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# (Knight)$^3$: A Graphical Perspective of the Knight's Tour on a Multi-Layered Chess Board

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## Abstract

The Knight's Tour is an interesting question related to the game of chess. In chess, the Knight must move two squares in one direction (forward, backward, left, right) followed by one square in a perpendicular direction. The question of the Knight's Tour follows: Does there exist a tour for the Knight that encompasses every single square on the chess board without revisiting any squares? The existence of Knight's Tours has been proven for the standard 8x8 chess board. Furthermore, the Knight's Tour can also exist on boards with different sizes and shapes. There has been a lot of research into tours on two-dimensional boards.

In this project, we explore the question of the Knight's Tour on multi-layered chess boards. In other words, would it still be possible for a Knight's Tour to exist on a chess board if there was a third dimension of movement that the Knight could take? This thesis will look at the Knight's Tour on a two-dimensional board, both standard and rectangular, and will then examine the existence of Knight's Tours on a multi-layered chess board. Finally, the Knight's Tour will also be explored on a three dimensional cube on which the Knight can only move on the face of the cube. Throughout the thesis, we will use concepts of Graph Theory to explore tours on the different types of boards.

## 1. Introduction

In the game of chess, two players have a set of pieces to move around an 8 x 8 grid of squares. Each piece in the game has its own unique movement that it can take in a turn. For example, the Bishop piece can move any number of squares in a diagonal direction. The Queen is able to move any number of squares in any one direction, while the King is restricted to moving to a single step in a given direction.

There are several different mathematical problems related to the game of chess, usually involving individual types of chess pieces, rather than the entire set of pieces altogether. The non-attacking piece problem involves placing a number of one type of piece around a board of a given size. The trick: none of the pieces can be in a position to attack one another from their current positions. For example, in the non-attacking Kings problem, any King placed on the board would require a ring of empty squares surrounding it, as the King is able to move into these squares, where it could attack another King. The non-attacking Queens problem would result in Queens being placed in separate rows, columns, and diagonals from each other, due to the movement of the Queen. The goal of this problem is to determine the maximum number of a given piece that can be placed on a chess board so that none of the pieces can attack each other from their given location. Another interesting problem related to chess involves finding tours, in which a given piece is able to make a series of movements so that it is able to travel to every square on a chess board exactly once.

In many of these problems, the Knight is one of the more interesting pieces to use. When moving around the board, the Knight's movement can be described as a simple L shape. In terms of the chess board, the Knight moves two squares in one direction (left, right, forward, backward), followed by one square in a perpendicular direction. If the Knight is in a corner square, there are only two possible moves for the Knight to make. When all surrounding squares are empty and open to movement, the Knight has up to 8 possible moves it can make from its current position. As such, on a completely open board with only a single Knight piece, there are anywhere from 2 to 8 possible moves that can be made. Figure 1 shows the possible movements for the Knight to make from a given square.

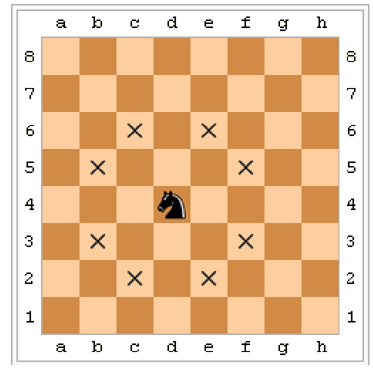



Figure 1: The Knight has up to 8 moves on an *m* x *n* chess board. The x's represent the possible moves that the Knight can make from its current position.

For this paper, I will be delving into the problem of the Knight's Tour. The question of the Knight's Tour is as follows: Does there exist a path of Knight's movements around the chess board such that every square is visited exactly once? This type of path is known as a Knight's Tour. Knight's Tours can start on any square of the chess board, since the path will encompass every square of the board. Because of the unique movement of the Knight, a path is not easily identified.

These tours can fall into one of two categories: open and closed. As shown in Figure 2, the example on the left shows a closed tour. In this tour, a complete loop is made, meaning that the tour begins on one square and moves to every other square before coming back to the starting position. On the right of Figure 2 is an example of an open tour. For this type of tour, the Knight cannot make a single move from the ending position to get back to the starting position.

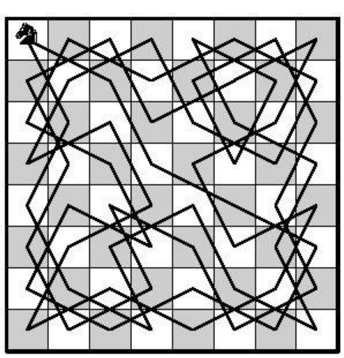
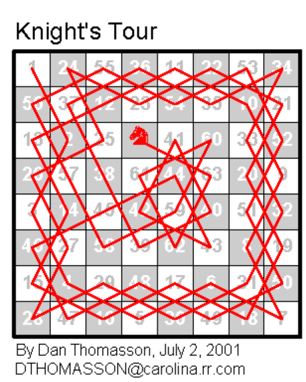


Figure 2: (left) A closed Knight's Tour, beginning and ending on the same square [13]. (right) An open Knight's Tour. Notice how the Knight cannot make a single move from the final square to the beginning square of the tour [6].



## 2. History of Knight's Tour

The study of Knight's Tours can be traced back to several different points in history. One of the first Knight's tours on the two-dimensional 8x8 chess board was given by Abraham de Moivre in the early 18$^{th}$ century [4]. The open tour was created by first making moves solely within the outer two rows of the chess board, moving inward only when necessary. By this method, the first 24 moves could be made on the outer edge of the board.

In 1759, Euler used a different technique to create a closed Knight's Tour on an 8x8 chess board from a special kind of open tour [4]. For this closed tour, four arbitrary squares on the chess board were missed. The open tour was created using the remaining 60 squares. In order to incorporate the missed squares into the tour, Euler reordered a portion of the tour. This reordering made it possible to make a move to a missed square, then move back into the rest of the tour. Through these numerous steps, Euler took the missing squares and integrated them into the tour, and was able to create a closed Knight's Tour in the process.

## 3. Graph Theory Concepts

To help explain the Knight's Tour, I will use several concepts from Graph Theory, and connect them to ideas and concepts from the Knight's Tour. In a general sense, a graph is a collection of nodes, or vertices, arranged in any way. The vertices in a graph are connected to one another through a set of edges. Graphs can be used to model different ideas and concepts, such as the movement of the Knight on the chess board.

Through the use of Graph Theory concepts, the Knight's Tour can be explained from a different perspective. The Knight's moves and the chess board can be modeled by a graph, with a set of vertices representing the squares of the chess board. Edges are placed between two different vertices if there exists a Knight's move between the corresponding squares of the chess board. In Figure 3, the 4 x 4 chess board is represented in both standard and graph form.

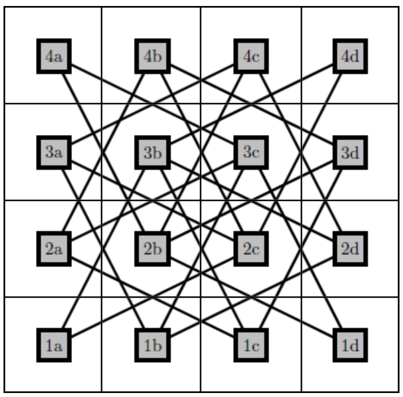
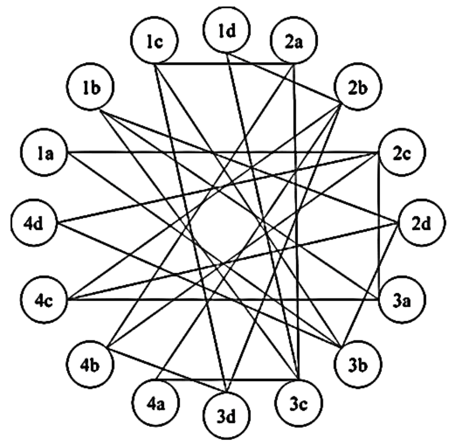



Figure 3: (left) A 4 x 4 chess board, with lines drawn between squares connected by a Knight's move [2]. (right) A graphical representation of the 4 x 4 chess board with Knight's moves.




For this thesis, we were interested in the paths on the chess board, along with the paths on the corresponding graph. A path in a graph is a set of movements over edges between connected vertices, where no edge is traversed twice. The path can begin and end on any two vertices of the graph. A special type of path is known as the Hamiltonian Path, where each vertex in the graph is included in the path exactly once. In a Hamiltonian Path, the beginning and ending vertex do not have to be the same. If, however, the Hamiltonian Path begins and ends on the same vertex, then it is an example of a Hamiltonian Cycle. Any graph that contains a Hamiltonian Cycle is called a Hamiltonian Graph.

The Knight's Tour on a chess board corresponds to a Hamiltonian Path or Hamiltonian Cycle on the graph. By completing a tour, the Knight is able to move to every single square on the chess board and thus an open Knight's Tour is an example of a Hamiltonian Path, as the beginning and ending square of the Knight's Tour is not the same. The closed Knight's Tour is an example of a Hamiltonian Cycle, where the beginning and ending square is the same, and the Knight's Tour creates a closed loop on the chess board.

## 4. Altering the Chess Board

Earlier mathematicians studied Knight's Tours on the standard 8x8 chess board, but later mathematicians worked on studying non-standard chess boards. If the board were to be altered, say by stretching or shrinking the dimensions of the board, would there be an effect on the existence of Knight's Tours on these new boards? In certain cases, the Knight's Tour would still exist on an $m$ x $n$ board, where $m$ and $n$ may or may not be equal. This opens up many new possibilities for the Knight's Tour. However, under some conditions, there may be no possible tour on a given board. Figure 4 gives two different examples of rectangular chess boards: one for which a Knight's Tour exists, and the other for which it is impossible for a Knight's Tour to exist.

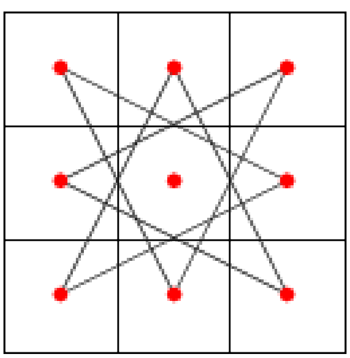
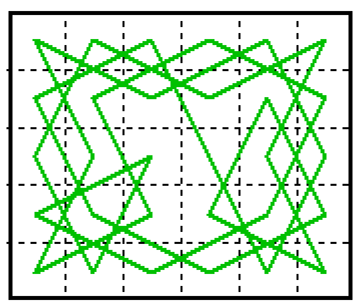


Figure 4: (left) A representation of a 3 x 3 rectangular chess board without a Knight's Tour, as the center square cannot be reached [5]. (right) A 5 x 6 chess board with a closed Knight's Tour [8].



Allen Schwenk gave conditions that can be used to determine the existence of a closed Knight's tour on an $m$ x $n$ board [1].

Theorem 1 (Schwenk [1]): Assume that $1 \leq m \leq n$, where $m, n \in \mathbb{N}$. Then there exists a closed Knight's tour on an $m$ x $n$ board unless one of these three conditions hold.

1. $m$ and $n$ are both odd,
2. $m \in \{1,2,4\}$,
3. $m = 3$ and $n \in \{4, 6, 8\}$

To explain why $m$ and $n$ cannot both be odd, we first prove a property of the graphs that represent the chess boards on which a closed tour exists. To show that a closed tour cannot exist on a board with an odd number of squares, we can show that the graph modeling the Knight's movement on a chess board is a bipartite graph, or a graph with two disjoint sets of vertices called partite sets, in which edges exist only between vertices in opposite partite sets This will show that there are no odd cycles, or cycles using an odd number of vertices. Once this is shown, we can then show that a closed Knight's Tour cannot exist on a chess board with an odd number of squares.

<u>Proposition</u>: Consider an $m$ x $n$ chess board for which a closed Knight's Tour exists. Then the graph representing the possible moves of the Knight on the chess board is bipartite.

Proof: We first define a labeling of the vertices of the graph, in particular, we label a vertex on the graph according to the location of the corresponding square on the chess board. Thus the vertex corresponding to the square in column a and row b of the chess board is labeled by the ordered pair (a, b), where $1 \leq a \leq n$, and $1 \leq b \leq m$, and (1, 1) represents the square in the upper left most corner of the $m$ x $n$ chess board. Note that two vertices labeled (a, b) and (c, d) are adjacent if the corresponding vertices are within a Knight's move of one another, that is if $(a - c, b - d) = (\pm 1, \pm 2)$ or $(a - c, b - d) = (\pm 2, \pm 1)$.

We partition the vertices into two sets X and Y of the chess board's squares based on the parity of the sum of the coordinates of the square. That is, $(a, b) \in X$ if $(a + b)$ is even and (a, b) ∈ Y if (a + b) is odd. Thus, two disjoint sets are created. Whenever a Knight makes a move, the net movement is three squares (two squares in one direction, followed by one square in a perpendicular direction), thus changing the parity of either a or b. In other words, the parity of the square is changed between moves, so the Knight can only move into a square with opposing parity. Thus, the graph is bipartite. ■

It is known that bipartite graphs have no odd cycles. Since this graph is bipartite, then the graph of Knight's movements can only have a cycle with an even number of squares. If *m* and *n* are both odd, then there are an odd number of squares in the chess board, thus there will be an uneven distribution of squares between the two partite sets. In order to create a closed Knight's Tour, the final Knight's move would have to be made between two squares in the same partite set. However, in a bipartite graph, no edges can exist between two vertices of the same partite set. Thus, it is impossible to have a closed Knight's Tour on an *m* x *n* chess board where *m* and *n* are both odd. One thing of note is that it is completely possible to have an open Knight's Tour with an odd number of squares; however, a closed Knight's Tour is not possible on a chess board with an odd number of squares because it would represent an odd cycle in the graph.

Condition 2 says that any chess board with a dimension of 1, 2, or 4 cannot contain a closed Knight's Tour. Consider first the case when $m = 1$. If we have a single row of squares on the chess board, it is impossible to make a movement in a perpendicular direction. Thus, a Knight's Tour cannot exist on a 1 x *n* board. When $m = 2$, there are only two rows of squares, and the Knight would only be able to move between alternating columns of the board. There would be no possible move that would allow the Knight to move into any adjacent column. For $m = 4$, the proof is completed by Schwenk in [1].

The proof of Condition 3 is also given by Schwenk in [1]. Condition 3 states that $m$ x $n$ chess boards are unable to have a closed Knight's Tour if $m = 3$ and $n \in \{4, 6, 8\}$. The proof uses a Graph Theory theorem that any $k$ vertices removed from a graph with a Hamiltonian Cycle leave the graph with at most $k$ connected components. When the vertices are removed, any edges connected to the removed vertices are also removed. The connected components represent the squares of the chess board connected by Knight's moves. So, by removing $k$ squares from the chess board, there can be at most $k$ disjoint sets of squares connected by Knight's moves. In the cases where $n$ is equal to 4, 6, or 8, it is possible to remove a certain number of squares from the board and create more disjoint sets of squares than the number of squares removed. Thus, a closed Knight's Tour cannot exist on these types of chess boards [1].

| n \ m | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | X | X | X | X | X | X | X | X | X | X | X | X |
| 2 | X | X | X | X | X | X | X | X | X | X | X | X |
| 3 | X | X | X | O | X | X | O | O | E | C | E | C |
| 4 | X | X | O | X | O | O | O | O | O | O | O | O |
| 5 | X | X | X | O | E | C | E | C | E | C | E | C |
| 6 | X | X | X | O | C | C | C | C | C | C | C | C |
| 7 | X | X | O | O | E | C | E | C | E | C | E | C |
| 8 | X | X | O | O | C | C | C | C | C | C | C | C |
| 9 | X | X | E | O | E | C | E | C | E | C | E | C |
| 10 | X | X | C | O | C | C | C | C | C | C | C | C |
| 11 | X | X | E | O | E | C | E | C | E | C | E | C |
| 12 | X | X | C | O | C | C | C | C | C | C | C | C |

Figure 5 (above): A chart of the different types of Knight's Tours on an $m$ x $n$ chess board.

Other authors have considered both closed and non-closed Knight's Tours. Shun-Shii Lin and Chung-Liang Wei studied closed and open tours, as well as a new type of tour, known as the Corner-missed closed tour, where a closed Knight's Tour is made without any of the four corner squares [10]. Figure 5 summarizes the results of their research for $m$ x $n$ chess boards, where $m$ and $n$ represent the integers between 1 and 12 The notation in the chart represents the different types of Knight's Tours that can exist on the board. X represents a chess board where there is no possible Knight's Tour, E represents a board with a Corner-missed closed tour, O represents a chess board where an open Knight's Tour is possible, and C represents a chess board with a closed Knight's Tour. For the purposes of the chart, if a closed Knight's tour exists for an $m$ x $n$ chess board, then an open Knight's Tour exists on the same board, as well as a Corner-missed closed tour. If there is an E, then an open tour and a Corner-missed closed tour exists, but not a full closed tour. If there is an O, then an open tour is possible, but not a closed tour or Corner-missed closed tour [10].

The results of the chart agree with the theorem given by Schwenk. As shown in the chart, a chess board where one of the dimensions is 1 or 2 cannot have any type of Knight's Tour. A closed tour also cannot exist when *m* or *n* is 4, though in certain cases, it is possible to have an open tour. In all entries of the chart representing chess boards with both dimensions being odd, it is impossible to have a closed Knight's Tour. Conversely, for $m, n \geq 5$, where *m* or *n* is even, it is possible to have a closed Knight's Tour. This chart also shows that the smallest possible board with an open Knight's Tour is the 3 x 4 chess board. In this case, only an open Knight's Tour exists on this size chess board due to the second condition of Theorem 1 [1]. Figure 6 shows an example of a 3 x 4 open Knight's tour [7].

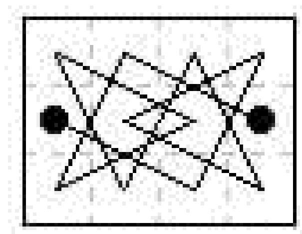


Figure 6 (above): Example of an open Knight's Tour on a 3 x 4 chess board.

## 5. Other Alterations of the Chess Board

Knight's Tours have also been studied on irregular chess boards. These irregular chess boards are non-rectangular, and include rectangular boards with squares removed. Here, the arrangement of the squares is important. In particular, the group of irregular chess boards does not include any chess boards where a portion of the board is only connected to the rest of the board diagonally. Figure 7 shows the difference between the types of chess boards that qualify as an irregular chess board, and those that aren't part of the category. As shown in the figure, the board on the left is within the category of irregular chess boards because all of the squares are connected to each other. The board on the right is not part of the category because there is a square that is only connected to the rest of the board by a diagonal.

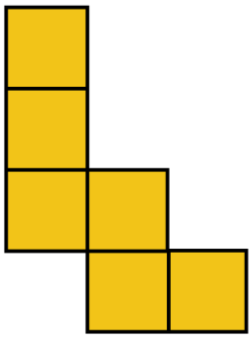
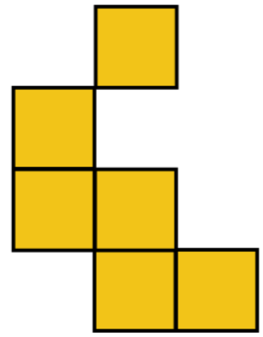


Figure 7: (left) An example of a connected irregular chess board. (right) A non-connected chess board, which is not considered part of the irregular chess board category.



Rather than focusing on the dimensions of the chess board, studies of Knight's Tours on irregular chess boards focus on the number of squares used to create the chess board, along with their arrangement. Using this type of chess board, it is possible to figure out the least number of squares needed for a Knight's Tour to exist on an irregular chess board.

From the chart in Figure 5, we see that there is no possible Knight's Tour on the 3 x 3 chess board. This is due to the fact that there is no way to move to or from the center square. However, it is possible to move around the eight surrounding squares. In this way, a closed Knight's Tour exists on the irregular board where the center square is removed (see Figure 8 below). By removing any one of the squares from that board, it is still possible to have an open Knight's Tour, but not a closed Knight's Tour. If any more squares are removed from the board, it will either create a disconnected board, or make it impossible for a Knight's Tour to exist. Figure 8 shows an example of an open Knight's Tour using 7 squares, and a closed Knight's Tour with 8 squares [11].

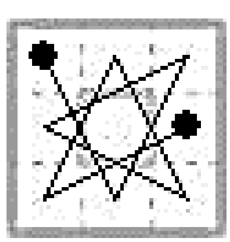
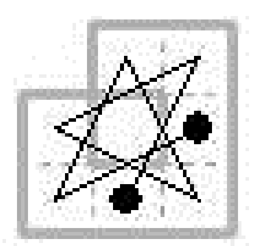


Figure 8: (left) Example of a closed Tour on the 3 x 3 irregular board without the center square. (right) An open Tour with a corner square removed.



## 6. Three Dimensional Chess Board

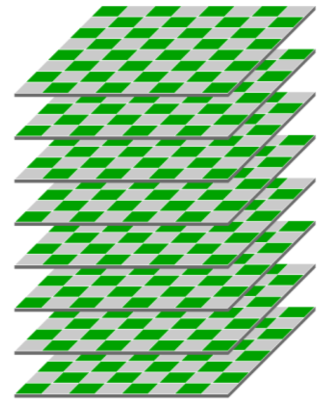


Figure 9: (above) A multi-layered chess board. In this case, the board consists of 8 layers of 8 x 8 chess boards.

For this thesis, we also looked at the three-dimensional chess board, considering conditions for the existence of a Knight's Tour on this type of chess board. In general, the three-dimensional chess board refers to multiple layers of an *m* x *n* chess board. In this way, the Knight is able to move between the layers of the chess board, in the same L-shape (see below). Figure 9 shows an example of a three dimensional chess board, with 8 layers of an 8 x 8 chess board [12]. Between each layer of the chess board, the coloring is altered to maintain the change in color whenever the Knight makes a move.

In the multi-layered chess board, the Knight is able to move between the layers of the three-dimensional chess board. So, if the Knight moves up one layer, the Knight can then move two squares in any given direction on the new layer of the chess board. If the Knight moves up two layers, then the Knight can move to one of four directly adjacent squares (left, right, forward, or backward) on the new layer. From a central position, the Knight has up to 24 possible moves. The set of 24 possible moves consists of the standard 8 moves on the original layer, 8 more moves from moving up or down one layer, and another 8 from moving up or down two layers. So, the number of Knight's moves is increased with the inclusion of multiple layers of chess boards.

## 7. Three Dimensional Knight's Tour

For this type of chess board, we can extend Theorem 1 to help determine the existence of closed Knight's Tours on a multi-layered chess board. These conditions, presented by Joe DeMaio and Bindia Mathew [3], can be compared to the conditions shown by Schwenk for the *m* x $n$ chess board, by also considering the number of layers of the $m$ x $n$ chess board. So, for condition 1 of Theorem 1, we can consider the number of layers of chess boards with odd dimensions to determine if a closed Knight's Tour can exist on all layers. With condition 2 of Theorem 1, we can consider the number of layers of a 2 x $n$ or a 4 x $n$ chess board. Multiple layers of a 1 x $n$ chess board can be explained using Theorem 1 alone, as it would just be a case of an $m$ x $n$ chess board, where $m$ represents the number of layers [3]. Condition 3 of Theorem 1 can be explored further using multiple layers [3]. Altogether, the conditions for the closed Knight's Tour on a multi-layered chess board are given in Theorem 2.

Theorem 2 (DeMaio, Mathew [3]): Assume $2 \leq i \leq j \leq k$ for $i, j, k \in \mathbb{N}$. Then there exists a closed Knight's Tour on an $i$ x $j$ x $k$ multi-layered chess board unless one of the following conditions holds:

1. $i, j,$ and $k$ are all odd;
2. $i = j = 2$;
3. $i = 2$ and $j = k = 3$

As previously mentioned, the conditions for a closed Knight's Tour on a multi-layered chess board outlined by DeMaio and Mathew can be compared to Theorem 1 from Schwenk. If $i$, $j$, and $k$ are all odd, then there are an odd number of squares. We can refer back to Theorem 1, where an $m$ x $n$ chess board is unable to have a closed Knight's Tour if $m$ and $n$ are both odd. Similarly, in the case of an $i$ x $j$ x $k$ multi-layered chess board, if $i, j,$ and $k$ are all odd, then there will be an odd number of squares altogether, which prevents the existence of a closed Knight's Tour on the $i$ x $j$ x $k$ chess board [3].

Condition 2 in Theorem 2 can be compared to the second condition of Theorem 1. Suppose we have a given number of layers of a 2 x 2 chess board. Since the Knight cannot move within the individual layer, the Knight is forced to move between layers. When the Knight makes a move to another square, it is only able to move between every other layer of the chess board. If the Knight were to move up or down one layer, there would be no way for the Knight to move two squares in any give direction on the new layer. So, the layer would have to be skipped over by the Knight. Thus, a closed Knight's Tour cannot exist on a multi-layered chess board consisting of 2 x 2 chess boards.

For condition 3 of Theorem 2, the proof is very simple. Suppose we have two layers of 3 x 3 chess boards. It is known that there is no Knight's Tour on a 3 x 3 chess board. This is due to the fact that no Knight's move can be made to or from the central square. By adding a second layer of a 3 x 3 chess board, there are still no possible movements to be made to or from the central square in each individual layer. Thus, a Knight's Tour cannot be made on a 2 x 3 x 3 multi-layered chess board. It is possible to extend this proof to the 3 x 3 x 3 multi-layered chess board. Aside from the fact that there would be an odd number of squares, there is no way to move to or from the center-most square in the chess board. However, it is possible for certain 2-dimensional chess boards without a closed Knight's Tour to have one when we add in multiple layers of the same chess board.

## 8. The *n*-Cube Knight's Tour

Finally, we consider a restricted three dimensional chess board. We define an *n*-cube to be a cube with an $n$ x $n$ chess board on each face. For example, an 8-cube is an 8 x 8 x 8 cube with an 8 x 8 chess board on each face. In this form, the Knight is able to move between the squares on each face of the cube as if they were adjacent to each other. There are no squares on the interior of the cube, as the Knight only moves around the outer faces of the cube. A closed Knight's Tour on the 8-cube can be seen in Figure 10, where the 8-cube is flattened. This tour was given by Miodrag Petković in [9]. The curved lines in the figure represent movement between the faces of the cube.

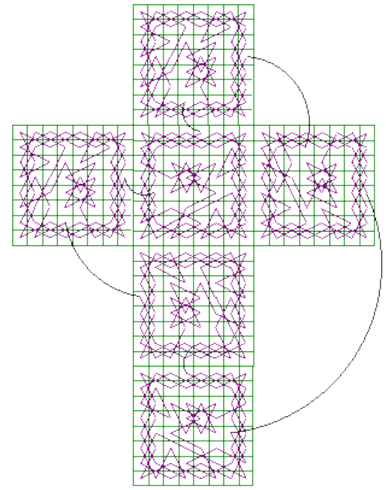


Figure 10: (above) An 8-cube with a Knight's Tour that uses every single square.

Petković discussed how to create a closed Knight's Tour on the 8-cube in [9]. The idea is to create an individual tour on each face of the cube, then link the tours together. To link two faces of the cube together, one edge is removed from the tour on each face. However, it must be possible to connect the resulting endpoints together via a Knight's move, thus creating a closed tour that includes both faces of the cube. This process is repeated for the remaining faces of the *n*-cube, so that there is a single closed Knight's Tour involving every square.

While the tour for the 8-cube was given in [9], there were no tours shown for other *n*-cubes. Figure 11 shows a portion of a closed Knight's Tour we created on the 6-cube and the 10-cube. These were created using known 6 x 6 and 10 x 10 closed tours respectively that were connected together using the method described above.

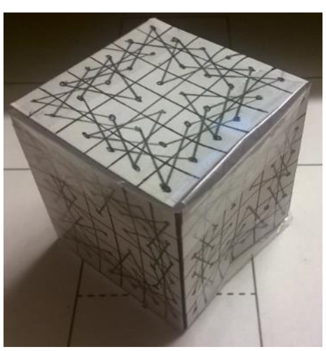


Figure 11: (left) A 6-cube with a closed Knight's Tour including all faces of the cube. (right): A 10-cube with a closed Knight's Tour including all of the squares.

Based on our work, we make the following conjecture about closed Knight's Tours on an *n*-cube.

Conjecture: If a closed $n$ x $n$ tour exists for $n \in \mathbb{N}$, then it is possible to create a closed Knight's Tour on an *n*-cube.

Although it may be possible for a closed tour to exist on an *n*-cube where an $n$ x $n$ closed Knight's Tour does not exist, this conjecture considers only the *n*-cubes where an $n$ x $n$ closed Knight's Tour exists. A proof of this conjecture could include describing how to piece together closed Knight's Tours on each face of the cube to create a single closed tour.

## 9. Conclusion

The Knight's Tour is an interesting problem related to the game of chess. The movement of the Knight makes it particularly interesting to study in different types of problems relating to the game of Chess. Since the Knight's Tour and the chess board can be modeled using graphs, we can use Graph Theory concepts to explore the properties of the Knight's Tour. In addition, these Graph Theory concepts can be applied to the Knight's Tour on both the multi-layered chess board and the *n*-cube.

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