 2 gives

$$\begin{aligned}\cos n\theta + i \sin n\theta - i \sin n\theta + \cos n\theta &= (\cos \theta + i \sin \theta)^n + (\cos \theta - i \sin \theta)^n \\ 2 \cos n\theta &= (\cos \theta + i \sin \theta)^n + (\cos \theta - i \sin \theta)^n \\ \cos n\theta &= \frac{(\cos \theta + i \sin \theta)^n + (\cos \theta - i \sin \theta)^n}{2}\end{aligned}$$

Expanding the right hand side using the binomial expansion Euler got

$$\cos n\theta = \left[\cos^n \theta - \frac{n(n-1)\cos^{n-2}\theta \sin^2 \theta}{2!} + \frac{n(n-1)(n-2)(n-3)\cos^{n-4}\theta \sin^4 \theta}{4!} + \dots \right]$$

Equation 2

Next Euler “went infinite” (Dunham 93). Letting $x=n\theta$, where n is infinitely large and thus θ is infinitely small he got $\cos \theta = 1$ and $\sin \theta = \theta = \frac{x}{n}$.

Which can be written as: $\lim_{\theta \rightarrow 0} \cos \theta = 1$ and $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$

Because n is infinitely large Euler recognized that there was no difference between $n-1$, $n-2$, $n-3$ and so on, therefore he replaced all them all by n . He also replaced $\cos \theta$ with 1 and $\sin \theta$ with $\frac{x}{n}$ in equation 2 obtaining

$$\cos x = 1^n - \frac{n \cdot n \cdot (1)^{n-2} \left(\frac{x}{n}\right)^2}{2!} + \frac{n \cdot n \cdot n \cdot n \cdot (1)^{n-4} \left(\frac{x}{n}\right)^4}{4!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Similarly Euler started with:

$$\sin n\theta = \frac{(\cos \theta + i \sin \theta)^n - (\cos \theta - i \sin \theta)^n}{2i}$$

Using a similar derivation he ultimately obtained the sine expansion:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad \text{Q.E.D.}$$

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