5-12-2016

Characterizations of Four Interval Wavelet Sets and Algorithms

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Submitted in Partial Completion of the
Requirements for Departmental Honors in Mathematics

Bridgewater State University

May 12, 2016

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May 10, 2016

Abstract

Wavelets are mathematical tools used to represent signals such as audio files, pictures, videos, and various other types of data. The theory of wavelets has recently attracted attention in Mathematics because of potential in applications. At this point, the field of wavelet theory is fairly mature, and the literature contains a body of techniques which are exploited to design wavelets. One of these techniques relies on the construction of wavelet sets. A wavelet set is a set whose successive translations and dilations partition a line. In practice, wavelet sets are tricky to construct. In fact, there is no known classification of wavelet sets consisting of more than three intervals available in the literature. In the present thesis, we will provide a complete characterization of wavelet sets of four intervals. Additionally, we will present two algorithms which are used to construct wavelet sets of four intervals.

Contents

1 Introduction

Wavelets are mathematical tools which are used to decompose signals into their most basic components. By definition, a signal can be regarded as a physical quantity that carries a specific type of information for communication. As such, wavelets are important tools used in signal transmission and data representation. There are a variety of techniques commonly used to construct wavelets. One very appealing method which has become popular in the last few years relies on the construction of wavelet sets. A wavelet set on $\mathbb{R}$ is a set whose successive dilations and translations partition the real line [1, 2, 3, 5]. A dilation is a transformation which either shrinks or stretches a given set, and a translation is a process by which a given set is moved from one place to another in a rigid motion. To be more precise, let x be a real number, and let h be a nonzero real number. The transformation $T_x : \mathbb{R} \to \mathbb{R}$ such that $T_x(t) = t + x$ is a translation map. Moreover, the map $D_h : \mathbb{R} \to \mathbb{R}$ defined such that $D_h(t) = ht$ acts on the real line by dilation. Now, let E be a non-empty subset of the real line. We say that E is a **translation-tiling** subset of the real line if the collection of sets $\{T_k(E) : k \in \mathbb{Z}\}$ forms a partition of $\mathbb{R}$. In a similar fashion, we say that E is a **dyadic dilation-tiling** subset of the real line if the collection of sets $\{D_{2^k}(E) : k \in \mathbb{Z}\}$ is a partition of the real line. Next, a set is a **wavelet set** if it is a translation-tiling subset as well as a dyadic dilation-tiling subset of the real line. In practice, it is not hard to construct a set which partitions a line by translations. For example, any unit interval is a translation-tiling subset of $\mathbb{R}$. It is also fairly easy to construct sets which partition $\mathbb{R}$ by dilations. For example, given real numbers a, b satisfying $a < 0 < b$, the set $\left[a, \frac{a}{2}\right) \cup \left[\frac{b}{2}, b\right)$ is a dyadic dilation-tiling of the real line. We observe that it is not possible for a unit interval to be a wavelet set because it does not satisfy the dilation-tiling condition. Moreover, it is not true that for any given real numbers $a < 0 < b$ the set $\left[a, \frac{a}{2}\right) \cup \left[\frac{b}{2}, b\right)$ is a wavelet set. However, when $a = -1$ and $b = 1$ we obtain the simplest example of a wavelet set known as the **Shannon-wavelet set**, which is described as follows:

$$\left[-1, -\frac{1}{2}\right) \cup \left[\frac{1}{2}, 1\right).$$

This set was named after Claude Shannon because it plays a central role in Shannon sampling theory for bandlimited spaces. Next, it is not very difficult to verify that the collections

$\{[-1+k,-\frac{1}{2}+k) \cup [\frac{1}{2}+k,1+k) : k \in \mathbb{Z}\}$ and

$$\left\{\left[-2^j,-\frac{2^j}{2}\right) \cup \left[\frac{2^j}{2},2^j\right) : j \in \mathbb{Z}\right\}$$

are two distinct partitions of the real line. The Shannon wavelet set is not the only wavelet set which is comprised of two intervals. In fact in the subsequent paragraphs, we will provide a complete descriptions of all two interval wavelet set.

Let now suppose that $\mathbf{W} = [a,b) \cup [c,d)$ is a wavelet set of two intervals where $a < b < 0 < c < d$. Since the integral translates of $\mathbf{W}$ produce a tiling of the real line, the following hold true: $(d-c)+(b-a) = 1$. Moreover, there exists an integer k such that $[a+k,b+k) \cup [c,d) = [u,u+1)$ where u is a real number. Next, since the collection of dyadic dilations of $\mathbf{W}$ must also tile the real line, it immediately follows that there exist integers j_1 and j_2 such that

$$\left[2^{j_1}a, 2^{j_1}b\right) = \left[-1,-\frac{1}{2}\right) \text{ and } \left[2^{j_2}c, 2^{j_2}d\right) = \left[\frac{1}{2},1\right).$$

Combining all the observations above we obtain the following system of equations

$$\begin{cases} (d-c)+(b-a) = 1 \\ a+k = u \\ b+k = c \\ d = u+1 \\ a = 2b \\ d = 2c \end{cases}$$

where $b < 0$ and $c > 0$. Solving the above system of equations together with the restrictions imposed on the variables b, c we obtain

$$a = 2b, c = 1+b, d = 2+2b, u = 1+2b, k = 1 \text{ and } b \in (-1,0).$$

Thus $\mathbf{W}$ is a two interval wavelet set if and only if there is a real number $b \in (-1,0)$ such that $\mathbf{W} = \mathbf{W}_b = [2b,b) \cup [1+b,2+2b)$. This complete description of two interval wavelet sets has been obtained by M. Bownik and K. Hoover (see Theorem 7.1 [1]). The authors of [1] also gave a complete description of three interval wavelet sets. It is worth pointing out that the case of three intervals is considerably more complicated than the case of wavelet sets consisting of two intervals. To the best of our knowledge, there is no classification of four interval wavelet sets available in the literature. As such, the chief objective of the present thesis is to undertake a thorough and careful investigation of four interval wavelet sets. In Section 2 we provide a rigorous definition of wavelet sets, and we prove several important properties enjoyed by wavelet sets. This section also contains a complete description of wavelet sets of four intervals. Our results show that the case of four interval wavelet sets is much more complex than the three interval case [1]. We also pay a special attention to **symmetric wavelet sets** (wavelet sets which are up to boundary points invariant under multiplication by -1). A complete parametrization of symmetric wavelet sets of four intervals is presented in the third section of this work. Finally, we introduce two algorithms which can be exploited to create wavelet sets in the fourth section. We have implemented one of these algorithms in Mathematica. An output of our program is contained in the appendix of the thesis.

2 Wavelet Sets

2.1 Some General Facts

Let $\mathbf{W}$ be a set of real numbers. We say that $\mathbf{W}$ is a **wavelet set** if and only if the collection of sets $\{\mathbf{W}+k : k \in \mathbb{Z}\}$ forms a partition of the real line, and the collection $\{2^j\mathbf{W} : j \in \mathbb{Z}\}$ partitions $\mathbb{R}$ as well. In other words, a wavelet set is a set which tiles the real line by both integral translations and dyadic dilations. Let $\mathbf{E}$ be a subset of real numbers. We define the indicator function of the set $\mathbf{E}$ as

$$\chi_{\mathbf{E}}(x) = \begin{cases} 1 \text{ if } x \in \mathbf{E} \\ 0 \text{ if } x \notin \mathbf{E} \end{cases}$$

From the preceding definition, it is easy to see that $\mathbf{W}$ is a wavelet set if and only if the following holds true

- $\sum_{k\in\mathbb{Z}} \chi_{\mathbf{W}}(x+k) = 1$ for all $x \in \mathbb{R}$
- $\sum_{j\in\mathbb{Z}} \chi_{\mathbf{W}}(2^j x) = 1$ for all $x \in \mathbb{R}$.

Appealing to the characterization above, we will derive several important properties of wavelet sets.

Definition 1 *Let* $\mathbf{E}$ *be the union of finitely many intervals. The* ***length*** *of* $\mathbf{E}$ *is denoted* $\text{Length}(\mathbf{E})$ *and is computed as follows:* $\text{Length}(\mathbf{E}) = \int_{\mathbf{E}} 1\, dx$.

Proposition 2 *Let* $\mathbf{W}$ *be a wavelet set. Assume that* $\mathbf{W}$ *is the union of finitely many intervals. Then*

1. $0 \notin \mathbf{W}$.
2. $\text{Length}(\mathbf{W} \cap (-\infty, 0)) > 0$ *and* $\text{Length}(\mathbf{W} \cap (0, \infty)) > 0$.
3. *The length of a wavelet set must be equal to one.*
4. *If* $\mathbf{W}$ *is a wavelet set then* $-\mathbf{W} = \{-x : x \in \mathbf{W}\}$ *is a wavelet set.*

Proof. For the first part, let us assume by contradiction that $\mathbf{W}$ is a wavelet set, and zero is an element of the wavelet set. Then for $x = 0$, the infinite sum $\sum_{j\in\mathbb{Z}} \chi_{\mathbf{W}}(2^j x)$ is equal to $\sum_{j\in\mathbb{Z}} \chi_{\mathbf{W}}(2^j 0) = \sum_{j\in\mathbb{Z}} \chi_{\mathbf{W}}(0)$. Next, since it is assumed that 0 is in the wavelet set, it must be the case that $\chi_{\mathbf{W}}(0) = 1$, and it follows that the series below $\sum_{j\in\mathbb{Z}} \chi_{\mathbf{W}}(0) = \sum_{j\in\mathbb{Z}} 1$ is divergent. This contradicts the fact that $\mathbf{W}$ is a wavelet set. For the second part, let us suppose that the wavelet set $\mathbf{W}$ is contained in the set $(0, \infty)$. In other words, the wavelet set is disjoint from the set of negative real numbers. Since the set of positive real numbers is invariant under multiplication by positive numbers, then it is clear that for any integer j the set $2^j\mathbf{W}$ is contained in $(0, \infty)$. As a result,

$$\left(\bigcup_{j\in\mathbb{Z}} \left(2^j \mathbf{W}\right) \right) \cap (-\infty, 0)$$

is an empty set. So, $\mathbf{W}$ cannot tile the real line by dilation. Using similar arguments, it is easy to see that if $\mathbf{W}$ is a wavelet set then $\mathbf{W} \cap (-\infty, 0)$ must have positive length. For the third part, let us assume by contradiction that the length of $\mathbf{W}$ is greater than one. Clearly,

$\mathbf{W} \cap (\mathbf{W}+1)$ is a nonempty set. As such,

$$\sum_{k\in\mathbb{Z}} \chi_{\mathbf{W}}(x+k) \geq \chi_{\mathbf{W}}(x) + \chi_{\mathbf{W}+1}(x) > 1$$

violates the facts that $\mathbf{W}$ tiles the real line by integral translations. Similarly, if we assume that the length of $\mathbf{W}$ is smaller than one, then it must be the case that

$$\sum_{k\in\mathbb{Z}} \chi_{\mathbf{W}}(x+k) < 1.$$

Thus, the length of $\mathbf{W}$ must be equal to one. To prove the last part, let us now suppose that $\mathbf{W}$ is a wavelet set. We check that for any given real number x,

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \chi_{-\mathbf{W}}(x+k) &= \sum_{k\in\mathbb{Z}} \chi_{\mathbf{W}}(-(x+k)) \\
&= \sum_{k\in\mathbb{Z}} \chi_{\mathbf{W}}(-x-k) \\
&= \sum_{j\in\mathbb{Z}} \chi_{\mathbf{W}}(-x+j) \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \chi_{-\mathbf{W}}\left(2^j x\right) &= \sum_{j\in\mathbb{Z}} \chi_{\mathbf{W}}\left(-2^j x\right) \\
&= \sum_{j\in\mathbb{Z}} \chi_{\mathbf{W}}\left(2^j(-x)\right) \\
&= 1.
\end{aligned}$$

So, $-\mathbf{W}$ is a wavelet set as well. ■

Definition 3 *Let E, F, G and H be subsets of the real line. We say that E, F are translation congruent modulo one if there is a bijection $\phi : E \longrightarrow F$ such that $\phi(s) - s$ in an integer for every s in E; or equivalently if there is a partition $\{E_n : n \in \mathbb{Z}\}$ of E such that $\{E_n + k : k \in \mathbb{Z}\}$ is a partition of F. In a similar fashion, we say that G and H are dilation congruent modulo two if there is a bijection $\tau : G \longrightarrow H$ such that for each s in G there* *is an integer n depending on s such that $\tau(s) = 2^n s$; or equivalently, if there is a partition $\{G_n : n \in \mathbb{Z}\}$ such that $\{2^n G : n \in \mathbb{Z}\}$ is a partition of H.*

The following result is of central importance and is proved in [6], Section 4.5.

Proposition 4 *Let $\mathbf{W}$ be a set of real numbers. Then $\mathbf{W}$ is a wavelet set if and only if $\mathbf{W}$ is both translation congruent modulo one to a unit interval, and dilation congruent modulo two to a set of the type $[2x, x) \cup [y, 2y)$ for some real numbers x, y satisfying $x < 0 < y$.*

Let $a_1, a_2, \cdots, a_{2n-1}, a_{2n}$ be real numbers satisfying $a_1 < a_2 < a_3 < \cdots < a_{2n-1} < a_{2n}$. Next, we define

$$\mathbf{W} = \bigcup_{k=0}^{n-1} [a_{2k+1}, a_{2k+2}) .$$

If $\mathbf{W}$ is a wavelet set, it is easy to see that there must exist integers $j_0, j_1, \cdots, j_{n-1}$ such that $\bigcup_{k=0}^{n-1} ([a_{2k+1}, a_{2k+2}) + j_k) = [u, u+1)$ for some real number u. Also, it is easy to verify that there must exist some integers $\ell_0, \ell_1, \cdots, \ell_{n-1}$ such that

$$\bigcup_{k=0}^{n-1} \left(2^{\ell_k} [a_{2k+1}, a_{2k+2})\right) = [2x, x) \cup [y, 2y)$$

where x and y are real numbers satisfying $x < 0 < y$. The observations above imply that the collections

$$\mathcal{C} = \{[a_{2k+1}, a_{2k+2}) + j_k : 0 \leq k \leq n-1\},$$
$$\mathcal{J} = \left\{2^{\ell_k} [a_{2k+1}, a_{2k+2}) : 0 \leq k \leq n-1\right\}$$

partition the sets $[u, u+1)$ and $[2x, x) \cup [y, 2y)$ respectively.

Let us consider the collection $\mathcal{C}$ as defined above. We observe that, in order to understand how this collection forms a partition of the unit interval $[u, u+1)$, it is not enough to simply list the elements of $\mathcal{C}$. In fact, there must be an ordering of the elements of $\mathcal{C}$ which provides $\mathcal{C}$ with its tiling property. In order to describe in a very precise way, how the elements of $\mathcal{C}$ tile the given unit interval, we shall introduce a refined concept of partition which we coin **partition ordered by a permutation**.

We recall that the set of all permutations of $\{1, 2, \cdots, n\}$ when endowed with the composition operation forms a group called the **symmetric group**. The symmetric group is commonly denoted S_n. This concept will play a central role in our results.

Definition 5 *Let $\{A_1, A_2, \cdots, A_n\}$ be a partition of a given set $A \subseteq \mathbb{R}$. If f is a permutation map $f : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ for the set $\{1, 2, \cdots, n\}$ such that*

$$\begin{cases} \inf A_{f(1)} = \inf A \\ \sup A_{f(1)} = \inf A_{f(2)} \\ \sup A_{f(2)} = \inf A_{f(3)} \\ \vdots \\ \sup A_{f(n-1)} = \inf A_{f(n)} \\ \sup A_{f(n)} = \sup A \end{cases}$$

*then we say that the n-tuple $\left(A_{f(1)}, A_{f(2)}, \cdots, A_{f(n)}\right)$ is an f-**ordered partition** of A or a **partition of** A **ordered by the permutation** f.*

Example 6 *Let $A = [0, 1)$. Put*

$$A_1 = \left[\frac{1}{2}, 1\right), A_2 = \left[0, \frac{1}{4}\right), A_3 = \left[\frac{1}{4}, \frac{1}{2}\right).$$

*Observe that the collection $\left\{\left[\frac{1}{2}, 1\right), \left[0, \frac{1}{4}\right), \left[\frac{1}{4}, \frac{1}{2}\right)\right\}$ is a partition of the set A. On one hand, we notice that $(A_1, A_2, A_3) = \left(\left[\frac{1}{2}, 1\right), \left[0, \frac{1}{4}\right), \left[\frac{1}{4}, \frac{1}{2}\right)\right)$ is **not** an identity-ordered partition of A. On the other, letting f be the permutation map described by $f(1) = 2, f(2) = 3$ and $f(3) = 1$, it is clear that the triple*

$$\left(A_{f(1)}, A_{f(2)}, A_{f(3)}\right) = (A_2, A_3, A_1) = \left(\left[0, \frac{1}{4}\right), \left[\frac{1}{4}, \frac{1}{2}\right), \left[\frac{1}{2}, 1\right)\right)$$

*an f-**ordered partition** of A.*

Remark 7 *In general, given a permutation f such that $f(k) = a_k$, we shall conveniently use the notation $(a_1, a_2, \cdots, a_n)$ when we refer to f.*

2.2 Wavelet Sets of Four Intervals

Definition 8 *Let i, j be two natural numbers less than four satisfying the condition $i+j = 4$. We say that $\mathbf{W}$ has the (i, j) configuration if the set $\mathbf{W} \cap (0, \infty)$ consists of exactly i intervals.*

From the definition above, it is clear that a four interval wavelet set can only have three configurations. It either has a $(2,2)$ configuration or a $(3,1)$ configuration or a $(1,3)$ configuration. Let us now fix our notation. Put

$$X_1 = [a_1, a_2), X_2 = [a_3, a_4),$$
$$X_3 = [a_5, a_6), X_4 = [a_7, a_8)$$

and let S_4 be the set of all permutation maps of the finite set $\{1,2,3,4\}$.

- Assume that $a_1 < a_2 < a_3 < a_4 < 0 < a_5 < a_6 < a_7 < a_8$. If there exist a permutation $f \in S_4$, and a real number u such that the quadruple

$$\left(X_{f(1)} + k_1, X_{f(2)}, X_{f(3)} + k_2, X_{f(4)} + k_3\right)$$

  is an f-**ordered partition** of the unit interval $[u, u+1)$ and if there exist integers p_1, p_2 such that

$$2^{p_1} a_3 = a_2, 2^{p_1} a_4 = \frac{a_1}{2}$$
$$2^{p_2} a_6 = a_7 2^{p_2} a_5 = \frac{a_8}{2}$$

  then we say that the set $\mathbf{W} = [a_1, a_2) \cup [a_3, a_4) \cup [a_5, a_6) \cup [a_7, a_8)$ is a $(2,2)$**-wavelet set ordered by** f. Fixing an element f of S_4, such that the quadruple consisting of $\left[a_{2(f(1))-1} + k_1, a_{2f(1)} + k_1\right)$, $\left[a_{2(f(2))-1}, a_{2f(2)}\right)$, $\left[a_{2(f(3))-1} + k_2, a_{2f(3)} + k_2\right)$ and $\left[a_{2(f(4))-1} + k_3, a_{2f(4)} + k_3\right)$ is an f-**ordered partition** for the interval $[u, u+1)$, then it must be the case that

$$\bigcup_{k=1}^{4} \left[a_{2(f(k))-1}, a_{2f(k)}\right) = [u, u+1)$$

and

$$\begin{cases} a_{2(f(1))-1} + k_1 = u \\ a_{2f(1)} + k_1 = a_{2(f(2))-1} \\ a_{2(f(3))-1} + k_2 = a_{2f(2)} \\ a_{2(f(4))-1} + k_3 = a_{2f(3)} + k_2 \\ a_{2f(4)} + k_3 = u + 1 \end{cases}$$

- Given real numbers $a_1, a_2, \cdots, a_8$ satisfying $a_1 < a_2 < a_3 < a_4 < a_5 < a_6 < 0 < a_7 < a_8$, if there exist a permutation $f \in S_4$, integers k_1, k_2, k_3 and a real number u such that the quadruple $\left(X_{f(1)} + k_1, X_{f(2)}, X_{f(3)} + k_2, X_{f(4)} + k_3\right)$ is an f**-ordered partition** of the unit interval $[u, u+1)$ and if additionally there exist integers p_1, p_2 such that either

$$2a_7 = a_8, 2^{p_1}a_5 = a_4, 2^{p_2}a_2 = a_3, 2^{p_1+1}a_6 = 2^{p_2}a_1$$

or

$$2a_7 = a_8, 2^{p_1}a_5 = a_2, 2^{p_2}a_4 = a_1, 2^{p_1+1}a_6 = 2^{p_2}a_3$$

holds true, then we say that the set

$$\mathbf{W} = [a_1, a_2) \cup [a_3, a_4) \cup [a_5, a_6) \cup [a_7, a_8)$$

is a (3,1)-**wavelet set ordered by** f. Fixing an element f of S_4 such that the quadruple

$$([a_{2(f(1))-1} + k_1, a_{2f(1)} + k_1), [a_{2(f(2))-1}, a_{2f(2)}), [a_{2(f(3))-1} + k_2, a_{2f(3)} + k_2),$$
$$[a_{2(f(4))-1} + k_3, a_{2f(4)} + k_3))$$

is an f**-ordered partition** for the interval $[u, u+1)$, it must be the case that

$$\bigcup_{k=1}^{4} [a_{2(f(k))-1}, a_{2f(k)}) = [u, u+1)$$

and

$$\begin{cases} a_{2(f(1))-1} + k_1 = u \\ a_{2f(1)} + k_1 = a_{2(f(2))-1} \\ a_{2(f(3))-1} + k_2 = a_{2f(2)} \\ a_{2(f(4))-1} + k_3 = a_{2f(3)} + k_2 \\ a_{2f(4)} + k_3 = u + 1 \end{cases}.$$

- Given real number $a_1, a_2, \cdots a_8$ such that

$$a_1 < a_2 < 0 < a_3 < a_4 < a_5 < a_6 < a_7 < a_8$$

if there exist a permutation $f \in S_4$, integers k_1, k_2, k_3 and a real number u such that the quadruple $\left(X_{f(1)} + k_1, X_{f(2)}, X_{f(3)} + k_2, X_{f(4)} + k_3\right)$ is an f-**ordered partition** of the unit interval $[u, u+1)$ and if there exist integers p_1, p_2 such that either

$$2a_2 = a_1, 2^{p_1} a_4 = a_5, 2^{p_2} a_7 = a_6, 2^{p_1+1} a_3 = 2^{p_2} a_8$$

or

$$2a_2 = a_1, 2^{p_1} a_4 = a_7, 2^{p_2} a_5 = a_8, 2^{p_1+1} a_3 = 2^{p_2} a_6,$$

then we say that the set $\mathbf{W} = [a_1, a_2) \cup [a_3, a_4) \cup [a_5, a_6) \cup [a_7, a_8)$ is a (1,3)-**wavelet set ordered by** f. Fixing an element f of S_4 such that the quadruple

$$([a_{2(f(1))-1} + k_1, a_{2f(1)} + k_1), [a_{2(f(2))-1}, a_{2f(2)}), [a_{2(f(3))-1} + k_2, a_{2f(3)} + k_2),$$

$$[a_{2(f(4))-1} + k_3, a_{2f(4)} + k_3))$$

is an f-**ordered partition** for the interval $[u, u+1)$, it must be the case that

$$\bigcup_{k=1}^{4} [a_{2(f(k))-1}, a_{2f(k)}) = [u, u+1)$$

and

$$\begin{cases} a_{2(f(1))-1} + k_1 = u \\ a_{2f(1)} + k_1 = a_{2(f(2))-1} \\ a_{2(f(3))-1} + k_2 = a_{2f(2)} \\ a_{2(f(4))-1} + k_3 = a_{2f(3)} + k_2 \\ a_{2f(4)} + k_3 = u + 1 \end{cases}.$$

From the discussion above, the following results are immediate.

Proposition 9 *Let* $\mathbf{W}$ *be a subset of the real line such that*

$$\mathbf{W} = [a_1, a_2) \cup [a_3, a_4) \cup [a_5, a_6) \cup [a_7, a_8).$$

$\mathbf{W}$ *is a* $(2,2)$*-wavelet set if and only if there exist a permutation* f*, a real number* u*, and integers* k_1*,* k_2*,* k_3*,* p_1*, and* p_2 *such that:*

$$\begin{cases} a_{2(f(1))-1} + k_1 = u \\ a_{2f(1)} + k_1 = a_{2(f(2))-1} \\ a_{2(f(3))-1} + k_2 = a_{2f(2)} \\ a_{2(f(4))-1} + k_3 = a_{2f(3)} + k_2 \\ a_{2f(4)} + k_3 = u + 1 \\ 2^{p_1} a_3 = a_2 \\ 2^{p_1} a_4 = a_1/2 \\ 2^{p_2} a_6 = a_7 \\ 2^{p_2} a_5 = a_8/2 \end{cases}$$

Proposition 10 *Let* $\mathbf{W}$ *be a subset of the real line such that*

$$\mathbf{W} = [a_1, a_2) \cup [a_3, a_4) \cup [a_5, a_6) \cup [a_7, a_8).$$

$\mathbf{W}$ *is a* $(3,1)$*-wavelet set if and only if there exist a permutation* f *and integers* k_1*,* k_2*,* k_3*,*

p_1, p_2 *and a real number* u *such that:*

$$\begin{cases} a_{2(f(1))-1} + k_1 = u \\ a_{2f(1)} + k_1 = a_{2(f(2))-1} \\ a_{2(f(3))-1} + k_2 = a_{2f(2)} \\ a_{2(f(4))-1} + k_3 = a_{2f(3)} + k_2 \\ a_{2f(4)} + k_3 = u + 1 \\ 2^{p_1} a_5 = a_4 \\ 2^{p_2} a_2 = a_3 \\ 2^{p_1+1} a_6 = 2^{p_2} a_1 \\ 2a_7 = a_8 \end{cases}$$

or

$$\begin{cases} a_{2(f(1))-1} + k_1 = u \\ a_{2f(1)} + k_1 = a_{2(f(2))-1} \\ a_{2(f(3))-1} + k_2 = a_{2f(2)} \\ a_{2(f(4))-1} + k_3 = a_{2f(3)} + k_2 \\ a_{2f(4)} + k_3 = u + 1 \\ 2^{p_1} a_5 = a_2 \\ 2^{p_2} a_4 = a_1 \\ 2^{p_1+1} a_6 = 2^{p_2} a_3 \\ 2a_7 = a_8 \end{cases}$$

Proposition 11 *Let* $\mathbf{W}$ *be a subset of the real line such that*

$$\mathbf{W} = [a_1, a_2) \cup [a_3, a_4) \cup [a_5, a_6) \cup [a_7, a_8) \, .$$

$\mathbf{W}$ *is a* $(1,3)$*-wavelet set if and only if there exist a permutation* f*, integers* k_1*,* k_2*,* k_3*,* p_1, p_2

and a real number u such that

$$\begin{cases} a_{2(f(1))-1} + k_1 = u \\ a_{2f(1)} + k_1 = a_{2(f(2))-1} \\ a_{2(f(3))-1} + k_2 = a_{2f(2)} \\ a_{2(f(4))-1} + k_3 = a_{2f(3)} + k_2 \\ a_{2f(4)} + k_3 = u + 1 \\ a_1 = 2a_2 \\ 2^{p_1} a_4 = a_5 \\ 2^{p_2} a_7 = a_6 \\ 2^{p_1+1} a_3 = 2^{p_2} a_8 \end{cases}$$

or

$$\begin{cases} a_{2(f(1))-1} + k_1 = u \\ a_{2f(1)} + k_1 = a_{2(f(2))-1} \\ a_{2(f(3))-1} + k_2 = a_{2f(2)} \\ a_{2(f(4))-1} + k_3 = a_{2f(3)} + k_2 \\ a_{2f(4)} + k_3 = u + 1 \\ a_1 = 2a_2 \\ 2^{p_1} a_4 = a_7 \\ 2^{p_2} a_5 = a_8 \\ 2^{p_1+1} a_3 = 2^{p_2} a_6 \end{cases}$$

2.3 Examples

Using the characterizations of wavelet sets provided above, we shall now present a few specific examples of four interval wavelet sets.

Example 12 *Let*

$$\mathbf{W} = \left[-\frac{32}{15}, -2\right) \cup \left[-\frac{1}{4}, -\frac{2}{15}\right) \cup \left[\frac{3}{7}, \frac{3}{4}\right) \cup \left[3, \frac{24}{7}\right)$$

Next, observe that

$$\left[3, \frac{24}{7}\right) - 3 = \left[0, \frac{3}{7}\right), \left[\frac{3}{7}, \frac{3}{4}\right) + 0 = \left[\frac{3}{7}, \frac{3}{4}\right)$$
$$\left[-\frac{1}{4}, -\frac{2}{15}\right) + 1 = \left[\frac{3}{4}, \frac{13}{15}\right), \left[-\frac{32}{15}, -2\right) + 3 = \left[\frac{13}{15}, 1\right)$$

Thus,

$$\left\{\left[3, \frac{24}{7}\right) - 3, \left[\frac{3}{7}, \frac{3}{4}\right) + 0, \left[-\frac{1}{4}, -\frac{2}{15}\right) + 1, \left[-\frac{32}{15}, -2\right) + 3\right\}$$

forms a tiling of the unit interval $[0, 1)$. *In other words, the integral translates of* $\mathbf{W}$ *is a partition of the real line. Next,*

$$2^{-3}\left[-\frac{32}{15}, -2\right) = \left[-\frac{4}{15}, -\frac{1}{4}\right) \text{ and } 2^2\left[\frac{3}{7}, \frac{3}{4}\right) = \left[\frac{12}{7}, 3\right).$$

Thus,

$$\left(2^{-3}\left[-\frac{32}{15}, -2\right) \cup \left[-\frac{1}{4}, -\frac{2}{15}\right)\right) \cup \left(2^2\left[\frac{3}{7}, \frac{3}{4}\right) \cup \left[3, \frac{24}{7}\right)\right)$$

is equal to

$$\left[-\frac{4}{15}, -\frac{1}{4}\right) \cup \left[-\frac{1}{4}, -\frac{2}{15}\right) \cup \left[\frac{12}{7}, 3\right) \cup \left[3, \frac{24}{7}\right) = \left[-\frac{4}{15}, -\frac{2}{15}\right) \cup \left[\frac{12}{7}, \frac{24}{7}\right)$$

and the dyadic dilations of $\mathbf{W}$ *gives a partition of the real line. Thus*

$$\mathbf{W} = \left[-\frac{32}{15}, -2\right) \cup \left[-\frac{1}{4}, -\frac{2}{15}\right) \cup \left[\frac{3}{7}, \frac{3}{4}\right) \cup \left[3, \frac{24}{7}\right)$$

is a four-interval wavelet set with the $(2, 2)$*-configuration.*

Next, let $\chi_{\mathbf{W}}$ *be the indicator function of the set*

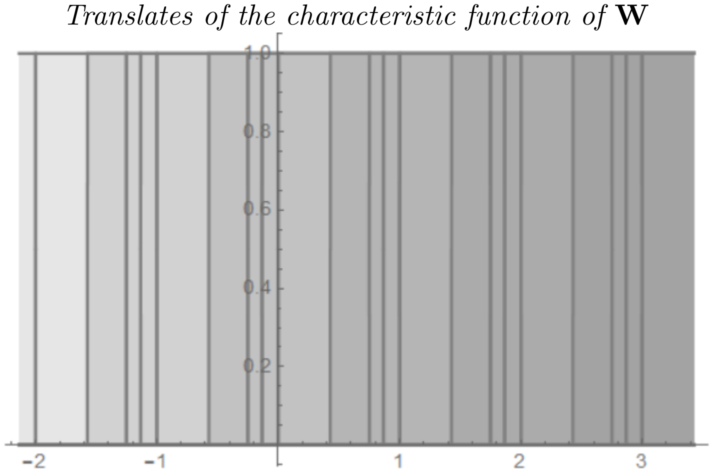
$$\mathbf{W} = \left[-\frac{32}{15}, -2\right) \cup \left[-\frac{1}{4}, -\frac{2}{15}\right) \cup \left[\frac{3}{7}, \frac{3}{4}\right) \cup \left[3, \frac{24}{7}\right).$$

That is $\chi_{\mathbf{W}}(x)$ is equal to one if x belongs to $\mathbf{W}$ or is equal to zero otherwise. Then for any real number x the following holds true

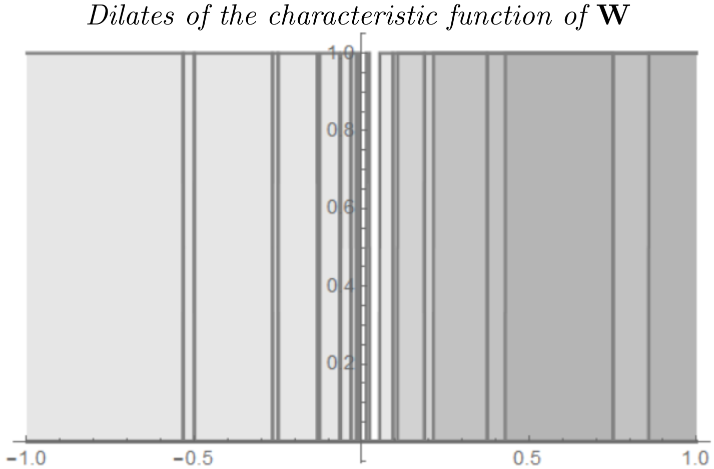
$$\sum_{k=-\infty}^{\infty} \chi_{\mathbf{W}}(x+k) = \sum_{j=-\infty}^{\infty} \chi_{\mathbf{W}}\left(2^j x\right) = 1.$$

We now present below some illustrations of translations and dilations of the characteristic function of the wavelet set $\mathbf{W}$.

Translates of the characteristic function of $\mathbf{W}$



Dilates of the characteristic function of $\mathbf{W}$



Example 13 *Let*

$$\mathbf{W} = \left[-\frac{640}{127}, -\frac{32}{7}\right) \cup \left[-\frac{1}{14}, -\frac{5}{127}\right) \cup \left[\frac{7}{15}, \frac{13}{14}\right) \cup \left[\frac{52}{7}, \frac{112}{15}\right)$$

It is easy to verify that

$$\left\{\left[\frac{52}{7},\frac{112}{15}\right)-7,\left[\frac{7}{15},\frac{13}{14}\right)+0,\left[-\frac{1}{14},-\frac{5}{127}\right)+1,\left[-\frac{640}{127},-\frac{32}{7}\right)+6\right\}$$

forms a partition of the unit interval $\left[\frac{3}{7},\frac{10}{7}\right)$. *Moreover,*

$$\left\{64^{-1}\left[-\frac{640}{127},-\frac{32}{7}\right),\left[-\frac{1}{14},-\frac{5}{127}\right),8\left[\frac{7}{15},\frac{13}{14}\right),\left[\frac{52}{7},\frac{112}{15}\right)\right\}$$

is a partition of

$$\left[-\frac{10}{127},-\frac{5}{127}\right)\cup\left[\frac{116}{15},\frac{112}{15}\right).$$

It follows that $\mathbf{W}$ *as described above is a four interval wavelet set having the* $(2,2)$*-configuration.*

Example 14 *Let Let*

$$\mathbf{W}=\left[-\frac{6}{31},-\frac{3}{31}\right)\cup\left[\frac{3}{7},\frac{25}{31}\right)\cup\left[\frac{10}{3},\frac{24}{7}\right)\cup\left[\frac{400}{31},\frac{40}{3}\right) \tag{1}$$

Next, observe that

$$\left\{\left[\frac{400}{31},\frac{40}{3}\right)-10,\left[\frac{10}{3},\frac{24}{7}\right)+0,\left[\frac{3}{7},\frac{25}{31}\right)+3,\left[-\frac{6}{31},-\frac{3}{31}\right)+4\right\}$$

is equal to

$$\left\{\left[\frac{90}{31},\frac{10}{3}\right),\left[\frac{10}{3},\frac{24}{7}\right),\left[\frac{24}{7},\frac{118}{31}\right),\left[\frac{118}{31},\frac{121}{31}\right)\right\}$$

and the latter is a partition of the unit interval $\left[\frac{90}{31},\frac{121}{31}\right)$. *Next, observe that*

$$\left[\frac{10}{3},\frac{24}{7}\right)\cup 2^{3}\left[\frac{3}{7},\frac{25}{31}\right)\cup 2^{-1}\left[\frac{400}{31},\frac{40}{3}\right)=\left[\frac{10}{3},\frac{24}{7}\right)\cup\left[\frac{24}{7},\frac{200}{31}\right)\cup\left[\frac{200}{31},\frac{20}{3}\right).$$

Consequently,

$$\left\{\left[-\frac{6}{31},-\frac{3}{31}\right),\left[\frac{10}{3},\frac{24}{7}\right),2^{3}\left[\frac{3}{7},\frac{25}{31}\right),2^{-1}\left[\frac{400}{31},\frac{40}{3}\right)\right\}$$

is a partition of the set $\left[-\frac{6}{31},-\frac{3}{31}\right)\cup\left[\frac{10}{3},\frac{20}{3}\right)$ *which a dilation-tiling set for the real line.*

Therefore,

$$\mathbf{W} = \left[-\frac{6}{31}, -\frac{3}{31}\right) \cup \left[\frac{3}{7}, \frac{25}{31}\right) \cup \left[\frac{10}{3}, \frac{24}{7}\right) \cup \left[\frac{400}{31}, \frac{40}{3}\right)$$

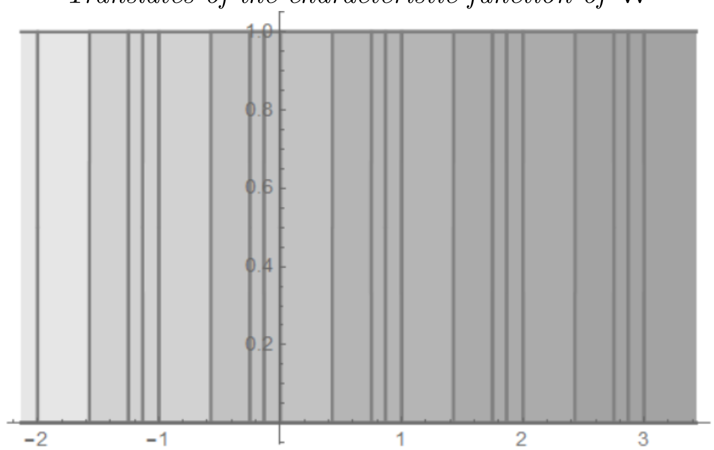
is a wavelet set with the $(1,3)$*-configuration.*

Example 15 *The interested reader is invited to verify that the following set*

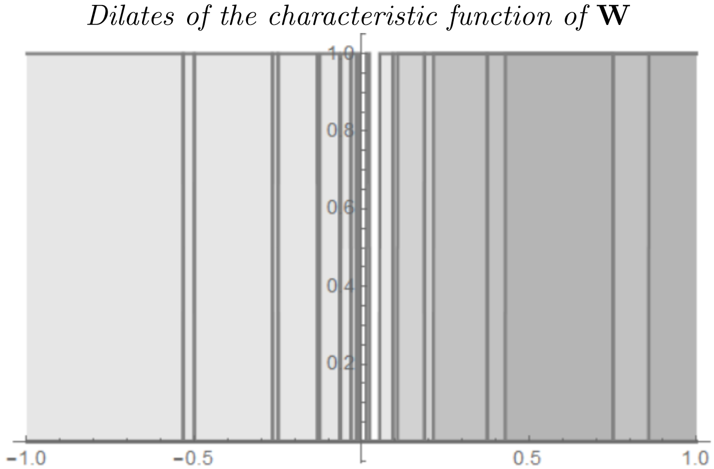
$$\mathbf{W} = \left[-\frac{4}{7}, -\frac{2}{7}\right) \cup \left[\frac{1}{3}, \frac{3}{7}\right) \cup \left[1, \frac{4}{3}\right) \cup \left[\frac{12}{7}, 2\right)$$

is indeed a wavelet set having the $(1,3)$*-configuration.*

Translates of the characteristic function of **W**



Dilates of the characteristic function of **W**



3 The Symmetric Case

Let W be a wavelet set and let $\text{int}(W)$ be its interior. We say that W is a **symmetric wavelet set** if and only if $-\text{int}(W) = \text{int}(W)$. In this section, we shall obtain a complete characterization of symmetric wavelet sets of four intervals.

3.1 Rotations of Permutations

In order to present our results in a clear and accessible manner, we will need to introduce a concept which we call the rotation of a permutation.

Definition 16 *Let $f = \left(f_{(1)}, f_{(2)}, f_{(3)}, f_{(4)}\right)$ be an element of S_4. We say that a permutation $\ell = \left(\ell_{(1)}, \ell_{(2)}, \ell_{(3)}, \ell_{(4)}\right) \in S_4$ is a k-rotation of f if and only if there exists $k \in \{0, 1, 2, 3\}$ such that*

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^k \begin{bmatrix} f_{(1)} \\ f_{(2)} \\ f_{(3)} \\ f_{(4)} \end{bmatrix} = \begin{bmatrix} \ell_{(1)} \\ \ell_{(2)} \\ \ell_{(3)} \\ \ell_{(4)} \end{bmatrix}.$$

Remark 17

1. *An arbitrary permutation is a zero-rotation of itself.*
2. *If ℓ is a k-rotation of f then f is a $(4-k) \bmod 4$-rotation of ℓ. If f is a k-rotation of ℓ, we shall simply say that f and ℓ are rotations of each other.*
3. *Generally, two arbitrary permutations are not rotations of each other.*
4. *The set of all permutations in S_4 contains 24 elements, which can be partitioned into six collections of permutations such that each collection contains exactly four elements which are all rotations of each other.*

Example 18 *Since*

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}^3 \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$

then $(3,2,1,4) \in S_4$ *is a* 3*-rotation of* $(4,3,2,1)$. *Moreover, notice that*

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}.$$

Thus, $(4,3,2,1) \in S_4$ *is a* 1*-rotation of* $(3,2,1,4)$.

Let A be the following circulant matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Lemma 19 *Let* X_1, X_2, X_3, X_4 *be a collection of four sets. There exist integers* k_1, k_2, k_3, k_4 *and a permutation* $f \in S_4$ *such that* $\bigcup_{i=1}^{4} \left(X_{f(i)} + k_i\right)$ *forms a partition of a unit interval of the type* $[u, u+1)$ *where* u *is a real number if and only if there exist integers* j_1, j_2, j_3, j_4 *such that for any* $\tau \in \{0,1,2,3\}$, $\bigcup_{i=1}^{4} \left(X_{A^{\tau} f(i)} + j_i\right)$ *is a partition of a unit interval.*

Proof. First we notice that the result above clearly holds if $\tau = 0$ since A^0 is just the identity matrix. Next let X_1, X_2, X_3, X_4 be a collection of four sets. Moreover, let us assume that there exist integers k_1, k_2, k_3, k_4 and a permutation $f \in S_4$ such that

$$\bigcup_{i=1}^{4} X_{f(i)} + k_i$$

forms a partition of a unit interval of the type $[u, u+1)$ where u is a real number. More precisely, there exist real numbers $\varepsilon_0 = 0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \varepsilon_4 = 1$ such that

$$\begin{aligned}
X_{f(1)} + k_1 &= [u + \varepsilon_0, u + \varepsilon_1) = [u, u + \varepsilon_1) \\
X_{f(2)} + k_2 &= [u + \varepsilon_1, u + \varepsilon_2) \\
X_{f(3)} + k_3 &= [u + \varepsilon_2, u + \varepsilon_3) \\
X_{f(4)} + k_4 &= [u + \varepsilon_3, u + \varepsilon_4) = [u + \varepsilon_3, u + 1)\,.
\end{aligned}$$

It is easy to check that

$$\bigcup_{i=1}^{4} X_{f(i)} + k_i = \bigcup_{i=1}^{4} [u + \varepsilon_{i-1}, u + \varepsilon_i) = [u, u+1)\,.$$

Now, we would like to separate the unit interval into two parts at either

$$u + \varepsilon_1 \text{ or } u + \varepsilon_2 \text{ or } u + \varepsilon_3.$$

Next, we shift the rightmost part of the interval backward by one unit. More precisely, we fix a natural number $m \in \{2, 3, 4\}$ and we write

$$[u, u+1) = \left(\bigcup_{i=1}^{m-1} [u + \varepsilon_{i-1}, u + \varepsilon_i) \right) \cup \left(\bigcup_{i=m}^{4} [u + \varepsilon_{i-1}, u + \varepsilon_i) \right).$$

Next, we observe that

$$\bigcup_{i=m}^{4} [u + \varepsilon_{i-1} - 1, u + \varepsilon_i - 1) = [u + \varepsilon_{m-1} - 1, u + \varepsilon_m - 1) \cup \cdots \cup [u + \varepsilon_3 - 1, u)\,.$$

It follows that the new set obtained by separating the unit interval $[u, u+1)$ into two parts and moving the rightmost part backward by one unit is a connected interval which is given by

$$\begin{aligned}
&\bigcup_{i=m}^{4} [u+\varepsilon_{i-1}-1, u+\varepsilon_i-1) \cup \bigcup_{i=1}^{m-1} [u+\varepsilon_{i-1}, u+\varepsilon_i) \\
&= ([u+\varepsilon_{m-1}-1, u+\varepsilon_m-1) \cup \cdots \cup [u+\varepsilon_3-1, u)) \cup \bigcup_{i=1}^{m-1} [u+\varepsilon_{i-1}, u+\varepsilon_i) \\
&= [u+\varepsilon_{m-1}-1, u+\varepsilon_{m-1}).
\end{aligned}$$

Thus, the following set

$$\begin{aligned}
&\left(\bigcup_{i=m}^{4} [u+\varepsilon_{i-1}-1, u+\varepsilon_i-1)\right) \cup \left(\bigcup_{i=1}^{m-1} [u+\varepsilon_{i-1}, u+\varepsilon_i)\right) \\
&= \left(\bigcup_{i=m}^{4} X_{f(i)} + (k_i-1)\right) \cup \left(\bigcup_{i=1}^{m-1} X_{f(i)} + k_i\right) \\
&= [u+\varepsilon_{m-1}-1, u+\varepsilon_{m-1})
\end{aligned}$$

is a unit interval which tiles the real line by the integers. Moreover, we note that if $m=2$ then

$$\left(X_{f(2)} + (k_2-1), X_{f(3)} + (k_3-1), X_{f(4)} + (k_4-1), X_{f(1)} + k_1\right)$$

is a partition of $[u+\varepsilon_1-1, u+\varepsilon_1)$ which is ordered by the permutation $(f_2, f_3, f_4, f_1) \in S_4$ which is obtained by rotating the permutation f. Additionally, there exist integers j_1, j_2, j_3, j_4 such that $[u+\varepsilon_1-1, u+\varepsilon_1) = \bigcup_{i=1}^{4} \left(X_{A^3 f(i)} + j_i\right)$. If $m=3$ then

$$\left(X_{f(3)} + (k_3-1), X_{f(4)} + (k_4-1), X_{f(1)} + k_1, X_{f(2)} + k_2\right)$$

is a partition of $[u+\varepsilon_2-1, u+\varepsilon_2)$ which is ordered by the permutation $(f_3, f_4, f_1, f_2) \in S_4$ which is obtained by rotating the permutation f. In this case, it is clear that there exist integers j_1, j_2, j_3, j_4 such that $[u+\varepsilon_2-1, u+\varepsilon_2) = \bigcup_{i=1}^{4} \left(X_{A^2 f(i)} + j_i\right)$. If $m=4$ then

$$\left(X_{f(4)} + (k_4-1), X_{f(1)} + k_1, X_{f(2)} + k_2, X_{f(3)} + k_3\right)$$

is a partition of $[u+\varepsilon_3-1, u+\varepsilon_3)$ which is ordered by the permutation $(f_4, f_1, f_2, f_3) \in S_4$ which is obtained by rotating the permutation f. Therefore, there exist integers j_1, j_2, j_3, j_4

such that $[u+\varepsilon_3-1, u+\varepsilon_3) = \bigcup_{i=1}^{4} \left(X_{Af(i)} + j_i\right)$. From the observation above, it immediately follows that if there exist integers k_1, k_2, k_3, k_4 and a permutation $f \in S_4$ such that $\bigcup_{i=1}^{4} \left(X_{f(i)} + k_i\right)$ is equal to a unit interval of the type $[u, u+1)$ where u is a real number then there exist integers j_1, j_2, j_3, j_4 such that for any $\tau \in \{0,1,2,3\}$, the set $\bigcup_{i=1}^{4} \left(X_{A^{\tau}f(i)} + j_i\right)$ is a unit interval. The converse can be proved by a similar approach, and we shall omit it. ■

From the lemma above, the following is immediate.

Lemma 20 *Let us suppose that f and ℓ are two permutations such that ℓ is a k-rotation of f. Let W be a four interval wavelet set. Then W tiles a unit interval by the permutation f if and only if W tiles a unit interval by the permutation ℓ.*

3.2 A Characterization of Four Interval Symmetric Wavelet Sets

We are now ready to present the main result of this section which is summarized as follows.

Theorem 21 *A four interval wavelet set is symmetric if and only if there exists a collection which consists of integral translates of the connected components of the wavelet set which is a $(4,3,2,1)$-ordered partition of a unit interval. Furthermore, the following collection exhausts all possible symmetric wavelet sets of four intervals:*

$$\left\{\left[-\frac{2^{2p}}{2^{p+1}-1}, -2^{p-1}\right) \cup \left[-\frac{1}{2}, -\frac{2^{p-1}}{2^{p+1}-1}\right) \cup \left[\frac{2^{p-1}}{2^{p+1}-1}, \frac{1}{2}\right) \cup \left[2^{p-1}, \frac{2^{2p}}{2^{p+1}-1}\right) : p \in \mathbb{N}\right\}.$$

In order to prove Theorem 21 we shall need several intermediate results.

Lemma 22 *It is not possible for a 4-tuple obtained by translating the connected components of a four interval symmetric wavelet set by integers to form a $(1,2,3,4)$-ordered partition of a unit interval.*

Proof. By contradiction, assume that there is a wavelet set

$$\mathbf{W} = [-a, -b) \cup [-c, -d) \cup [d, c) \cup [b, a)$$

where $0 < d < c < b < a$ such that the 4-tuple obtained by translating the connected components of $\mathbf{W}$ by integers form a $(1, 2, 3, 4)$-ordered partition of a unit interval. Using our characterization of $(2, 2)$-wavelet sets, there must exist an integer p and a real number u such that

$$a = -2^p(1 + 2u), b = -\frac{2^p\left(2^p - u + 2^{1+p}u\right)}{-1 + 2^p},$$

$$c = -\frac{2^p - u + 2^{1+p}u}{-1 + 2^p}, d = \frac{1}{2}(-1 - 2u).$$

Additionally, there exist integers $k_1 > 0, k_2 < 0, k_3 < 0$ such that

$$k_1 = -2^p + u - 2^{1+p}u, k_2 = 1 + 2u, k_3 = 1 + 2^p + u + 2^{1+p}u.$$

Since the length of any wavelet set must be equal to one, and since it is assumed that $\mathbf{W}$ is symmetric then it must be the case that $0 < a - b < \frac{1}{2}$. Using the values of a and b obtained above, it immediately follows that $0 < \dfrac{2^p(1+u)}{2^p - 1} < \dfrac{1}{2}$. Solving the above inequality for u yields $-1 < u < \dfrac{-2^p - 1}{2^{p+1}} < -\dfrac{1}{2}$. Since $k_2 = 1 + 2u$, u must be equal to $\dfrac{k_2 - 1}{2}$. Therefore u must belong to $\dfrac{1}{2}\mathbb{Z} \cap \left(-1, -\dfrac{1}{2}\right)$ which is clearly empty.

■

Lemma 23 *It is not possible for a 4-tuple obtained by translating the connected components of a four interval symmetric wavelet set by integers to form a $(1, 2, 4, 3)$-ordered partition of a unit interval.*

Proof. By contradiction, let us suppose that there is such a wavelet set. Appealing to the characterization of $(2, 2)$-wavelet sets provided above, it must be the case that there exist variables $k_1, k_2, k_3 \in \mathbb{Z}, p \in \mathbb{N}$ and $u \in \mathbb{R}$ satisfying the following conditions:

$$a = -2^p(1 + 2u), b = -\frac{2^p\left(2^p - u + 2^{1+p}u\right)}{-1 + 2^p},$$

$$c = -\frac{2^p - u + 2^{1+p}u}{-1 + 2^p}, d = \frac{1}{2}(-1 - 2u), k_1 = -2^p + u - 2^{1+p}u$$

and

$$k_2 = \frac{-1 + 2^p + 2^{1+2p} - 2u + 2^{2+2p}u}{2\left(-1 + 2^p\right)}, k_3 = \frac{-1 + 2^{1+p} - 2u + 3 \times 2^p u}{-1 + 2^p}.$$

According to the stated assumptions, we have $k_1 > 0, k_2 < 0$, and $k_3 < 0$. Let us consider the quantity

$$k_3 = \frac{-1 + 2^{1+p} - 2u + 3 \times 2^p u}{-1 + 2^p}.$$

First, we would like to show that $|k_3| < 1$. Using similar computations to the ones given in the proof of Lemma 22 it is not hard to verify that $u \in \left(-1, -\frac{1}{2}\right)$. Next, we verify that for $-1 < u < \frac{-1}{2}$ and for any natural number p the following holds true:

$$|-1 + 2^{1+p} - 2u + 3 \times 2^p u| < 2^p - 1.$$

Indeed, it suffices to check that

$$-2^p + 1 < -1 + 2^{1+p} - 2u + 3 \times 2^p u < 2^{p-1} < 2^p - 1.$$

Now, since k_3 is an integer, it must be equal to zero and this contradicts the fact that $k_3 < 0$.

■

Lemma 24 *It is not possible for a 4-tuple obtained by translating the connected components of a four interval symmetric wavelet set by integers to form a* $(1, 3, 2, 4)$*-ordered partition of a unit interval.*

Proof. Using our characterization of $(2, 2)$-wavelet sets, we obtain that

$$a = \frac{(2^p(2^p - 2u + 2^{1+p}u))}{(-1 + 2^{1+p})}, b = 2^{-1+p}(1 + 2u), c = \frac{1 + 2u}{2}, d = \frac{(2^p - 2u + 2^{1+p}u)}{(2(-1 + 2^{1+p}))}$$

and

$$k_1 = u + \frac{2^p(2^p - 2u + 2^{1+p}u)}{-1 + 2^{1+p}}, k_2 = 1 + 2u, k_3 = 1 + u - \frac{(2^p(2^p - 2u + 2^{1+p}u))}{(-1 + 2^{1+p})},$$

where u is a real number, p is a natural number and k_1, k_2, k_3 are integers. With some formal calculations, using the fact that

$$2^{-1+p}(1 + 2u) < \frac{(2^p(2^p - 2u + 2^{1+p}u))}{(-1 + 2^{1+p})}$$

together with the inequality $0 < \frac{1+2u}{2}$ it is easy to see that $-\frac{1}{2} < u < \frac{1}{2}$. Solving the equation $k_2 = 1 + 2u$ we obtain $u = \frac{k_2 - 1}{2}$. Thus, $u \in \frac{1}{2}\mathbb{Z} \cap \left(-\frac{1}{2}, \frac{1}{2}\right)$ and it follows that u must be equal to zero. Substituting $u = 0$ into the following equations

$$a = \frac{(2^p(2^p - 2u + 2^{1+p}u))}{(-1 + 2^{1+p})}, b = 2^{-1+p}(1 + 2u), k_1 = u + \frac{2^p(2^p - 2u + 2^{1+p}u)}{-1 + 2^{1+p}},$$

we obtain that

$$a = \frac{2^{2p}}{2^{p+1} - 1}, b = \frac{2^p}{2}, \text{ and } k_1 = a.$$

This implies that a and b are both integers. As a result, $a - b$ must be a nonzero integer. That is $a - b \geq 1$ which is absurd since the length of a wavelet set must be equal to one. ■

Lemma 25 *It is not possible for a 4-tuple obtained by translating the connected components of a four interval symmetric wavelet set by integers to form a* $(1, 3, 4, 2)$*-ordered partition of a unit interval.*

Proof. Let us assume by contradiction that we can construct such a wavelet set. Using our characterization of $(2, 2)$-wavelet sets, we obtain that

$$a = \frac{2^p(2^p - 2u + 2^{1+p}u)}{-1 + 2^{1+p}}, b = 2^{-1+p}(1 + 2u), c = \frac{1}{2}(1 + 2u),$$

$$d = \frac{2^p - 2u + 2^{1+p}u}{2(-1 + 2^{1+p})}, k_1 = \frac{2^{2p} - u + 2^{1+2p}u}{-1 + 2^{1+p}}, k_2 = -(1/2)(-1 + 2^p)(1 + 2u),$$

and

$$k_3 = \frac{-2 + 5 \times 2^p - 4u + 3 \times 2^{1+p}u}{2(-1 + 2^{1+p})}$$

where u is a real number, p is a natural number, and k_1, k_2, k_3 are integers. Again, we appeal to the fact that $0 < d < c < b < a$ and we easily see that u must be an element of the open interval $(-1/2, 1/2)$. Next, it is not hard to check that

$$0 \leq k_3 = \frac{-2 + 5 \times 2^p - 4u + 3 \times 2^{1+p}u}{2(-1 + 2^{1+p})} < 2.$$

In other words, $k_3 \in \left(\frac{1}{2}, 2\right) \cap \mathbb{Z}$ and this implies that $k_3 = 1$. Now, $k_3 = 1$ if and only if $-2 + 5 \times 2^p - 4u + 3 \times 2^{1+p}u = 2(-1 + 2^{1+p})$. Solving this equation for u, we obtain

$u=-\dfrac{2^{-1+p}}{-2+3\times 2^{p}}$. It follows that

$$k_2=-(1/2)(-1+2^p)(1+2u)=-\frac{(-1+2^p)^2}{-2+3\times 2^p}.$$

Note that $(-1+2^p)^2$ is odd and $-2+3\times 2^p$ is even. Therefore, k_2 cannot be an integer, and this is clearly a contradiction of our assumption. This completes the proof. ■

Lemma 26 *It is not possible for a* 4*-tuple obtained by translating the connected components of a four interval symmetric wavelet set by integers to form a* $(1,4,2,3)$*-ordered partition of a unit interval.*

Proof. Let us assume by contradiction that we can construct such a wavelet set. Using our characterization of $(2,2)$-wavelet sets, using our characterization of four interval wavelet sets, there exists a positive integer p such that

$$\begin{cases} a=1+u-\frac{u}{2^p} \\ b=\frac{1}{2}+u-\frac{u}{2^{1+p}} \\ c=\frac{1}{2^{1+p}}-\frac{u}{2^{1+2p}}+\frac{u}{2^p} \\ d=\frac{1}{2^{1+p}}-\frac{u}{2^{1+2p}}+\frac{u}{2^{1+p}} \end{cases}$$

for some real number u. Moreover, it must be the case that

$$\begin{aligned} k_1&=1+2u-\frac{u}{2^p}, \\ k_2&=1+\frac{1}{2^{1+p}}+u-\frac{u}{2^{1+2p}}, \\ k_3&=1-\frac{1}{2^{1+p}}+u+\frac{u}{2^{1+2p}}-\frac{u}{2^p}. \end{aligned}$$

Solving the first equation above for u we obtain

$$u=-\frac{k_1-1}{\frac{1}{2^p}-2}=\frac{1-k_1}{\frac{1-2^{1+p}}{2^p}}=\frac{2^p\,(1-k_1)}{1-2^{1+p}}.$$

Now,

$$k_2 = \frac{2 \times 2^{2p} k_1 - k_1 + 2 \times 2^{2p}}{2 \times 2^p \left(2^{p+1} - 1\right)}$$
$$k_3 = \frac{\left(k_1 + 2 \times 2^{2p} k_1 + 2 \times 2^{2p} - 2 \times 2^p k_1 - 2 \times 2^p\right)}{2 \times 2^p \left(2^{p+1} - 1\right)}.$$

Additionally,

$$a = \frac{2^p k_1 - k_1 + 2^p}{2^{p+1} - 1} \text{ and } b = \frac{k_1}{2}.$$

Since

$$k_3 - k_2 = \frac{2^p + \left(2^p - 1\right) k_1}{2^p \left(1 - 2^{1+p}\right)} = j \in \mathbb{Z} \tag{2}$$

then

$$k_1 = -\frac{-2^p j + 2 \times 2^{2p} j + 2^p}{2^p - 1} = \frac{2f}{2h+1}$$

for some integers f and h. Thus, $2f = k_1 \left(2h+1\right)$ and it must be the case that k_1 is an even integer. Appealing to Equality (2), it immediately follows that $2^p \left(k_3 - k_2\right) = \frac{2^p + (2^p - 1)k_1}{(1 - 2^{1+p})} \in \mathbb{Z}$. This implies that the numbers

$$a = \frac{2^p k_1 - k_1 + 2^p}{2^{p+1} - 1} \text{ and } b = \frac{1}{2} k_1$$

are integers and this is not possible since the quantity $b - a$ must be less than $\frac{1}{2}$. We have reached a contradiction and this completes the proof. ■

We are now ready to present the proof of Theorem 21 as stated above.

3.3 Proof of Theorem 21

First, the following lemma will be useful.

Lemma 27 *Let $a_1 = -k_1 + a_4, a_2 = -k_1 + u$, and $a_1 > a_2$ for $a_1, a_2, a_4, u \in \mathbb{R}$ and $k_1 \in \mathbb{Z}$. If $a_4 = \dfrac{2^{p-1} + u(2^p - 1)}{2^{p+1} - 1}$, where p is a positive integer, then $u < \dfrac{1}{2}$.*

Proof. Let a_1, a_2, a_4, k_1, u and p be as given above. By assumption, $a_4 = \dfrac{2^{p-1} + u(2^p - 1)}{2^{p+1} - 1}$. Since $a_1 = -k_1 + a_4$ which is strictly larger than $-k_1 + u = a_2$ we obtain that a_4 is greater than u. It follows that

$$a_4 = \frac{2^{p-1} + u(2^p - 1)}{2^{p+1} - 1} > u.$$

Next, multiplying each side of the previous inequality by $2^{p+1} - 1$ which is positive, we obtain $2^{p-1} + u(2^p - 1) > u(2^{p+1} - 1)$. With straight forward computations, the previous statement implies that $\frac{1}{2} > u$.

■

3.3.1 Proof of the Theorem

Let $\mathbf{W}$ be a symmetric wavelet set of four intervals. Furthermore, let us suppose that there is a collection consisting of integral translations of its connected components which form a $(4, 3, 2, 1)$-ordered partition of some unit interval. In other words,

$$\mathbf{W} = [-a_1, -a_2) \cup [-a_3, -a_4) \cup [a_4, a_3) \cup [a_2, a_1)$$

where $0 < a_4 < a_3 < a_2 < a_1$. By assumption, there exists a real number u such that

$$([a_2, a_1) + k_1, [a_4, a_3), [-a_3, -a_4) + k_2, [-a_1, -a_2) + k_3)$$

defines a $(4, 3, 2, 1)$-ordered partition for $[u, u + 1)$. Solving the system of equations:

$$\begin{cases} a_2 + k_1 = u \\ a_1 + k_1 = a_4 \\ -a_3 + k_2 = a_3 \\ -a_1 + k_3 = -a_4 + k_2 \\ -a_2 + k_3 = u + 1 \\ 2^p a_4 = \dfrac{a_1}{2} \\ 2^p a_3 = a_2 \end{cases}$$

for the unknowns $a_1, a_2, a_3, a_4, k_1, k_2, k_3$, we obtain $a_1 = \dfrac{2^{2p}(2^p - 2u + 2^{p+1}u)}{2^{p+1} - 1}$, $a_2 = 2^{p-1}(1 + 2u)$, $a_3 = \frac{1}{2}(1 + 2u)$, and $a_4 = \dfrac{2^{p-1} - u + 2^p u}{2^{p+1} - 1}$, $k_1 = \frac{1}{2}(-2^p + 2u - 2^{p+1}u$, $k_2 = 1 + 2u$, $k_3 = 1 + 2^{p-1} + u + 2^p u$. It is not hard to verify that the following facts hold true:

- $k_2 = 2a_3$
- a_2 must be an integer because $2^p a_3 = a_2$
- u is also an integer because $a_2 + k_1 = u$
- Since $a_1 > a_2$ and appealing to Lemma 27, we have $u < \frac{1}{2}$
- Since $a_3 > a_4$, we have $u > \frac{-2^p+1}{2^{p+1}} > \frac{-2^p}{2^{p+1}} = -\frac{1}{2}$

It follows from the observations above that u must be equal to zero because zero is the only integer in the open interval $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Substituting $u = 0$ into the system of equations gives us the following: $a_1 = \frac{2^{2p}}{2^{p+1}-1}, a_2 = 2^{p-1}, a_3 = \frac{1}{2}$ and $a_4 = \frac{2^{p-1}}{2^{p+1}-1}, k_1 = -2^{p-1}, k_2 = 1, k_3 = 2^{p-1}+1$. Next, we check that $\frac{2^{p-1}}{2^{p+1}-1} > 0$ for $p \in \mathbb{N}$ and $\frac{2^{p-1}}{2^{p+1}-1} < \frac{1}{2} < 2^{p-1} < \frac{2^{2p}}{2^{p+1}-1}$. Therefore for any $p \in \mathbb{N}$, the interval

$$\left[-\frac{2^{2p}}{2^{p+1}-1}, -2^{p-1}\right) \cup \left[-\frac{1}{2}, -\frac{2^{p-1}}{2^{p+1}-1}\right) \cup \left[\frac{2^{p-1}}{2^{p+1}-1}, \frac{1}{2}\right) \cup \left[2^{p-1}, \frac{2^{2p}}{2^{p+1}-1}\right)$$

makes sense. Next, by definition 7, the rotations of the permutation $(4, 3, 2, 1)$ are:

$$(3, 2, 1, 4), (2, 1, 4, 3), \text{ and } (1, 4, 3, 2).$$

By Lemma 14, since the permutation $(4, 3, 2, 1)$ provides a family of symmetric wavelet sets, each of its rotations gives rise to the same family of symmetric wavelet sets. Also, it follows from Lemma 7, Lemma 8, Lemma 9, Lemma 10, and Lemma 11 that there is no symmetric wavelet set parametrized by each of the other 20 permutations. This is due to the fact that each of the five permutations in the lemmas do not provide any symmetric wavelet sets. Therefore, the other three permutations that are rotations of each of those five permutations cannot provide symmetric wavelet sets. Thus, the only possible symmetric wavelet sets of four intervals are ordered by rotations of the permutation $(4, 3, 2, 1)$.

4 Algorithms

We will now describe the process of creating algorithms for the construction of four interval wavelet sets. In particular, we will present an algorithm which can be exploited to compute

(2, 2) wavelet sets. First, we appeal to the characterization of (2, 2) wavelet sets given in Proposition 9. Fixing f to be the permutation $(4, 3, 2, 1)$, we obtain the following system of equations:

$$\begin{cases} u = b_3 + k_1 \\ b_1 = b_4 + k_1 \\ b_2 = a_3 + k_2 \\ a_1 + k_3 = a_4 + k_2 \\ a_2 + k_3 = u + 1 \\ 2^p b_2 = b_3 \\ 2^{p+1} b_1 = b_4 \\ 2^q a_3 = a_2 \\ 2^{q+1} a_4 = a_1 \end{cases}$$

Next, solving the system of equations above for the variables $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$, and k_1, we obtain the following:

$$a_1 = \frac{2^{1+q}(k_2 - k_3)}{2^{1+q} - 1}, a_2 = 1 + k_3 + u, a_3 = -2^q(-1 - k_3 - u), a_4 = \frac{(k_2 - k_3)}{2^{1+q} - 1},$$

$$b_1 = \frac{2^{-q}(2^p + 2^{p+q}k_2 - 2^p k_3 + 2^p u - 2^q u)}{2^{1+p} - 1}, b_2 = 2^{-q}(1 + 2^q k_2 - k_3 + u),$$

$$b_3 = 2^{p-q}(1 + 2^q k_2 - k_3 + u),$$

$$b_4 = \frac{2^{1+p-q}(2^p + 2^{p+q}k_2 - 2^p k_3 + 2^p u - 2^q u)}{2^{1+p} - 1}, \text{ and } k_1 = -2^p k_2 + 2^{p-q}(k_3 - u - 1) + u.$$

Observe that $a_1, a_2, a_3, a_4, b_1, b_2, b_3$ and b_4 are functions of $k_2, k_3, p, q \in \mathbb{Z}$ and u which is not necessarily an integer. Since our objective is to construct an algorithm which will generate boundary points of a wavelet set, it is important that they are strictly dependent on variables which are all elements of a discrete set. As such, we solve the equation $k_1 = -2^p k_2 + 2^{p-q}(k_3 - u - 1) + u$ for u, and we obtain

$$u = \frac{2^p + 2^q k_1 + 2^{p+q} k_2 - 2^p k_3}{2^q - 2^p}.$$

Next, substituting the above into the formula obtained for each boundary point yields

$$\begin{cases} a_1 = \frac{2^{1+q}(k_2-k_3)}{2^{1+q}-1} \\ a_2 = \frac{2^q(1+k_1+2^pk_2-k_3)}{2^q-2^p} \\ a_3 = \frac{1+k_1+2^pk_2-k_3}{2^q-2^p} \\ a_4 = \frac{k_2-k_3}{2^{1+q}-1} \\ b_1 = \frac{k_1}{1-2^{1+p}} \\ b_2 = \frac{1+k_1+2^qk_2-k_3}{2^q-2^p} \\ b_3 = \frac{2^p(1+k_1+2^qk_2-k_3)}{2^q-2^p} \\ b_4 = \frac{2^{1+p}k_1}{1-2^{1+p}} \end{cases}$$

The above, together with the restriction that $a_1 < a_2 < a_3 < a_4 < 0 < b_1 < b_2 < b_3 < b_4$ yields the following algorithm.

$2-2$-**Wavelet Set**

Input: m (upper bound for p, q, k_1, k_2, k_3)

Output: $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ (set of ordered 8-tuples)

Set *Collector* as an empty array

For $1 \le p \le m$

For $1 \le q \le m$

For $-m \le k_1 \le 0$

For $1 \le k_2 \le m$

For $1 \le k_3 \le m$

If

$$\frac{2^{1+q}(k_2-k_3)}{-1+2^{1+q}} < \frac{2^q(1+k_1+2^pk_2-k_3)}{-2^p+2^q} < \frac{1+k_1+2^pk_2-k_3}{-2^p+2^q} < \frac{k_2-k_3}{-1+2^{1+q}} < 0$$

$$< \frac{k_1}{1-2^{1+p}} < \frac{1+k_1+2^qk_2-k_3}{-2^p+2^q} < \frac{2^p(1+k_1+2^qk_2-k_3)}{-2^p+2^q} < \frac{2^{1+p}k_1}{1-2^{1+p}}$$

Then

$$a_1 = \frac{2^{1+q}(k_2 - k_3)}{-1 + 2^{1+q}},\ a_2 = \frac{2^q(1 + k_1 + 2^p k_2 - k_3)}{-2^p + 2^q},\ a_3 = \frac{1 + k_1 + 2^p k_2 - k_3}{-2^p + 2^q},\ a_4 = \frac{k_2 - k_3}{-1 + 2^{1+q}},$$

$$b_1 = \frac{k_1}{1 - 2^{1+p}},\ b_2 = \frac{1 + k_1 + 2^q k_2 - k_3}{-2^p + 2^q},\ b_3 = \frac{2^p(1 + k_1 + 2^q k_2 - k_3)}{-2^p + 2^q},\ b_4 = \frac{2^{1+p} k_1}{1 - 2^{1+p}}$$

End if

End for

End for

End for

End for

End for

Output *Collector*

Using an approach which is similar to the one used to produce an algorithm for the construction of $2-2$-wavelet set, we obtain an algorithm for $3-1$ wavelet set which we present below

3-1 Wavelet Set

Input: m (upper bound for p, q, k_1, k_2, k_3)

Output: (a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2) (set of ordered 8-tuples)

Set $Collector$ as an empty array

For $1 \leq p \leq m$

For $1 \leq q \leq m$

For $-m \leq k_1 \leq 0$

For $1 \leq k_2 \leq m$

For $1 \leq k_3 \leq m$

If

$$\frac{2^q(k_2-k_3)}{-1+2^q} < \frac{2^p(2+k_1-2k_3)}{-1+2^{1+p}} < \frac{2^{1+p}k_2}{-2^{1+p}+2^q} < \frac{k_2-k_3}{-1+2^q}$$

$$< \frac{2+k_1-2k_3}{-1+2^{1+p}} < \frac{2^q k_2}{-2^{1+p}+2^q} < 0 < \frac{1+k_1-2^p k_1-k_3}{-1+2^{1+p}} < \frac{2(1+k_1-2^p k_1-k_3)}{-1+2^{1+p}}$$

Then

$$a_1 = \frac{2^q(k_2-k_3)}{-1+2^q},\ a_2 = \frac{2^p(2+k_1-2k_3)}{-1+2^{1+p}},\ a_3 = \frac{2^{1+p}k_2}{-2^{1+p}+2^q},\ a_4 = \frac{k_2-k_3}{-1+2^q},$$

$$a_5 = \frac{2+k_1-2k_3}{-1+2^{1+p}},\ a_6 = \frac{2^q k_2}{-2^{1+p}+2^q},\ b_1 = \frac{1+k_1-2^p k_1-k_3}{-1+2^{1+p}},\ b_2 = \frac{2(1+k_1-2^p k_1-k_3)}{-1+2^{1+p}}$$

End if

End for

End for

End for

End for

End for

Output *Collector*

References

[1] M. Bownik, K. Hoover, Dimension Functions of Rationally Dilated GMRAs and Wavelets, J. Fourier Anal. Appl. 15 (2009), 585-615

[2] X. Dai, D.R. Larson, and D. Speegle, "Wavelet Sets in R^n", J. Fourier Analysis and Applications, 4(1997), 451-456

[3] X. Dai, D.R. Larson, and D. Speegle, "Wavelet Sets in R^n II", Contemporary Math. 216(1998), 15-40

[4] D. Han and D.R. Larson, Wandering Vector Multipliers for Unitary Groups, Transactions AMS, 353(2001), 3347-3370

[5] E. Ionascu, D.R. Larson and C. Pearcy, "On Wavelet Sets", J. Fourier Analysis and Applications, 4(1998), 711-721

[6] D.R. Larson, Unitary systems and wavelet sets, Wavelet Analysis and Applications, Applied and Numerical Harmonic Analysis, Birkhauser Verlan Basel (2006), 143-171.

5 Appendix

```
In[124]:= WaveletSet22p1[m0_] := Module[{m = m0, p, q, k1, k2, k3, Collector}, Collector = {};
  For[p = 1, p ≤ m, p++, For[q = 1, q ≤ m, q++,
    For[k1 = -m, k1 ≤ 0, k1++, For[k2 = 1, k2 ≤ m, k2++, For[k3 = 1, k3 ≤ m, k3++,
      If[Quiet[(2^(1+q) (k2 - k3))/(-1 + 2^(1+q)) < (2^q (1 + k1 + 2^p k2 - k3))/(-2^p + 2^q) < (1 + k1 + 2^p k2 - k3)/(-2^p + 2^q) < (k2 - k3)/(-1 + 2^(1+q)) <
         0 < k1/(1 - 2^(1+p)) < (1 + k1 + 2^q k2 - k3)/(-2^p + 2^q) < (2^p (1 + k1 + 2^q k2 - k3))/(-2^p + 2^q) < (2^(1+p) k1)/(1 - 2^(1+p))],
       Collector = Join[{{{(2^(1+q) (k2 - k3))/(-1 + 2^(1+q)), (2^q (1 + k1 + 2^p k2 - k3)) / (-2^p + 2^q)},
           {(1 + k1 + 2^p k2 - k3)/(-2^p + 2^q), (k2 - k3)/(-1 + 2^(1+q))}, {k1/(1 - 2^(1+p)), (1 + k1 + 2^q k2 - k3)/(-2^p + 2^q)},
           {(2^p (1 + k1 + 2^q k2 - k3)) / (-2^p + 2^q), (2^(1+p) k1)/(1 - 2^(1+p))}}}, Collector]]]]]]];
  Collector]

In[125]:= WaveletSet22p1[25]

Out[125]= {{{-480/31, -479/31}, {-479/496, -15/31}, {6/341, 17/496}, {544/31, 6144/341}},
  {{-480/31, -478/31}, {-239/248, -15/31}, {19/1023, 9/248}, {576/31, 19456/1023}},
  {{-480/31, -477/31}, {-477/496, -15/31}, {20/1023, 19/496}, {608/31, 20480/1023}},
  {{-480/31, -476/31}, {-119/124, -15/31}, {7/341, 5/124}, {640/31, 7168/341}},
  {{-480/31, -475/31}, {-475/496, -15/31}, {2/93, 21/496}, {672/31, 2048/93}},
  {{-480/31, -474/31}, {-237/248, -15/31}, {23/1023, 11/248}, {704/31, 23552/1023}},
  {{-480/31, -473/31}, {-473/496, -15/31}, {8/341, 23/496}, {736/31, 8192/341}},
  {{-480/31, -472/31}, {-59/62, -15/31}, {25/1023, 3/62}, {768/31, 25600/1023}},
  {{-480/31, -232/15}, {-29/30, -15/31}, {9/511, 1/30}, {128/15, 4608/511}},
  {{-480/31, -77/5}, {-77/80, -15/31}, {10/511, 3/80}, {48/5, 5120/511}},
  {{-480/31, -46/3}, {-23/24, -15/31}, {11/511, 1/24}, {32/3, 5632/511}},
  {{-480/31, -229/15}, {-229/240, -15/31}, {12/511, 11/240}, {176/15, 6144/511}},
  {{-480/31, -76/5}, {-19/20, -15/31}, {13/511, 1/20}, {64/5, 6656/511}},
  {{-480/31, -227/15}, {-227/240, -15/31}, {2/73, 13/240}, {208/15, 1024/73}},
  {{-480/31, -226/15}, {-113/120, -15/31}, {15/511, 7/120}, {224/15, 7680/511}},
  {{-480/31, -15}, {-15/16, -15/31}, {16/511, 1/16}, {16, 8192/511}},
```

$\{\{-\frac{112}{15}, -\frac{231}{31}\}, \{-\frac{231}{248}, -\frac{7}{15}\}, \{\frac{18}{511}, \frac{17}{248}\}, \{\frac{544}{31}, \frac{9216}{511}\}\},$

$\{\{-\frac{112}{15}, -\frac{230}{31}\}, \{-\frac{115}{124}, -\frac{7}{15}\}, \{\frac{19}{511}, \frac{9}{124}\}, \{\frac{576}{31}, \frac{9728}{511}\}\},$

$\{\{-\frac{112}{15}, -\frac{229}{31}\}, \{-\frac{229}{248}, -\frac{7}{15}\}, \{\frac{20}{511}, \frac{19}{248}\}, \{\frac{608}{31}, \frac{10240}{511}\}\},$

$\{\{-\frac{112}{15}, -\frac{228}{31}\}, \{-\frac{57}{62}, -\frac{7}{15}\}, \{\frac{3}{73}, \frac{5}{62}\}, \{\frac{640}{31}, \frac{1536}{73}\}\},$

$\{\{-\frac{112}{15}, -\frac{227}{31}\}, \{-\frac{227}{248}, -\frac{7}{15}\}, \{\frac{22}{511}, \frac{21}{248}\}, \{\frac{672}{31}, \frac{11264}{511}\}\},$

$\{\{-\frac{112}{15}, -\frac{226}{31}\}, \{-\frac{113}{124}, -\frac{7}{15}\}, \{\frac{23}{511}, \frac{11}{124}\}, \{\frac{704}{31}, \frac{11776}{511}\}\},$

$\{\{-\frac{112}{15}, -\frac{225}{31}\}, \{-\frac{225}{248}, -\frac{7}{15}\}, \{\frac{24}{511}, \frac{23}{248}\}, \{\frac{736}{31}, \frac{12288}{511}\}\},$

$\{\{-\frac{112}{15}, -\frac{224}{31}\}, \{-\frac{28}{31}, -\frac{7}{15}\}, \{\frac{25}{511}, \frac{3}{31}\}, \{\frac{768}{31}, \frac{12800}{511}\}\},$

$\{\{-\frac{480}{31}, -\frac{108}{7}\}, \{-\frac{27}{28}, -\frac{15}{31}\}, \{\frac{1}{51}, \frac{1}{28}\}, \{\frac{32}{7}, \frac{256}{51}\}\},$

$\{\{-\frac{480}{31}, -\frac{107}{7}\}, \{-\frac{107}{112}, -\frac{15}{31}\}, \{\frac{2}{85}, \frac{5}{112}\}, \{\frac{40}{7}, \frac{512}{85}\}\},$

$\{\{-\frac{480}{31}, -\frac{106}{7}\}, \{-\frac{53}{56}, -\frac{15}{31}\}, \{\frac{7}{255}, \frac{3}{56}\}, \{\frac{48}{7}, \frac{1792}{255}\}\},$

$\{\{-\frac{480}{31}, -15\}, \{-\frac{15}{16}, -\frac{15}{31}\}, \{\frac{8}{255}, \frac{1}{16}\}, \{8, \frac{2048}{255}\}\},$

$\{\{-\frac{448}{31}, -\frac{101}{7}\}, \{-\frac{101}{112}, -\frac{14}{31}\}, \{\frac{13}{255}, \frac{11}{112}\}, \{\frac{88}{7}, \frac{3328}{255}\}\},$

$\{\{-\frac{448}{31}, -\frac{100}{7}\}, \{-\frac{25}{28}, -\frac{14}{31}\}, \{\frac{14}{255}, \frac{3}{28}\}, \{\frac{96}{7}, \frac{3584}{255}\}\},$

$\{\{-\frac{448}{31}, -\frac{99}{7}\}, \{-\frac{99}{112}, -\frac{14}{31}\}, \{\frac{1}{17}, \frac{13}{112}\}, \{\frac{104}{7}, \frac{256}{17}\}\},$

$\{\{-\frac{448}{31}, -14\}, \{-\frac{7}{8}, -\frac{14}{31}\}, \{\frac{16}{255}, \frac{1}{8}\}, \{16, \frac{4096}{255}\}\},$

$\{\{-\frac{416}{31}, -\frac{93}{7}\}, \{-\frac{93}{112}, -\frac{13}{31}\}, \{\frac{22}{255}, \frac{19}{112}\}, \{\frac{152}{7}, \frac{5632}{255}\}\},$

$\{\{-\frac{416}{31}, -\frac{92}{7}\}, \{-\frac{23}{28}, -\frac{13}{31}\}, \{\frac{23}{255}, \frac{5}{28}\}, \{\frac{160}{7}, \frac{5888}{255}\}\},$

$\{\{-\frac{416}{31}, -13\}, \{-\frac{13}{16}, -\frac{13}{31}\}, \{\frac{8}{85}, \frac{3}{16}\}, \{24, \frac{2048}{85}\}\},$

$\{\{-\frac{112}{15}, -\frac{37}{5}\}, \{-\frac{37}{40}, -\frac{7}{15}\}, \{\frac{2}{51}, \frac{3}{40}\}, \{\frac{48}{5}, \frac{512}{51}\}\},$

$\{\{-\frac{112}{15}, -\frac{22}{3}\}, \{-\frac{11}{12}, -\frac{7}{15}\}, \{\frac{11}{255}, \frac{1}{12}\}, \{\frac{32}{3}, \frac{2816}{255}\}\},$

$\{\{-\frac{112}{15}, -\frac{109}{15}\}, \{-\frac{109}{120}, -\frac{7}{15}\}, \{\frac{4}{85}, \frac{11}{120}\}, \{\frac{176}{15}, \frac{1024}{85}\}\},$

$\{\{-\frac{112}{15}, -\frac{36}{5}\}, \{-\frac{9}{10}, -\frac{7}{15}\}, \{\frac{13}{255}, \frac{1}{10}\}, \{\frac{64}{5}, \frac{3328}{255}\}\},$

$\{\{-\frac{112}{15}, -\frac{107}{15}\}, \{-\frac{107}{120}, -\frac{7}{15}\}, \{\frac{14}{255}, \frac{13}{120}\}, \{\frac{208}{15}, \frac{3584}{255}\}\},$

$\{\{-\frac{112}{15}, -\frac{106}{15}\}, \{-\frac{53}{60}, -\frac{7}{15}\}, \{\frac{1}{17}, \frac{7}{60}\}, \{\frac{224}{15}, \frac{256}{17}\}\},$

$\{\{-\frac{112}{15}, -7\}, \{-\frac{7}{8}, -\frac{7}{15}\}, \{\frac{16}{255}, \frac{1}{8}\}, \{16, \frac{4096}{255}\}\},$

$\{\{-\frac{24}{7}, -\frac{106}{31}\}, \{-\frac{53}{62}, -\frac{3}{7}\}, \{\frac{19}{255}, \frac{9}{62}\}, \{\frac{576}{31}, \frac{4864}{255}\}\},$

$\{\{-\frac{24}{7}, -\frac{105}{31}\}, \{-\frac{105}{124}, -\frac{3}{7}\}, \{\frac{4}{51}, \frac{19}{124}\}, \{\frac{608}{31}, \frac{1024}{51}\}\},$

$\{\{-\frac{24}{7}, -\frac{104}{31}\}, \{-\frac{26}{31}, -\frac{3}{7}\}, \{\frac{7}{85}, \frac{5}{31}\}, \{\frac{640}{31}, \frac{1792}{85}\}\},$

$\{\{-\frac{24}{7}, -\frac{103}{31}\}, \{-\frac{103}{124}, -\frac{3}{7}\}, \{\frac{22}{255}, \frac{21}{124}\}, \{\frac{672}{31}, \frac{5632}{255}\}\},$

$\{\{-\frac{24}{7}, -\frac{102}{31}\}, \{-\frac{51}{62}, -\frac{3}{7}\}, \{\frac{23}{255}, \frac{11}{62}\}, \{\frac{704}{31}, \frac{5888}{255}\}\},$

$\{\{-\frac{24}{7}, -\frac{101}{31}\}, \{-\frac{101}{124}, -\frac{3}{7}\}, \{\frac{8}{85}, \frac{23}{124}\}, \{\frac{736}{31}, \frac{2048}{85}\}\},$

$\{\{-\frac{24}{7}, -\frac{100}{31}\}, \{-\frac{25}{31}, -\frac{3}{7}\}, \{\frac{5}{51}, \frac{6}{31}\}, \{\frac{768}{31}, \frac{1280}{51}\}\},$

$\{\{-\frac{512}{21}, -24\}, \{-\frac{3}{4}, -\frac{8}{21}\}, \{\frac{16}{127}, \frac{1}{4}\}, \{16, \frac{2048}{127}\}\},$

$\{\{-\frac{1472}{63}, -23\}, \{-\frac{23}{32}, -\frac{23}{63}\}, \{\frac{18}{127}, \frac{9}{32}\}, \{18, \frac{2304}{127}\}\},$

$\{\{-\frac{1408}{63}, -22\}, \{-\frac{11}{16}, -\frac{22}{63}\}, \{\frac{20}{127}, \frac{5}{16}\}, \{20, \frac{2560}{127}\}\},$

$\{\{-\frac{64}{3}, -21\}, \{-\frac{21}{32}, -\frac{1}{3}\}, \{\frac{22}{127}, \frac{11}{32}\}, \{22, \frac{2816}{127}\}\},$

$\{\{-\frac{1280}{63}, -20\}, \{-\frac{5}{8}, -\frac{20}{63}\}, \{\frac{24}{127}, \frac{3}{8}\}, \{24, \frac{3072}{127}\}\},$

$\{\{-\frac{480}{31}, -\frac{46}{3}\}, \{-\frac{23}{24}, -\frac{15}{31}\}, \{\frac{3}{127}, \frac{1}{24}\}, \{\frac{8}{3}, \frac{384}{127}\}\},$

$\{\{-\frac{480}{31}, -15\}, \{-\frac{15}{16}, -\frac{15}{31}\}, \{\frac{4}{127}, \frac{1}{16}\}, \{4, \frac{512}{127}\}\},$

$\{\{-\frac{448}{31}, -\frac{43}{3}\}, \{-\frac{43}{48}, -\frac{14}{31}\}, \{\frac{7}{127}, \frac{5}{48}\}, \{\frac{20}{3}, \frac{896}{127}\}\},$

$\{\{-\frac{448}{31}, -14\}, \{-\frac{7}{8}, -\frac{14}{31}\}, \{\frac{8}{127}, \frac{1}{8}\}, \{8, \frac{1024}{127}\}\},$

$\{\{-\frac{416}{31}, -\frac{40}{3}\}, \{-\frac{5}{6}, -\frac{13}{31}\}, \{\frac{11}{127}, \frac{1}{6}\}, \{\frac{32}{3}, \frac{1408}{127}\}\},$

$\{\{-\frac{416}{31}, -13\}, \{-\frac{13}{16}, -\frac{13}{31}\}, \{\frac{12}{127}, \frac{3}{16}\}, \{12, \frac{1536}{127}\}\},$

$\{\{-\frac{384}{31}, -\frac{37}{3}\}, \{-\frac{37}{48}, -\frac{12}{31}\}, \{\frac{15}{127}, \frac{11}{48}\}, \{\frac{44}{3}, \frac{1920}{127}\}\},$

$\{\{-\frac{384}{31}, -12\}, \{-\frac{3}{4}, -\frac{12}{31}\}, \{\frac{16}{127}, \frac{1}{4}\}, \{16, \frac{2048}{127}\}\},$

$\{\{-\frac{352}{31}, -\frac{34}{3}\}, \{-\frac{17}{24}, -\frac{11}{31}\}, \{\frac{19}{127}, \frac{7}{24}\}, \{\frac{56}{3}, \frac{2432}{127}\}\},$

$\{\{-\frac{352}{31}, -11\}, \{-\frac{11}{16}, -\frac{11}{31}\}, \{\frac{20}{127}, \frac{5}{16}\}, \{20, \frac{2560}{127}\}\},$

$\{\{-\frac{320}{31}, -10\}, \{-\frac{5}{8}, -\frac{10}{31}\}, \{\frac{24}{127}, \frac{3}{8}\}, \{24, \frac{3072}{127}\}\},$

$\{\{-\frac{112}{15}, -\frac{52}{7}\}, \{-\frac{13}{14}, -\frac{7}{15}\}, \{\frac{5}{127}, \frac{1}{14}\}, \{\frac{32}{7}, \frac{640}{127}\}\},$

$\{\{-\frac{112}{15}, -\frac{51}{7}\}, \{-\frac{51}{56}, -\frac{7}{15}\}, \{\frac{6}{127}, \frac{5}{56}\}, \{\frac{40}{7}, \frac{768}{127}\}\},$
$\{\{-\frac{112}{15}, -\frac{50}{7}\}, \{-\frac{25}{28}, -\frac{7}{15}\}, \{\frac{7}{127}, \frac{3}{28}\}, \{\frac{48}{7}, \frac{896}{127}\}\},$
$\{\{-\frac{112}{15}, -7\}, \{-\frac{7}{8}, -\frac{7}{15}\}, \{\frac{8}{127}, \frac{1}{8}\}, \{8, \frac{1024}{127}\}\},$
$\{\{-\frac{32}{5}, -\frac{44}{7}\}, \{-\frac{11}{14}, -\frac{2}{5}\}, \{\frac{14}{127}, \frac{3}{14}\}, \{\frac{96}{7}, \frac{1792}{127}\}\},$
$\{\{-\frac{32}{5}, -\frac{43}{7}\}, \{-\frac{43}{56}, -\frac{2}{5}\}, \{\frac{15}{127}, \frac{13}{56}\}, \{\frac{104}{7}, \frac{1920}{127}\}\},$
$\{\{-\frac{32}{5}, -6\}, \{-\frac{3}{4}, -\frac{2}{5}\}, \{\frac{16}{127}, \frac{1}{4}\}, \{16, \frac{2048}{127}\}\},$
$\{\{-\frac{16}{3}, -\frac{37}{7}\}, \{-\frac{37}{56}, -\frac{1}{3}\}, \{\frac{22}{127}, \frac{19}{56}\}, \{\frac{152}{7}, \frac{2816}{127}\}\},$
$\{\{-\frac{16}{3}, -\frac{36}{7}\}, \{-\frac{9}{14}, -\frac{1}{3}\}, \{\frac{23}{127}, \frac{5}{14}\}, \{\frac{160}{7}, \frac{2944}{127}\}\},$
$\{\{-\frac{16}{3}, -5\}, \{-\frac{5}{8}, -\frac{1}{3}\}, \{\frac{24}{127}, \frac{3}{8}\}, \{24, \frac{3072}{127}\}\},$
$\{\{-\frac{16}{3}, -\frac{34}{7}\}, \{-\frac{17}{28}, -\frac{1}{3}\}, \{\frac{25}{127}, \frac{11}{28}\}, \{\frac{176}{7}, \frac{3200}{127}\}\},$
$\{\{-\frac{24}{7}, -\frac{17}{5}\}, \{-\frac{17}{20}, -\frac{3}{7}\}, \{\frac{10}{127}, \frac{3}{20}\}, \{\frac{48}{5}, \frac{1280}{127}\}\},$
$\{\{-\frac{24}{7}, -\frac{10}{3}\}, \{-\frac{5}{6}, -\frac{3}{7}\}, \{\frac{11}{127}, \frac{1}{6}\}, \{\frac{32}{3}, \frac{1408}{127}\}\},$
$\{\{-\frac{24}{7}, -\frac{49}{15}\}, \{-\frac{49}{60}, -\frac{3}{7}\}, \{\frac{12}{127}, \frac{11}{60}\}, \{\frac{176}{15}, \frac{1536}{127}\}\},$
$\{\{-\frac{24}{7}, -\frac{16}{5}\}, \{-\frac{4}{5}, -\frac{3}{7}\}, \{\frac{13}{127}, \frac{1}{5}\}, \{\frac{64}{5}, \frac{1664}{127}\}\},$
$\{\{-\frac{24}{7}, -\frac{47}{15}\}, \{-\frac{47}{60}, -\frac{3}{7}\}, \{\frac{14}{127}, \frac{13}{60}\}, \{\frac{208}{15}, \frac{1792}{127}\}\},$
$\{\{-\frac{24}{7}, -\frac{46}{15}\}, \{-\frac{23}{30}, -\frac{3}{7}\}, \{\frac{15}{127}, \frac{7}{30}\}, \{\frac{224}{15}, \frac{1920}{127}\}\},$
$\{\{-\frac{24}{7}, -3\}, \{-\frac{3}{4}, -\frac{3}{7}\}, \{\frac{16}{127}, \frac{1}{4}\}, \{16, \frac{2048}{127}\}\},$
$\{\{-\frac{24}{7}, -\frac{44}{15}\}, \{-\frac{11}{15}, -\frac{3}{7}\}, \{\frac{17}{127}, \frac{4}{15}\}, \{\frac{256}{15}, \frac{2176}{127}\}\},$
$\{\{-\frac{24}{7}, -\frac{43}{15}\}, \{-\frac{43}{60}, -\frac{3}{7}\}, \{\frac{18}{127}, \frac{17}{60}\}, \{\frac{272}{15}, \frac{2304}{127}\}\},$
$\{\{-\frac{4}{3}, -\frac{41}{31}\}, \{-\frac{41}{62}, -\frac{1}{3}\}, \{\frac{22}{127}, \frac{21}{62}\}, \{\frac{672}{31}, \frac{2816}{127}\}\},$
$\{\{-\frac{4}{3}, -\frac{40}{31}\}, \{-\frac{20}{31}, -\frac{1}{3}\}, \{\frac{23}{127}, \frac{11}{31}\}, \{\frac{704}{31}, \frac{2944}{127}\}\},$
$\{\{-\frac{4}{3}, -\frac{39}{31}\}, \{-\frac{39}{62}, -\frac{1}{3}\}, \{\frac{24}{127}, \frac{23}{62}\}, \{\frac{736}{31}, \frac{3072}{127}\}\},$
$\{\{-\frac{4}{3}, -\frac{38}{31}\}, \{-\frac{19}{31}, -\frac{1}{3}\}, \{\frac{25}{127}, \frac{12}{31}\}, \{\frac{768}{31}, \frac{3200}{127}\}\},$
$\{\{-\frac{3072}{127}, -24\}, \{-\frac{3}{8}, -\frac{24}{127}\}, \{\frac{20}{63}, \frac{5}{8}\}, \{20, \frac{1280}{63}\}\},$
$\{\{-\frac{2816}{127}, -22\}, \{-\frac{11}{32}, -\frac{22}{127}\}, \{\frac{1}{3}, \frac{21}{32}\}, \{21, \frac{64}{3}\}\},$

$\{\{-\frac{2560}{127}, -20\}, \{-\frac{5}{16}, -\frac{20}{127}\}, \{\frac{22}{63}, \frac{11}{16}\}, \{22, \frac{1408}{63}\}\},$

$\{\{-\frac{2304}{127}, -18\}, \{-\frac{9}{32}, -\frac{18}{127}\}, \{\frac{23}{63}, \frac{23}{32}\}, \{23, \frac{1472}{63}\}\},$

$\{\{-\frac{2048}{127}, -16\}, \{-\frac{1}{4}, -\frac{16}{127}\}, \{\frac{8}{21}, \frac{3}{4}\}, \{24, \frac{512}{21}\}\},$

$\{\{-\frac{1792}{127}, -14\}, \{-\frac{7}{32}, -\frac{14}{127}\}, \{\frac{25}{63}, \frac{25}{32}\}, \{25, \frac{1600}{63}\}\},$

$\{\{-\frac{480}{31}, -15\}, \{-\frac{15}{16}, -\frac{15}{31}\}, \{\frac{2}{63}, \frac{1}{16}\}, \{2, \frac{128}{63}\}\},$

$\{\{-\frac{448}{31}, -14\}, \{-\frac{7}{8}, -\frac{14}{31}\}, \{\frac{4}{63}, \frac{1}{8}\}, \{4, \frac{256}{63}\}\},$

$\{\{-\frac{416}{31}, -13\}, \{-\frac{13}{16}, -\frac{13}{31}\}, \{\frac{2}{21}, \frac{3}{16}\}, \{6, \frac{128}{21}\}\},$

$\{\{-\frac{384}{31}, -12\}, \{-\frac{3}{4}, -\frac{12}{31}\}, \{\frac{8}{63}, \frac{1}{4}\}, \{8, \frac{512}{63}\}\},$

$\{\{-\frac{352}{31}, -11\}, \{-\frac{11}{16}, -\frac{11}{31}\}, \{\frac{10}{63}, \frac{5}{16}\}, \{10, \frac{640}{63}\}\},$

$\{\{-\frac{320}{31}, -10\}, \{-\frac{5}{8}, -\frac{10}{31}\}, \{\frac{4}{21}, \frac{3}{8}\}, \{12, \frac{256}{21}\}\},$

$\{\{-\frac{288}{31}, -9\}, \{-\frac{9}{16}, -\frac{9}{31}\}, \{\frac{2}{9}, \frac{7}{16}\}, \{14, \frac{128}{9}\}\},$

$\{\{-\frac{256}{31}, -8\}, \{-\frac{1}{2}, -\frac{8}{31}\}, \{\frac{16}{63}, \frac{1}{2}\}, \{16, \frac{1024}{63}\}\},$

$\{\{-\frac{224}{31}, -7\}, \{-\frac{7}{16}, -\frac{7}{31}\}, \{\frac{2}{7}, \frac{9}{16}\}, \{18, \frac{128}{7}\}\},$

$\{\{-\frac{192}{31}, -6\}, \{-\frac{3}{8}, -\frac{6}{31}\}, \{\frac{20}{63}, \frac{5}{8}\}, \{20, \frac{1280}{63}\}\},$

$\{\{-\frac{160}{31}, -5\}, \{-\frac{5}{16}, -\frac{5}{31}\}, \{\frac{22}{63}, \frac{11}{16}\}, \{22, \frac{1408}{63}\}\},$

$\{\{-\frac{128}{31}, -4\}, \{-\frac{1}{4}, -\frac{4}{31}\}, \{\frac{8}{21}, \frac{3}{4}\}, \{24, \frac{512}{21}\}\},$

$\{\{-\frac{112}{15}, -\frac{22}{3}\}, \{-\frac{11}{12}, -\frac{7}{15}\}, \{\frac{1}{21}, \frac{1}{12}\}, \{\frac{8}{3}, \frac{64}{21}\}\},$

$\{\{-\frac{112}{15}, -7\}, \{-\frac{7}{8}, -\frac{7}{15}\}, \{\frac{4}{63}, \frac{1}{8}\}, \{4, \frac{256}{63}\}\},$

$\{\{-\frac{32}{5}, -\frac{19}{3}\}, \{-\frac{19}{24}, -\frac{2}{5}\}, \{\frac{1}{9}, \frac{5}{24}\}, \{\frac{20}{3}, \frac{64}{9}\}\},$

$\{\{-\frac{32}{5}, -6\}, \{-\frac{3}{4}, -\frac{2}{5}\}, \{\frac{8}{63}, \frac{1}{4}\}, \{8, \frac{512}{63}\}\},$

$\{\{-\frac{16}{3}, -5\}, \{-\frac{5}{8}, -\frac{1}{3}\}, \{\frac{4}{21}, \frac{3}{8}\}, \{12, \frac{256}{21}\}\},$

$\{\{-\frac{64}{15}, -4\}, \{-\frac{1}{2}, -\frac{4}{15}\}, \{\frac{16}{63}, \frac{1}{2}\}, \{16, \frac{1024}{63}\}\},$

$\{\{-\frac{16}{5}, -3\}, \{-\frac{3}{8}, -\frac{1}{5}\}, \{\frac{20}{63}, \frac{5}{8}\}, \{20, \frac{1280}{63}\}\},$

$\{\{-\frac{32}{15}, -2\}, \{-\frac{1}{4}, -\frac{2}{15}\}, \{\frac{8}{21}, \frac{3}{4}\}, \{24, \frac{512}{21}\}\},$

$\{\{-\frac{32}{15}, -\frac{5}{3}\}, \{-\frac{5}{24}, -\frac{2}{15}\}, \{\frac{25}{63}, \frac{19}{24}\}, \{\frac{76}{3}, \frac{1600}{63}\}\},$

$\{\{-\frac{24}{7}, -\frac{23}{7}\}, \{-\frac{23}{28}, -\frac{3}{7}\}, \{\frac{2}{21}, \frac{5}{28}\}, \{\frac{40}{7}, \frac{128}{21}\}\},$

$\{\{-\frac{24}{7}, -\frac{22}{7}\}, \{-\frac{11}{14}, -\frac{3}{7}\}, \{\frac{1}{9}, \frac{3}{14}\}, \{\frac{48}{7}, \frac{64}{9}\}\},$

$\{\{-\frac{24}{7}, -3\}, \{-\frac{3}{4}, -\frac{3}{7}\}, \{\frac{8}{63}, \frac{1}{4}\}, \{8, \frac{512}{63}\}\},$

$\{\{-\frac{16}{7}, -\frac{15}{7}\}, \{-\frac{15}{28}, -\frac{2}{7}\}, \{\frac{5}{21}, \frac{13}{28}\}, \{\frac{104}{7}, \frac{320}{21}\}\},$

$\{\{-\frac{16}{7}, -2\}, \{-\frac{1}{2}, -\frac{2}{7}\}, \{\frac{16}{63}, \frac{1}{2}\}, \{16, \frac{1024}{63}\}\},$

$\{\{-\frac{16}{7}, -\frac{13}{7}\}, \{-\frac{13}{28}, -\frac{2}{7}\}, \{\frac{17}{63}, \frac{15}{28}\}, \{\frac{120}{7}, \frac{1088}{63}\}\},$

$\{\{-\frac{8}{7}, -1\}, \{-\frac{1}{4}, -\frac{1}{7}\}, \{\frac{8}{21}, \frac{3}{4}\}, \{24, \frac{512}{21}\}\},$

$\{\{-\frac{8}{7}, -\frac{6}{7}\}, \{-\frac{3}{14}, -\frac{1}{7}\}, \{\frac{25}{63}, \frac{11}{14}\}, \{\frac{176}{7}, \frac{1600}{63}\}\},$

$\{\{-\frac{4}{3}, -\frac{19}{15}\}, \{-\frac{19}{30}, -\frac{1}{3}\}, \{\frac{4}{21}, \frac{11}{30}\}, \{\frac{176}{15}, \frac{256}{21}\}\},$

$\{\{-\frac{4}{3}, -\frac{6}{5}\}, \{-\frac{3}{5}, -\frac{1}{3}\}, \{\frac{13}{63}, \frac{2}{5}\}, \{\frac{64}{5}, \frac{832}{63}\}\},$

$\{\{-\frac{4}{3}, -\frac{17}{15}\}, \{-\frac{17}{30}, -\frac{1}{3}\}, \{\frac{2}{9}, \frac{13}{30}\}, \{\frac{208}{15}, \frac{128}{9}\}\},$

$\{\{-\frac{4}{3}, -\frac{16}{15}\}, \{-\frac{8}{15}, -\frac{1}{3}\}, \{\frac{5}{21}, \frac{7}{15}\}, \{\frac{224}{15}, \frac{320}{21}\}\},$

$\{\{-\frac{4}{3}, -1\}, \{-\frac{1}{2}, -\frac{1}{3}\}, \{\frac{16}{63}, \frac{1}{2}\}, \{16, \frac{1024}{63}\}\},$

$\{\{-\frac{4}{3}, -\frac{14}{15}\}, \{-\frac{7}{15}, -\frac{1}{3}\}, \{\frac{17}{63}, \frac{8}{15}\}, \{\frac{256}{15}, \frac{1088}{63}\}\},$

$\{\{-\frac{4}{3}, -\frac{13}{15}\}, \{-\frac{13}{30}, -\frac{1}{3}\}, \{\frac{2}{7}, \frac{17}{30}\}, \{\frac{272}{15}, \frac{128}{7}\}\},$

$\{\{-\frac{4}{3}, -\frac{4}{5}\}, \{-\frac{2}{5}, -\frac{1}{3}\}, \{\frac{19}{63}, \frac{3}{5}\}, \{\frac{96}{5}, \frac{1216}{63}\}\},$

$\{\{-\frac{4}{3}, -\frac{11}{15}\}, \{-\frac{11}{30}, -\frac{1}{3}\}, \{\frac{20}{63}, \frac{19}{30}\}, \{\frac{304}{15}, \frac{1280}{63}\}\},$

$\{\{-\frac{8192}{341}, -\frac{736}{31}\}, \{-\frac{23}{496}, -\frac{8}{341}\}, \{\frac{15}{31}, \frac{473}{496}\}, \{\frac{473}{31}, \frac{480}{31}\}\},$

$\{\{-\frac{23552}{1023}, -\frac{704}{31}\}, \{-\frac{11}{248}, -\frac{23}{1023}\}, \{\frac{15}{31}, \frac{237}{248}\}, \{\frac{474}{31}, \frac{480}{31}\}\},$

$\{\{-\frac{2048}{93}, -\frac{672}{31}\}, \{-\frac{21}{496}, -\frac{2}{93}\}, \{\frac{15}{31}, \frac{475}{496}\}, \{\frac{475}{31}, \frac{480}{31}\}\},$

$\{\{-\frac{7168}{341}, -\frac{640}{31}\}, \{-\frac{5}{124}, -\frac{7}{341}\}, \{\frac{15}{31}, \frac{119}{124}\}, \{\frac{476}{31}, \frac{480}{31}\}\},$

$\{\{-\frac{20480}{1023}, -\frac{608}{31}\}, \{-\frac{19}{496}, -\frac{20}{1023}\}, \{\frac{15}{31}, \frac{477}{496}\}, \{\frac{477}{31}, \frac{480}{31}\}\},$

$\{\{-\frac{19456}{1023}, -\frac{576}{31}\}, \{-\frac{9}{248}, -\frac{19}{1023}\}, \{\frac{15}{31}, \frac{239}{248}\}, \{\frac{478}{31}, \frac{480}{31}\}\},$

$\{\{-\frac{6144}{341}, -\frac{544}{31}\}, \{-\frac{17}{496}, -\frac{6}{341}\}, \{\frac{15}{31}, \frac{479}{496}\}, \{\frac{479}{31}, \frac{480}{31}\}\},$

$\{\{-\frac{8192}{511}, -16\}, \{-\frac{1}{16}, -\frac{16}{511}\}, \{\frac{15}{31}, \frac{15}{16}\}, \{15, \frac{480}{31}\}\},$

$$\{\{-\frac{7680}{511}, -\frac{224}{15}\}, \{-\frac{7}{120}, -\frac{15}{511}\}, \{\frac{15}{31}, \frac{113}{120}\}, \{\frac{226}{15}, \frac{480}{31}\}\},$$
$$\{\{-\frac{1024}{73}, -\frac{208}{15}\}, \{-\frac{13}{240}, -\frac{2}{73}\}, \{\frac{15}{31}, \frac{227}{240}\}, \{\frac{227}{15}, \frac{480}{31}\}\},$$
$$\{\{-\frac{6656}{511}, -\frac{64}{5}\}, \{-\frac{1}{20}, -\frac{13}{511}\}, \{\frac{15}{31}, \frac{19}{20}\}, \{\frac{76}{5}, \frac{480}{31}\}\},$$
$$\{\{-\frac{6144}{511}, -\frac{176}{15}\}, \{-\frac{11}{240}, -\frac{12}{511}\}, \{\frac{15}{31}, \frac{229}{240}\}, \{\frac{229}{15}, \frac{480}{31}\}\},$$
$$\{\{-\frac{5632}{511}, -\frac{32}{3}\}, \{-\frac{1}{24}, -\frac{11}{511}\}, \{\frac{15}{31}, \frac{23}{24}\}, \{\frac{46}{3}, \frac{480}{31}\}\},$$
$$\{\{-\frac{5120}{511}, -\frac{48}{5}\}, \{-\frac{3}{80}, -\frac{10}{511}\}, \{\frac{15}{31}, \frac{77}{80}\}, \{\frac{77}{5}, \frac{480}{31}\}\},$$
$$\{\{-\frac{4608}{511}, -\frac{128}{15}\}, \{-\frac{1}{30}, -\frac{9}{511}\}, \{\frac{15}{31}, \frac{29}{30}\}, \{\frac{232}{15}, \frac{480}{31}\}\},$$
$$\{\{-\frac{2048}{85}, -24\}, \{-\frac{3}{16}, -\frac{8}{85}\}, \{\frac{13}{31}, \frac{13}{16}\}, \{13, \frac{416}{31}\}\},$$
$$\{\{-\frac{5888}{255}, -\frac{160}{7}\}, \{-\frac{5}{28}, -\frac{23}{255}\}, \{\frac{13}{31}, \frac{23}{28}\}, \{\frac{92}{7}, \frac{416}{31}\}\},$$
$$\{\{-\frac{5632}{255}, -\frac{152}{7}\}, \{-\frac{19}{112}, -\frac{22}{255}\}, \{\frac{13}{31}, \frac{93}{112}\}, \{\frac{93}{7}, \frac{416}{31}\}\},$$
$$\{\{-\frac{4096}{255}, -16\}, \{-\frac{1}{8}, -\frac{16}{255}\}, \{\frac{14}{31}, \frac{7}{8}\}, \{14, \frac{448}{31}\}\},$$
$$\{\{-\frac{256}{17}, -\frac{104}{7}\}, \{-\frac{13}{112}, -\frac{1}{17}\}, \{\frac{14}{31}, \frac{99}{112}\}, \{\frac{99}{7}, \frac{448}{31}\}\},$$
$$\{\{-\frac{3584}{255}, -\frac{96}{7}\}, \{-\frac{3}{28}, -\frac{14}{255}\}, \{\frac{14}{31}, \frac{25}{28}\}, \{\frac{100}{7}, \frac{448}{31}\}\},$$
$$\{\{-\frac{3328}{255}, -\frac{88}{7}\}, \{-\frac{11}{112}, -\frac{13}{255}\}, \{\frac{14}{31}, \frac{101}{112}\}, \{\frac{101}{7}, \frac{448}{31}\}\},$$
$$\{\{-\frac{2048}{255}, -8\}, \{-\frac{1}{16}, -\frac{8}{255}\}, \{\frac{15}{31}, \frac{15}{16}\}, \{15, \frac{480}{31}\}\},$$
$$\{\{-\frac{1792}{255}, -\frac{48}{7}\}, \{-\frac{3}{56}, -\frac{7}{255}\}, \{\frac{15}{31}, \frac{53}{56}\}, \{\frac{106}{7}, \frac{480}{31}\}\},$$
$$\{\{-\frac{512}{85}, -\frac{40}{7}\}, \{-\frac{5}{112}, -\frac{2}{85}\}, \{\frac{15}{31}, \frac{107}{112}\}, \{\frac{107}{7}, \frac{480}{31}\}\},$$
$$\{\{-\frac{256}{51}, -\frac{32}{7}\}, \{-\frac{1}{28}, -\frac{1}{51}\}, \{\frac{15}{31}, \frac{27}{28}\}, \{\frac{108}{7}, \frac{480}{31}\}\},$$
$$\{\{-\frac{3072}{127}, -24\}, \{-\frac{3}{8}, -\frac{24}{127}\}, \{\frac{10}{31}, \frac{5}{8}\}, \{10, \frac{320}{31}\}\},$$
$$\{\{-\frac{2560}{127}, -20\}, \{-\frac{5}{16}, -\frac{20}{127}\}, \{\frac{11}{31}, \frac{11}{16}\}, \{11, \frac{352}{31}\}\},$$
$$\{\{-\frac{2432}{127}, -\frac{56}{3}\}, \{-\frac{7}{24}, -\frac{19}{127}\}, \{\frac{11}{31}, \frac{17}{24}\}, \{\frac{34}{3}, \frac{352}{31}\}\},$$
$$\{\{-\frac{2048}{127}, -16\}, \{-\frac{1}{4}, -\frac{16}{127}\}, \{\frac{12}{31}, \frac{3}{4}\}, \{12, \frac{384}{31}\}\},$$
$$\{\{-\frac{1920}{127}, -\frac{44}{3}\}, \{-\frac{11}{48}, -\frac{15}{127}\}, \{\frac{12}{31}, \frac{37}{48}\}, \{\frac{37}{3}, \frac{384}{31}\}\},$$
$$\{\{-\frac{1536}{127}, -12\}, \{-\frac{3}{16}, -\frac{12}{127}\}, \{\frac{13}{31}, \frac{13}{16}\}, \{13, \frac{416}{31}\}\},$$
$$\{\{-\frac{1408}{127}, -\frac{32}{3}\}, \{-\frac{1}{6}, -\frac{11}{127}\}, \{\frac{13}{31}, \frac{5}{6}\}, \{\frac{40}{3}, \frac{416}{31}\}\},$$

$\{\{-\frac{1024}{127}, -8\}, \{-\frac{1}{8}, -\frac{8}{127}\}, \{\frac{14}{31}, \frac{7}{8}\}, \{14, \frac{448}{31}\}\}$,
$\{\{-\frac{896}{127}, -\frac{20}{3}\}, \{-\frac{5}{48}, -\frac{7}{127}\}, \{\frac{14}{31}, \frac{43}{48}\}, \{\frac{43}{3}, \frac{448}{31}\}\}$,
$\{\{-\frac{512}{127}, -4\}, \{-\frac{1}{16}, -\frac{4}{127}\}, \{\frac{15}{31}, \frac{15}{16}\}, \{15, \frac{480}{31}\}\}$,
$\{\{-\frac{384}{127}, -\frac{8}{3}\}, \{-\frac{1}{24}, -\frac{3}{127}\}, \{\frac{15}{31}, \frac{23}{24}\}, \{\frac{46}{3}, \frac{480}{31}\}\}$,
$\{\{-\frac{512}{21}, -24\}, \{-\frac{3}{4}, -\frac{8}{21}\}, \{\frac{4}{31}, \frac{1}{4}\}, \{4, \frac{128}{31}\}\}$,
$\{\{-\frac{1408}{63}, -22\}, \{-\frac{11}{16}, -\frac{22}{63}\}, \{\frac{5}{31}, \frac{5}{16}\}, \{5, \frac{160}{31}\}\}$,
$\{\{-\frac{1280}{63}, -20\}, \{-\frac{5}{8}, -\frac{20}{63}\}, \{\frac{6}{31}, \frac{3}{8}\}, \{6, \frac{192}{31}\}\}$,
$\{\{-\frac{128}{7}, -18\}, \{-\frac{9}{16}, -\frac{2}{7}\}, \{\frac{7}{31}, \frac{7}{16}\}, \{7, \frac{224}{31}\}\}$,
$\{\{-\frac{1024}{63}, -16\}, \{-\frac{1}{2}, -\frac{16}{63}\}, \{\frac{8}{31}, \frac{1}{2}\}, \{8, \frac{256}{31}\}\}$,
$\{\{-\frac{128}{9}, -14\}, \{-\frac{7}{16}, -\frac{2}{9}\}, \{\frac{9}{31}, \frac{9}{16}\}, \{9, \frac{288}{31}\}\}$,
$\{\{-\frac{256}{21}, -12\}, \{-\frac{3}{8}, -\frac{4}{21}\}, \{\frac{10}{31}, \frac{5}{8}\}, \{10, \frac{320}{31}\}\}$,
$\{\{-\frac{640}{63}, -10\}, \{-\frac{5}{16}, -\frac{10}{63}\}, \{\frac{11}{31}, \frac{11}{16}\}, \{11, \frac{352}{31}\}\}$,
$\{\{-\frac{512}{63}, -8\}, \{-\frac{1}{4}, -\frac{8}{63}\}, \{\frac{12}{31}, \frac{3}{4}\}, \{12, \frac{384}{31}\}\}$,
$\{\{-\frac{128}{21}, -6\}, \{-\frac{3}{16}, -\frac{2}{21}\}, \{\frac{13}{31}, \frac{13}{16}\}, \{13, \frac{416}{31}\}\}$,
$\{\{-\frac{256}{63}, -4\}, \{-\frac{1}{8}, -\frac{4}{63}\}, \{\frac{14}{31}, \frac{7}{8}\}, \{14, \frac{448}{31}\}\}$,
$\{\{-\frac{128}{63}, -2\}, \{-\frac{1}{16}, -\frac{2}{63}\}, \{\frac{15}{31}, \frac{15}{16}\}, \{15, \frac{480}{31}\}\}$,
$\{\{-\frac{112}{15}, -7\}, \{-\frac{7}{8}, -\frac{7}{15}\}, \{\frac{2}{31}, \frac{1}{8}\}, \{2, \frac{64}{31}\}\}$,
$\{\{-\frac{32}{5}, -6\}, \{-\frac{3}{4}, -\frac{2}{5}\}, \{\frac{4}{31}, \frac{1}{4}\}, \{4, \frac{128}{31}\}\}$,
$\{\{-\frac{16}{3}, -5\}, \{-\frac{5}{8}, -\frac{1}{3}\}, \{\frac{6}{31}, \frac{3}{8}\}, \{6, \frac{192}{31}\}\}$,
$\{\{-\frac{64}{15}, -4\}, \{-\frac{1}{2}, -\frac{4}{15}\}, \{\frac{8}{31}, \frac{1}{2}\}, \{8, \frac{256}{31}\}\}$,
$\{\{-\frac{16}{5}, -3\}, \{-\frac{3}{8}, -\frac{1}{5}\}, \{\frac{10}{31}, \frac{5}{8}\}, \{10, \frac{320}{31}\}\}$,
$\{\{-\frac{32}{15}, -2\}, \{-\frac{1}{4}, -\frac{2}{15}\}, \{\frac{12}{31}, \frac{3}{4}\}, \{12, \frac{384}{31}\}\}$,
$\{\{-\frac{16}{15}, -1\}, \{-\frac{1}{8}, -\frac{1}{15}\}, \{\frac{14}{31}, \frac{7}{8}\}, \{14, \frac{448}{31}\}\}$,
$\{\{-\frac{24}{7}, -\frac{10}{3}\}, \{-\frac{5}{6}, -\frac{3}{7}\}, \{\frac{3}{31}, \frac{1}{6}\}, \{\frac{8}{3}, \frac{96}{31}\}\}$,
$\{\{-\frac{24}{7}, -3\}, \{-\frac{3}{4}, -\frac{3}{7}\}, \{\frac{4}{31}, \frac{1}{4}\}, \{4, \frac{128}{31}\}\}$,

$\{\{-\frac{16}{7}, -2\}, \{-\frac{1}{2}, -\frac{2}{7}\}, \{\frac{8}{31}, \frac{1}{2}\}, \{8, \frac{256}{31}\}\},$

$\{\{-\frac{8}{7}, -1\}, \{-\frac{1}{4}, -\frac{1}{7}\}, \{\frac{12}{31}, \frac{3}{4}\}, \{12, \frac{384}{31}\}\},$

$\{\{-\frac{8}{7}, -\frac{2}{3}\}, \{-\frac{1}{6}, -\frac{1}{7}\}, \{\frac{13}{31}, \frac{5}{6}\}, \{\frac{40}{3}, \frac{416}{31}\}\},$

$\{\{-\frac{4}{3}, -\frac{9}{7}\}, \{-\frac{9}{14}, -\frac{1}{3}\}, \{\frac{6}{31}, \frac{5}{14}\}, \{\frac{40}{7}, \frac{192}{31}\}\},$

$\{\{-\frac{4}{3}, -\frac{8}{7}\}, \{-\frac{4}{7}, -\frac{1}{3}\}, \{\frac{7}{31}, \frac{3}{7}\}, \{\frac{48}{7}, \frac{224}{31}\}\},$

$\{\{-\frac{4}{3}, -1\}, \{-\frac{1}{2}, -\frac{1}{3}\}, \{\frac{8}{31}, \frac{1}{2}\}, \{8, \frac{256}{31}\}\},$

$\{\{-\frac{4}{3}, -\frac{6}{7}\}, \{-\frac{3}{7}, -\frac{1}{3}\}, \{\frac{9}{31}, \frac{4}{7}\}, \{\frac{64}{7}, \frac{288}{31}\}\},$

$\{\{-\frac{4}{3}, -\frac{5}{7}\}, \{-\frac{5}{14}, -\frac{1}{3}\}, \{\frac{10}{31}, \frac{9}{14}\}, \{\frac{72}{7}, \frac{320}{31}\}\},$

$\{\{-\frac{12288}{511}, -\frac{736}{31}\}, \{-\frac{23}{248}, -\frac{24}{511}\}, \{\frac{7}{15}, \frac{225}{248}\}, \{\frac{225}{31}, \frac{112}{15}\}\},$

$\{\{-\frac{11776}{511}, -\frac{704}{31}\}, \{-\frac{11}{124}, -\frac{23}{511}\}, \{\frac{7}{15}, \frac{113}{124}\}, \{\frac{226}{31}, \frac{112}{15}\}\},$

$\{\{-\frac{11264}{511}, -\frac{672}{31}\}, \{-\frac{21}{248}, -\frac{22}{511}\}, \{\frac{7}{15}, \frac{227}{248}\}, \{\frac{227}{31}, \frac{112}{15}\}\},$

$\{\{-\frac{1536}{73}, -\frac{640}{31}\}, \{-\frac{5}{62}, -\frac{3}{73}\}, \{\frac{7}{15}, \frac{57}{62}\}, \{\frac{228}{31}, \frac{112}{15}\}\},$

$\{\{-\frac{10240}{511}, -\frac{608}{31}\}, \{-\frac{19}{248}, -\frac{20}{511}\}, \{\frac{7}{15}, \frac{229}{248}\}, \{\frac{229}{31}, \frac{112}{15}\}\},$

$\{\{-\frac{9728}{511}, -\frac{576}{31}\}, \{-\frac{9}{124}, -\frac{19}{511}\}, \{\frac{7}{15}, \frac{115}{124}\}, \{\frac{230}{31}, \frac{112}{15}\}\},$

$\{\{-\frac{9216}{511}, -\frac{544}{31}\}, \{-\frac{17}{248}, -\frac{18}{511}\}, \{\frac{7}{15}, \frac{231}{248}\}, \{\frac{231}{31}, \frac{112}{15}\}\},$

$\{\{-\frac{4096}{255}, -16\}, \{-\frac{1}{8}, -\frac{16}{255}\}, \{\frac{7}{15}, \frac{7}{8}\}, \{7, \frac{112}{15}\}\},$

$\{\{-\frac{256}{17}, -\frac{224}{15}\}, \{-\frac{7}{60}, -\frac{1}{17}\}, \{\frac{7}{15}, \frac{53}{60}\}, \{\frac{106}{15}, \frac{112}{15}\}\},$

$\{\{-\frac{3584}{255}, -\frac{208}{15}\}, \{-\frac{13}{120}, -\frac{14}{255}\}, \{\frac{7}{15}, \frac{107}{120}\}, \{\frac{107}{15}, \frac{112}{15}\}\},$

$\{\{-\frac{3328}{255}, -\frac{64}{5}\}, \{-\frac{1}{10}, -\frac{13}{255}\}, \{\frac{7}{15}, \frac{9}{10}\}, \{\frac{36}{5}, \frac{112}{15}\}\},$

$\{\{-\frac{1024}{85}, -\frac{176}{15}\}, \{-\frac{11}{120}, -\frac{4}{85}\}, \{\frac{7}{15}, \frac{109}{120}\}, \{\frac{109}{15}, \frac{112}{15}\}\},$

$\{\{-\frac{2816}{255}, -\frac{32}{3}\}, \{-\frac{1}{12}, -\frac{11}{255}\}, \{\frac{7}{15}, \frac{11}{12}\}, \{\frac{22}{3}, \frac{112}{15}\}\},$

$\{\{-\frac{512}{51}, -\frac{48}{5}\}, \{-\frac{3}{40}, -\frac{2}{51}\}, \{\frac{7}{15}, \frac{37}{40}\}, \{\frac{37}{5}, \frac{112}{15}\}\},$

$\{\{-\frac{3072}{127}, -24\}, \{-\frac{3}{8}, -\frac{24}{127}\}, \{\frac{1}{3}, \frac{5}{8}\}, \{5, \frac{16}{3}\}\},$

$\{\{-\frac{2944}{127}, -\frac{160}{7}\}, \{-\frac{5}{14}, -\frac{23}{127}\}, \{\frac{1}{3}, \frac{9}{14}\}, \{\frac{36}{7}, \frac{16}{3}\}\},$

$\{\{-\frac{2816}{127}, -\frac{152}{7}\}, \{-\frac{19}{56}, -\frac{22}{127}\}, \{\frac{1}{3}, \frac{37}{56}\}, \{\frac{37}{7}, \frac{16}{3}\}\},$

$\{\{-\frac{2048}{127}, -16\}, \{-\frac{1}{4}, -\frac{16}{127}\}, \{\frac{2}{5}, \frac{3}{4}\}, \{6, \frac{32}{5}\}\},$

$\{\{-\frac{1920}{127}, -\frac{104}{7}\}, \{-\frac{13}{56}, -\frac{15}{127}\}, \{\frac{2}{5}, \frac{43}{56}\}, \{\frac{43}{7}, \frac{32}{5}\}\},$

$\{\{-\frac{1792}{127}, -\frac{96}{7}\}, \{-\frac{3}{14}, -\frac{14}{127}\}, \{\frac{2}{5}, \frac{11}{14}\}, \{\frac{44}{7}, \frac{32}{5}\}\},$

$\{\{-\frac{1024}{127}, -8\}, \{-\frac{1}{8}, -\frac{8}{127}\}, \{\frac{7}{15}, \frac{7}{8}\}, \{7, \frac{112}{15}\}\},$

$\{\{-\frac{896}{127}, -\frac{48}{7}\}, \{-\frac{3}{28}, -\frac{7}{127}\}, \{\frac{7}{15}, \frac{25}{28}\}, \{\frac{50}{7}, \frac{112}{15}\}\},$

$\{\{-\frac{768}{127}, -\frac{40}{7}\}, \{-\frac{5}{56}, -\frac{6}{127}\}, \{\frac{7}{15}, \frac{51}{56}\}, \{\frac{51}{7}, \frac{112}{15}\}\},$

$\{\{-\frac{640}{127}, -\frac{32}{7}\}, \{-\frac{1}{14}, -\frac{5}{127}\}, \{\frac{7}{15}, \frac{13}{14}\}, \{\frac{52}{7}, \frac{112}{15}\}\},$

$\{\{-\frac{512}{21}, -24\}, \{-\frac{3}{4}, -\frac{8}{21}\}, \{\frac{2}{15}, \frac{1}{4}\}, \{2, \frac{32}{15}\}\},$

$\{\{-\frac{1280}{63}, -20\}, \{-\frac{5}{8}, -\frac{20}{63}\}, \{\frac{1}{5}, \frac{3}{8}\}, \{3, \frac{16}{5}\}\},$

$\{\{-\frac{1024}{63}, -16\}, \{-\frac{1}{2}, -\frac{16}{63}\}, \{\frac{4}{15}, \frac{1}{2}\}, \{4, \frac{64}{15}\}\},$

$\{\{-\frac{256}{21}, -12\}, \{-\frac{3}{8}, -\frac{4}{21}\}, \{\frac{1}{3}, \frac{5}{8}\}, \{5, \frac{16}{3}\}\},$

$\{\{-\frac{512}{63}, -8\}, \{-\frac{1}{4}, -\frac{8}{63}\}, \{\frac{2}{5}, \frac{3}{4}\}, \{6, \frac{32}{5}\}\},$

$\{\{-\frac{64}{9}, -\frac{20}{3}\}, \{-\frac{5}{24}, -\frac{1}{9}\}, \{\frac{2}{5}, \frac{19}{24}\}, \{\frac{19}{3}, \frac{32}{5}\}\},$

$\{\{-\frac{256}{63}, -4\}, \{-\frac{1}{8}, -\frac{4}{63}\}, \{\frac{7}{15}, \frac{7}{8}\}, \{7, \frac{112}{15}\}\},$

$\{\{-\frac{64}{21}, -\frac{8}{3}\}, \{-\frac{1}{12}, -\frac{1}{21}\}, \{\frac{7}{15}, \frac{11}{12}\}, \{\frac{22}{3}, \frac{112}{15}\}\},$

$\{\{-\frac{448}{31}, -14\}, \{-\frac{7}{8}, -\frac{14}{31}\}, \{\frac{1}{15}, \frac{1}{8}\}, \{1, \frac{16}{15}\}\},$

$\{\{-\frac{384}{31}, -12\}, \{-\frac{3}{4}, -\frac{12}{31}\}, \{\frac{2}{15}, \frac{1}{4}\}, \{2, \frac{32}{15}\}\},$

$\{\{-\frac{320}{31}, -10\}, \{-\frac{5}{8}, -\frac{10}{31}\}, \{\frac{1}{5}, \frac{3}{8}\}, \{3, \frac{16}{5}\}\},$

$\{\{-\frac{256}{31}, -8\}, \{-\frac{1}{2}, -\frac{8}{31}\}, \{\frac{4}{15}, \frac{1}{2}\}, \{4, \frac{64}{15}\}\},$

$\{\{-\frac{192}{31}, -6\}, \{-\frac{3}{8}, -\frac{6}{31}\}, \{\frac{1}{3}, \frac{5}{8}\}, \{5, \frac{16}{3}\}\},$

$\{\{-\frac{128}{31}, -4\}, \{-\frac{1}{4}, -\frac{4}{31}\}, \{\frac{2}{5}, \frac{3}{4}\}, \{6, \frac{32}{5}\}\},$

$\{\{-\frac{64}{31}, -2\}, \{-\frac{1}{8}, -\frac{2}{31}\}, \{\frac{7}{15}, \frac{7}{8}\}, \{7, \frac{112}{15}\}\},$

$\{\{-\frac{24}{7}, -3\}, \{-\frac{3}{4}, -\frac{3}{7}\}, \{\frac{2}{15}, \frac{1}{4}\}, \{2, \frac{32}{15}\}\},$

$\{\{-\frac{16}{7}, -2\}, \{-\frac{1}{2}, -\frac{2}{7}\}, \{\frac{4}{15}, \frac{1}{2}\}, \{4, \frac{64}{15}\}\},$

$\{\{-\frac{8}{7}, -1\}, \{-\frac{1}{4}, -\frac{1}{7}\}, \{\frac{2}{5}, \frac{3}{4}\}, \{6, \frac{32}{5}\}\}, \{\{-\frac{4}{3}, -1\}, \{-\frac{1}{2}, -\frac{1}{3}\}, \{\frac{4}{15}, \frac{1}{2}\}, \{4, \frac{64}{15}\}\},$

$\{\{-\frac{2048}{85}, -\frac{736}{31}\}, \{-\frac{23}{124}, -\frac{8}{85}\}, \{\frac{3}{7}, \frac{101}{124}\}, \{\frac{101}{31}, \frac{24}{7}\}\},$

$\{\{-\frac{5888}{255}, -\frac{704}{31}\}, \{-\frac{11}{62}, -\frac{23}{255}\}, \{\frac{3}{7}, \frac{51}{62}\}, \{\frac{102}{31}, \frac{24}{7}\}\},$

$\{\{-\frac{5632}{255}, -\frac{672}{31}\}, \{-\frac{21}{124}, -\frac{22}{255}\}, \{\frac{3}{7}, \frac{103}{124}\}, \{\frac{103}{31}, \frac{24}{7}\}\},$

$\{\{-\frac{1792}{85}, -\frac{640}{31}\}, \{-\frac{5}{31}, -\frac{7}{85}\}, \{\frac{3}{7}, \frac{26}{31}\}, \{\frac{104}{31}, \frac{24}{7}\}\},$

$\{\{-\frac{1024}{51}, -\frac{608}{31}\}, \{-\frac{19}{124}, -\frac{4}{51}\}, \{\frac{3}{7}, \frac{105}{124}\}, \{\frac{105}{31}, \frac{24}{7}\}\},$

$\{\{-\frac{4864}{255}, -\frac{576}{31}\}, \{-\frac{9}{62}, -\frac{19}{255}\}, \{\frac{3}{7}, \frac{53}{62}\}, \{\frac{106}{31}, \frac{24}{7}\}\},$

$\{\{-\frac{2304}{127}, -\frac{272}{15}\}, \{-\frac{17}{60}, -\frac{18}{127}\}, \{\frac{3}{7}, \frac{43}{60}\}, \{\frac{43}{15}, \frac{24}{7}\}\},$

$\{\{-\frac{2176}{127}, -\frac{256}{15}\}, \{-\frac{4}{15}, -\frac{17}{127}\}, \{\frac{3}{7}, \frac{11}{15}\}, \{\frac{44}{15}, \frac{24}{7}\}\},$

$\{\{-\frac{2048}{127}, -16\}, \{-\frac{1}{4}, -\frac{16}{127}\}, \{\frac{3}{7}, \frac{3}{4}\}, \{3, \frac{24}{7}\}\},$

$\{\{-\frac{1920}{127}, -\frac{224}{15}\}, \{-\frac{7}{30}, -\frac{15}{127}\}, \{\frac{3}{7}, \frac{23}{30}\}, \{\frac{46}{15}, \frac{24}{7}\}\},$

$\{\{-\frac{1792}{127}, -\frac{208}{15}\}, \{-\frac{13}{60}, -\frac{14}{127}\}, \{\frac{3}{7}, \frac{47}{60}\}, \{\frac{47}{15}, \frac{24}{7}\}\},$

$\{\{-\frac{1664}{127}, -\frac{64}{5}\}, \{-\frac{1}{5}, -\frac{13}{127}\}, \{\frac{3}{7}, \frac{4}{5}\}, \{\frac{16}{5}, \frac{24}{7}\}\},$

$\{\{-\frac{1536}{127}, -\frac{176}{15}\}, \{-\frac{11}{60}, -\frac{12}{127}\}, \{\frac{3}{7}, \frac{49}{60}\}, \{\frac{49}{15}, \frac{24}{7}\}\},$

$\{\{-\frac{1408}{127}, -\frac{32}{3}\}, \{-\frac{1}{6}, -\frac{11}{127}\}, \{\frac{3}{7}, \frac{5}{6}\}, \{\frac{10}{3}, \frac{24}{7}\}\},$

$\{\{-\frac{1280}{127}, -\frac{48}{5}\}, \{-\frac{3}{20}, -\frac{10}{127}\}, \{\frac{3}{7}, \frac{17}{20}\}, \{\frac{17}{5}, \frac{24}{7}\}\},$

$\{\{-\frac{512}{21}, -24\}, \{-\frac{3}{4}, -\frac{8}{21}\}, \{\frac{1}{7}, \frac{1}{4}\}, \{1, \frac{8}{7}\}\},$

$\{\{-\frac{1088}{63}, -\frac{120}{7}\}, \{-\frac{15}{28}, -\frac{17}{63}\}, \{\frac{2}{7}, \frac{13}{28}\}, \{\frac{13}{7}, \frac{16}{7}\}\},$

$\{\{-\frac{1024}{63}, -16\}, \{-\frac{1}{2}, -\frac{16}{63}\}, \{\frac{2}{7}, \frac{1}{2}\}, \{2, \frac{16}{7}\}\},$

$\{\{-\frac{320}{21}, -\frac{104}{7}\}, \{-\frac{13}{28}, -\frac{5}{21}\}, \{\frac{2}{7}, \frac{15}{28}\}, \{\frac{15}{7}, \frac{16}{7}\}\},$

$\{\{-\frac{512}{63}, -8\}, \{-\frac{1}{4}, -\frac{8}{63}\}, \{\frac{3}{7}, \frac{3}{4}\}, \{3, \frac{24}{7}\}\},$

$\{\{-\frac{64}{9}, -\frac{48}{7}\}, \{-\frac{3}{14}, -\frac{1}{9}\}, \{\frac{3}{7}, \frac{11}{14}\}, \{\frac{22}{7}, \frac{24}{7}\}\},$

$\{\{-\frac{128}{21}, -\frac{40}{7}\}, \{-\frac{5}{28}, -\frac{2}{21}\}, \{\frac{3}{7}, \frac{23}{28}\}, \{\frac{23}{7}, \frac{24}{7}\}\},$

$\{\{-\frac{416}{31}, -\frac{40}{3}\}, \{-\frac{5}{6}, -\frac{13}{31}\}, \{\frac{1}{7}, \frac{1}{6}\}, \{\frac{2}{3}, \frac{8}{7}\}\},$

$\{\{-\frac{384}{31}, -12\}, \{-\frac{3}{4}, -\frac{12}{31}\}, \{\frac{1}{7}, \frac{1}{4}\}, \{1, \frac{8}{7}\}\},$

$\{\{-\frac{256}{31}, -8\}, \{-\frac{1}{2}, -\frac{8}{31}\}, \{\frac{2}{7}, \frac{1}{2}\}, \{2, \frac{16}{7}\}\},$

$\{\{-\frac{128}{31}, -4\}, \{-\frac{1}{4}, -\frac{4}{31}\}, \{\frac{3}{7}, \frac{3}{4}\}, \{3, \frac{24}{7}\}\}$,
$\{\{-\frac{96}{31}, -\frac{8}{3}\}, \{-\frac{1}{6}, -\frac{3}{31}\}, \{\frac{3}{7}, \frac{5}{6}\}, \{\frac{10}{3}, \frac{24}{7}\}\}$,
$\{\{-\frac{32}{5}, -6\}, \{-\frac{3}{4}, -\frac{2}{5}\}, \{\frac{1}{7}, \frac{1}{4}\}, \{1, \frac{8}{7}\}\}, \{\{-\frac{64}{15}, -4\}, \{-\frac{1}{2}, -\frac{4}{15}\}, \{\frac{2}{7}, \frac{1}{2}\}, \{2, \frac{16}{7}\}\}$,
$\{\{-\frac{32}{15}, -2\}, \{-\frac{1}{4}, -\frac{2}{15}\}, \{\frac{3}{7}, \frac{3}{4}\}, \{3, \frac{24}{7}\}\}, \{\{-\frac{4}{3}, -1\}, \{-\frac{1}{2}, -\frac{1}{3}\}, \{\frac{2}{7}, \frac{1}{2}\}, \{2, \frac{16}{7}\}\}$,
$\{\{-\frac{3072}{127}, -\frac{736}{31}\}, \{-\frac{23}{62}, -\frac{24}{127}\}, \{\frac{1}{3}, \frac{39}{62}\}, \{\frac{39}{31}, \frac{4}{3}\}\}$,
$\{\{-\frac{2944}{127}, -\frac{704}{31}\}, \{-\frac{11}{31}, -\frac{23}{127}\}, \{\frac{1}{3}, \frac{20}{31}\}, \{\frac{40}{31}, \frac{4}{3}\}\}$,
$\{\{-\frac{2816}{127}, -\frac{672}{31}\}, \{-\frac{21}{62}, -\frac{22}{127}\}, \{\frac{1}{3}, \frac{41}{62}\}, \{\frac{41}{31}, \frac{4}{3}\}\}$,
$\{\{-\frac{1280}{63}, -\frac{304}{15}\}, \{-\frac{19}{30}, -\frac{20}{63}\}, \{\frac{1}{3}, \frac{11}{30}\}, \{\frac{11}{15}, \frac{4}{3}\}\}$,
$\{\{-\frac{1216}{63}, -\frac{96}{5}\}, \{-\frac{3}{5}, -\frac{19}{63}\}, \{\frac{1}{3}, \frac{2}{5}\}, \{\frac{4}{5}, \frac{4}{3}\}\}$,
$\{\{-\frac{128}{7}, -\frac{272}{15}\}, \{-\frac{17}{30}, -\frac{2}{7}\}, \{\frac{1}{3}, \frac{13}{30}\}, \{\frac{13}{15}, \frac{4}{3}\}\}$,
$\{\{-\frac{1088}{63}, -\frac{256}{15}\}, \{-\frac{8}{15}, -\frac{17}{63}\}, \{\frac{1}{3}, \frac{7}{15}\}, \{\frac{14}{15}, \frac{4}{3}\}\}$,
$\{\{-\frac{1024}{63}, -16\}, \{-\frac{1}{2}, -\frac{16}{63}\}, \{\frac{1}{3}, \frac{1}{2}\}, \{1, \frac{4}{3}\}\}$,
$\{\{-\frac{320}{21}, -\frac{224}{15}\}, \{-\frac{7}{15}, -\frac{5}{21}\}, \{\frac{1}{3}, \frac{8}{15}\}, \{\frac{16}{15}, \frac{4}{3}\}\}$,
$\{\{-\frac{128}{9}, -\frac{208}{15}\}, \{-\frac{13}{30}, -\frac{2}{9}\}, \{\frac{1}{3}, \frac{17}{30}\}, \{\frac{17}{15}, \frac{4}{3}\}\}$,
$\{\{-\frac{832}{63}, -\frac{64}{5}\}, \{-\frac{2}{5}, -\frac{13}{63}\}, \{\frac{1}{3}, \frac{3}{5}\}, \{\frac{6}{5}, \frac{4}{3}\}\}$,
$\{\{-\frac{256}{21}, -\frac{176}{15}\}, \{-\frac{11}{30}, -\frac{4}{21}\}, \{\frac{1}{3}, \frac{19}{30}\}, \{\frac{19}{15}, \frac{4}{3}\}\}$,
$\{\{-\frac{320}{31}, -\frac{72}{7}\}, \{-\frac{9}{14}, -\frac{10}{31}\}, \{\frac{1}{3}, \frac{5}{14}\}, \{\frac{5}{7}, \frac{4}{3}\}\}$,
$\{\{-\frac{288}{31}, -\frac{64}{7}\}, \{-\frac{4}{7}, -\frac{9}{31}\}, \{\frac{1}{3}, \frac{3}{7}\}, \{\frac{6}{7}, \frac{4}{3}\}\}$,
$\{\{-\frac{256}{31}, -8\}, \{-\frac{1}{2}, -\frac{8}{31}\}, \{\frac{1}{3}, \frac{1}{2}\}, \{1, \frac{4}{3}\}\}$,
$\{\{-\frac{224}{31}, -\frac{48}{7}\}, \{-\frac{3}{7}, -\frac{7}{31}\}, \{\frac{1}{3}, \frac{4}{7}\}, \{\frac{8}{7}, \frac{4}{3}\}\}$,
$\{\{-\frac{192}{31}, -\frac{40}{7}\}, \{-\frac{5}{14}, -\frac{6}{31}\}, \{\frac{1}{3}, \frac{9}{14}\}, \{\frac{9}{7}, \frac{4}{3}\}\}$,
$\{\{-\frac{64}{15}, -4\}, \{-\frac{1}{2}, -\frac{4}{15}\}, \{\frac{1}{3}, \frac{1}{2}\}, \{1, \frac{4}{3}\}\}, \{\{-\frac{16}{7}, -2\}, \{-\frac{1}{2}, -\frac{2}{7}\}, \{\frac{1}{3}, \frac{1}{2}\}, \{1, \frac{4}{3}\}\}\}$