Modeling Weak Gravitational Lensing Through C++

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Chris is a junior majoring in Physics with a minor in Chemistry. His research was funded with an Adrian Tinsley Program summer grant under the mentorship of Dr. Thomas Kling of the Physics department. This work was also accepted for presentation at the 2009 National Conference of Undergraduate Research. Chris plans to attend graduate school in Applied Physics.

Gravitational Lensing has become an integral part of astrophysics and the study of matter in the universe. In weak gravitational lensing, an object appears distorted when viewed from the observer's perspective. This is caused by the bundle of light being distorted by an object of considerable mass. Because the distance to each part of this bundle is not the same, the rays of light are affected differently causing a shearing and magnification of the image. This alteration in the image we see can be modeled using the thin lens approximation. By integrating the equations for light travel based on general relativity, a more accurate model can be created. Over the summer of 2008, my mentor and I have worked on creating a program through this method. By coding numerous mathematical operations and rewriting existing code to support a new coordinate system we have created a full program.

I. Introduction

In space, there is a phenomenon that occurs when an object of exceptionally large clump of matter is positioned in the direct line of sight between an observer and a source of light. Under these circumstances, the gravitational pull of the matter will alter the path taken by light. The resulting path is deflected from its initial straight path into and curves from the source to the observer. This unique event is known as Gravitational Lensing.

There are two types of gravitational lensing: strong lensing and weak lensing. Strong gravitational lensing is the effect caused when the interfering matter has a very large gravitational pull, and is situated close to the source. The gravitational pull is so strong that it causes multiple images to appear around the area of the lens. It can also cause the appearance of an Einstein ring, a distortion of a set of images into a connected ring centered on the lens. Regardless of what happens to the image, the results are of such a magnitude that the distortion itself can be used in calculations.

Our research involves the second type of gravitation lensing called weak lensing. Weak gravitational lensing occurs when the large mass near the line of sight. The light rays under normal circumstances would travel straight to the observer, but with a lens so close to the line of sight the rays are curved into a different direction, while other rays are bent towards the observer. The resulting image that arrives at the observer can be affected in a variety of ways. In some cases, it may be elongated and warped. In other cases it may be magnified, or appear in a different location than it should. The biggest

difference between the distortions of weak lensing and strong lensing is that weak lensing causes altering of the image that are not significant enough to make calculations with. Furthermore, they may be so slight that without knowledge of its initial state, it is impossible to tell where and how it was affected. Therefore, in weak gravitational lensing, we must generally use statistical analyses of many observed objects.

The basic purpose of our research is to model weak gravitational lensing exactly for the first time– with none of the usual approximations. To do this, we will write a computer program in c++ that will model this phenomenon. By making use of the Runge-Kutta Adaptive Step Size process (RFK Step), we can integrate a differential equation to find the path of the light. This particular method of calculating the location of light as it travels is superior to the current technique being used, which only calculates it at three points in its path.

In addition, our program will also need to account for the distortion of the light ray as it is traveling closer and farther from the lensing mass. To do this, we consider the fact that light travels in curved space during gravitational lensing. As such, the difference in the paths that light rays take can be described by the geodesic deviation equation. By coding this into our program, we can calculate how the image is affected.

II. General Relativity

In General Relativity, there are specific equations that describe the path a ray of light travels. Through the use of the calculus of variations, the geodesic equations for light can be derived from these equations. In short, geodesic equations describe the shortest path between two points, while accounting for any factors that affect the distance. For example, the geodesic describing the shortest path between two points on the surface of a sphere will account for the curved surface it is on. It is these equations that will be used to find the length of the path, and the gravitational potential of the lensing object.

Our goal in this project is to determine the accuracy of the current approach. This approach, known as the thin lens approximation, considers the area where light is affected by the lens to be so thin that it is essentially its namesake, a thin lens. The effect gravity has on light is considered to occur only within a small area where the lens is considered to be, and has a much larger effect than normal to keep it accurate for the small area. Rays of light are considered to travel undisturbed until they reach this particular area and are then redirected, as if shining a light through a lens made of glass. The advantage this has is that the calculation will be solved geometrically. The problem with it is that in actuality, the light is affected the entire time it is within range of the lensing object, and so it changes continually until it leaves the influence of the lensing object. Since the path changes so constantly, it makes sense to use integration to account for the changes. This is where the RFK Step program comes into play.

The idea behind the step size method is that before the light is within the range of the gravitational field, it travels with very little change. Then, when the rays come within range of the field, their paths change drastically. Once it leaves the gravitational field, the path is then changes very little again. The advantage of the adaptive step size is that when it integrates it takes two steps, followed by one large step that should be equal to the first two, and then calculates the difference between them. If the difference is within a predefined range, the program will consider it a good step and proceeds to take another of the same size. If it is outside of the range, the program then cuts down the step sizes and repeats the process. This allows for the program to quickly integrate the consistent areas the light travels, and then accurately integrate the areas where the rays are affected by gravity by cutting its step size down.

This method will calculate the change at all points along the geodesic, as opposed to only within a small area. This approach is closer to the actual process that light undergoes, and should yield more accurate results when making calculations. In addition to the path of light, we are creating code that describes the distortion of the light rays as they travel.

The distortion in the image is caused by different particles of light in these rays being pulled away from each other. The result is that the ray itself extends and rotates as the lens pulls on it. Mathematically, this distortion of light is called geodesic deviation. We model it by creating vectors that travel with the ray as it progresses through the geodesic. These vectors describe how two rays of light move in tandem with one another as time passes. By including them in the program, we will be able to examine how the rays of light change before they reach the observer. For the mathematical equations, please refer to the appendix.

III. Computer Code

Our program main file is similar to a previous program that was written by Dr. Kling, with some adjustments made to it. The basis of the old program is that it uses the RFK- Adaptive Step size method to integrate along the light path from the observer to the source of light. We have changed it to the more appropriate scenario for this study of integrating from the source to the observer. The program is designed to output the angle theta, between the observer and the apparent source, and the bisection value. The next change we made was to the

coordinate system. It is necessary for us to calculate in a system of three coordinates: φ, r, and t.

To calculate the geodesic of the light ray and its root bisection value, we use the calculus of variations. Basically, we take an equation known as the Lagrangian, and we insert it into the Euler-Lagrange equation. This calculation is coded using the RFK Step size method. The program calculates the gravitational potential and the root bisection value, outputting both for the user. In order to use the RKF step process, we coded the necessary derivatives from the calculus of variations into functions for use in the integration file. This is necessary because the program cannot easily calculate multiple variables at once. The functions find specific variables based on other initial constants input by the user. In addition, we have coded the program that calls these functions to find the amount of error generated when taking the steps.

The next alteration to be coded was the creation of a function that calculates the geodesic deviation. By using the geodesic deviation equation, we can account for the distortion of light rays as they travel through space. This program is called by the main file as it runs a for loop. The bisection and angle values already calculated by the program are passed into the function to ensure that when the geodesic deviation is calculated, it is for the same ray of light we are examining. After this, the for loop causes the program to repeat its actions. Before combining the two parts into one program, there was one more file to be coded. We needed to create four functions that would be used to calculate the deviation with predetermined values.

Once complete, our program will integrate the geodesic of a specific light ray and output the angle between the observer's line of sight to the lens and the source of light. It will also output a bisection value. These values will then be passed into a function that calculates the geodesic deviation of that particular light ray, and will output the result of that calculation onto the screen. This completes one loop in the main file, and the program will change the position of the source and make the same calculation again, up to ten times.

IV. Mathematical Equations

The Lagrangian (1.1) is defined in the (φ, r, t) coordinates system, and is inserted into the Euler-Lagrange Equation (1.2).

$$L = \frac{1}{2}\left(\left(1+2\varphi\right)\varsigma^2 - \left(1-2\varphi\right)\left(r^2 + r^2\phi^2\right)\right) \qquad (0.1)$$

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \qquad (0.2)$$

Because computer code cannot solve for more than one variable, it was necessary for us to create the following four functions (1.3 – 1.6).

$$\phi = \frac{b}{r^2}\left(1+2\varphi\right) \qquad (0.3)$$

$$\varsigma = \frac{b}{r^2}\left(1+2\varphi\right) \qquad (1.4)$$

$$r = -2\varphi_{jr} + 2\left(1+2\varphi\right)\varphi_{jr}r^2 + \frac{b^2}{r^3}\left(1+4\varphi\right) \qquad (0.4)$$

$$r = v_{r_o} = \sqrt{1-\frac{b^2}{r^2}\left(1+4\varphi\right)} \qquad (0.5)$$

These functions are parts of the derivative calculation that are called by the RFK step program where at different points in its operation to take a complete derivative.

The geodesic deviation equation is calculated using the Riemann Curvature Tensor. The Riemann Curvature Tensor is made up of first and second derivatives of the gravitational potential. In our study, we used the truncated NFW potential introduced by Baltz et al.:

$$\varphi = \left(\frac{GM_0}{r_s}\right)\left(\frac{\tau^2}{\left(1+\tau^2\right)^2}\right)\times F \qquad (2.1)$$

where

$$\begin{aligned} F = {} & \arctan\left(x/\tau\right)\left(\frac{1}{\tau} - \tau - 2\frac{\tau}{x}\right) \\ & + \ln\left(\frac{1+\left(x/\tau\right)^2}{\left(1+x\right)^2}\right)\left(\frac{\tau^2-1}{2x} - 1\right) \\ & + \frac{\pi\left(\tau^2-1\right)}{2\tau} - 2\ln\tau \end{aligned}$$

where $x = r / r_s$ and $\tau = r_t / r_s$, for the radial coordinate r, and the "scale radius" rs and "tidal radius" rt.

After taking the x, y, and z derivatives (dx, dy, dz) we are left with the Reimann Curvature Tensor, which indicates the presence of spacetime curvature caused by a massive lens.

$$R^a_{bcd} = g^{ga} R_{abcd}$$

The following equations show the components of the Reimann Tensor:

$$ds^2 = (1 + 2\varphi)dt^2 - (1 - 2\varphi)(dx^2 + dy^2 + dz^2) \tag{2.2}$$

$$R_{oioi} = -\varphi_{ii} \tag{2.3}$$

$$R_{oioj} = -\varphi_{ij} \tag{2.4}$$

$$R_{ijij} = -\varphi_{ii} - \varphi_{jj} \tag{2.5}$$

$$R_{ijik} = -\varphi_{jk} \tag{2.6}$$

Where $j \neq k$

After computing each Riemann tensor term, we used the geodesic deviation equation, which explains how a vector Y changes as it is carried along the light rays. The geodesic deviation equation for Y is a second order differential equation,

$$\ddot{Y}^a = -R^a_{bcd}\ {}^b\ {}^c Y^d\ ,$$

where here we are using the standard Einstein Summation Convention of general relativity. When we write out the full equations, using the definitions of the Riemann tensor terms above, we get four, very complicated differential equations (3.1 – 3.4), which together will calculate the geodesic deviation.

$$\begin{aligned}
\ddot{Y}^o =& -\varphi_{xx} l^x l^o Y^x + \varphi_{xx} l^x l^x Y^o - \varphi_{yy} l^y l^o Y^y + \varphi_{yy} l^y l^y Y^o - \varphi_{zz} l^z l^o Y^z + \varphi_{zz} l^z l^z Y^o \\
& -\varphi_{xy} l^x l^o Y^y + \varphi_{xy} l^x l^y Y^o - \varphi_{xz} l^o l^x Y^z + \varphi_{xz} l^x l^z Y^o - \varphi_{yz} l^y l^o Y^z + \varphi_{yz} l^y l^z Y^o \\
& -\varphi_{vx} l^y l^o Y^x + \varphi_{vx} l^y l^x Y^o - \varphi_{zx} l^z l^o Y^x + \varphi_{zx} l^z l^x Y^o - \varphi_{zv} l^z l^o Y^y + \varphi_{zv} l^z l^y Y^o
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
\ddot{Y}^x =& \varphi_{xx} l^o l^x Y^o - \varphi_{xx} l^o l^o Y^x + \varphi_{xy} l^o l^y Y^o - \varphi_{xy} l^o l^o Y^y + \varphi_{xz} l^o l^z Y^o - \varphi_{xz} l^o l^o Y^z \\
& +(\varphi_{xx} + \varphi_{yy}) l^y l^x Y^y - (\varphi_{xx} + \varphi_{yy}) l^y l^y Y^x + (\varphi_{xx} + \varphi_{zz}) l^z l^x Y^z - (\varphi_{xx} + \varphi_{zz}) l^z l^z Y^x \\
& +\varphi_{yz} l^y l^x Y^z - \varphi_{yz} l^y l^z Y^x + \varphi_{zy} l^z l^x Y^y - \varphi_{zy} l^z l^y Y^x - \varphi_{xz} l^y l^y Y^z + \varphi_{xz} l^y l^z Y^y \\
& -\varphi_{xv} l^z l^z Y^y + \varphi_{xv} l^z l^y Y^z
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\ddot{Y}^y =& \varphi_{yy} l^o l^y Y^o - \varphi_{yy} l^o l^o Y^y + \varphi_{yx} l^o l^x Y^o - \varphi_{yx} l^o l^o Y^x + \varphi_{yz} l^o l^z Y^o - \varphi_{yz} l^o l^o Y^z \\
& +(\varphi_{yy} + \varphi_{xx}) l^x l^y Y^x - (\varphi_{yy} + \varphi_{xx}) l^x l^x Y^y + (\varphi_{yy} + \varphi_{zz}) l^z l^y Y^z - (\varphi_{yy} + \varphi_{zz}) l^z l^z Y^y \\
& +\varphi_{xz} l^x l^y Y^z - \varphi_{xz} l^x l^z Y^y + \varphi_{zx} l^z l^y Y^x - \varphi_{zx} l^z l^x Y^y - \varphi_{yx} l^x l^x Y^z + \varphi_{yz} l^x l^z Y^x \\
& -\varphi_{vx} l^z l^z Y^x + \varphi_{vx} l^z l^x Y^z
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\ddot{Y}^z =& \varphi_{zx} l^o l^x Y^o - \varphi_{zx} l^o l^o Y^x + \varphi_{zy} l^o l^y Y^o - \varphi_{zy} l^o l^o Y^y + \varphi_{zz} l^o l^z Y^o - \varphi_{zz} l^o l^o Y^z \\
& +(\varphi_{zz} + \varphi_{xx}) l^x l^z Y^x - (\varphi_{zz} + \varphi_{xx}) l^x l^x Y^z + (\varphi_{zz} + \varphi_{yy}) l^y l^z Y^y - (\varphi_{zz} + \varphi_{yy}) l^y l^y Y^z \\
& +\varphi_{xy} l^x l^z Y^y - \varphi_{xy} l^x l^y Y^z + \varphi_{yx} l^y l^z Y^x - \varphi_{yx} l^y l^x Y^z - \varphi_{zy} l^x l^x Y^y + \varphi_{zy} l^x l^y Y^x \\
& -\varphi_{zx} l^y l^y Y^x + \varphi_{zx} l^y l^x Y^y
\end{aligned} \tag{3.4}$$

The subscripts represent partial derivatives, while the superscripts represent vectors in the matrix. Each of these equations describes a particular component of the geodesic deviation. For example the third equation describes the amount of change along the y direction of two rays of light.

In order to find the potential we used, it was necessary to derive the numerical derivative for use in the computer code. The most accurate way to do this is to find the Taylor Series expansion.

$$f(x+\varsigma) = f(x) + f'(x)\varsigma + \frac{1}{2} f''(x)\varsigma^2$$

$$f(x-\varsigma) = f(x) - f'(x)\varsigma + \frac{1}{2} f''(x)\varsigma^2$$

By combining these two expansions, we are left with the derivative we need.

$$f''(x) = \frac{f(x+\varsigma) + f(x-\varsigma) - 2f(x)}{\varsigma^2}$$

We have also taken a second expansion resulting in four equations.

$$f(x+\varsigma, y+\varsigma) = f + f_x\varsigma + f_y\varsigma + \frac{1}{2} f_{xx}\varsigma^2 + \frac{1}{2} f_{yy}\varsigma^2 + f_{xy}\varsigma^2$$

$$f(x+\varsigma, y-\varsigma) = f + f_x\varsigma - f_y\varsigma + \frac{1}{2} f_{xx}\varsigma^2 + \frac{1}{2} f_{yy}\varsigma^2 - f_{xy}\varsigma^2$$

$$f(x-\varsigma, y+\varsigma) = f - f_x\varsigma + f_y\varsigma + \frac{1}{2} f_{xx}\varsigma^2 + \frac{1}{2} f_{yy}\varsigma^2 - f_{xy}\varsigma^2$$

$$f(x-\varsigma, y-\varsigma) = f - f_x\varsigma - f_y\varsigma + \frac{1}{2} f_{xx}\varsigma^2 + \frac{1}{2} f_{yy}\varsigma^2 + f_{xy}\varsigma^2$$

The subscripts once again represent partial derivatives. These equations were combined in the same manner as before, resulting in:

$$4 f_{xy} \varsigma^2$$

V. Research Activities

Over the summer, we have made a great amount of progress toward our goals. There were many tasks we set about in order to allow us to use the most accurate method to model thin lensing. The following is a table of completed objectives for the research.

Our project is in the final stages of the process we set out to achieve. Presently, we have finished combining each segment of code we have written into a complete computer code. While each particular segment ran properly before combining them, the complete file does not function correctly. When we manage to run the file, it outputs a geodesic deviation that is too large to be correct. Because of this, we must search through the code to find the error that is causing the problem. This error could be a simple bug in the program, or it could be as serious as a conceptual error in the calculations. As such, the future of this project is to find the source of the error in this computer code and fix it.

Works Cited

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Item Description	Status
Introduced more physical truncated NFW potential	Complete
Changed equations of motion to be time independent	Complete
Changed from integrating backwards in time to forwards	Complete
Confirmed code working to find true path	Complete
Coded the equations for geodesic deviation	Coded and checked with maple
Derived and wrote code for numerical derivatives	Complete
Modified RKF code to integrate path and geodesic deviation	Code written, tested, but not fully implemented
Examine shape of objects with geodesic vector	Incomplete