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Analysis and Dynamics of Laser Models

Kassaundra Przelomski

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Kassaundra Przelomski

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______________________________ ____________
Dr. Laura K. Gross, Thesis Director Date

______________________________ ____________
Dr. T. Kling, Committee Member Date

______________________________ ____________
Dr. E. Deveney, Committee Member Date

______________________________ ____________
Dr. U. Shama, Committee Member Date

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By

Kassaundra Przelomski

Bridgewater State University

Bridgewater, MA

May 12, 2015

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Kassaundra Przelomski

Bridgewater State University
Bridgewater, MA
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ABSTRACT

Lasers produce a strong focused ray of light that is useful in many industries, including medicine and technology. For example, they can be used for guiding surgical instruments or playing compact discs. In this thesis I characterize the dynamics of a fundamental laser model. A mathematical goal is to determine the number of photons in the laser cavity as time evolves. Different choices of parameter values produce qualitatively different behaviors of the system. I perform a bifurcation analysis in order to capture the different physical situations, identifying equilibrium solutions and their stability properties. I sketch representative solutions using the bifurcation points as a guide. Some parameter regimes produce a stable equilibrium solution, meaning that the laser action persists given any nearby initial conditions. In other situations the same equilibrium solution is unstable, while a zero solution is stable, meaning the number of photons tends to zero. A model consisting of a system of differential equations is also presented and analyzed. Both models predict the behavior of the laser.

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Chapter 1
Introduction

1.1 What is a Laser?

LASER is the acronym for Light Amplification by Stimulated Emission of Radiation. A laser is set up in a gain medium where the property of the material allows a particular wavelength of light to be amplified. In a two energy state laser the electrons have two options, either the ground state or one excited state. Two energy levels are depicted in Figure 1.1.

In the figure photons enter the active region of a laser from the left they can be resonantly absorbed by the atom (stimulated absorption). Electrons are excited to the higher level , in a two state model the higher level is the 2nd state, conserving energy. The atom with an electron in an excited state has a lifetime on the order of $10^{-}14$ seconds. After this point the atom relaxes back to the electron in the ground state but in this case to conserve energy a photon is emitted. This process is called spontaneous emission. It is possible to accumulate a net number of atoms with electrons in the excited state and this is called population inversion. In this inverted state, a resonant energy photon can stimulate the atom to emit a photon (stimulated emission instead of stimulated absorption) that has the same energy and phase (coherent) as the phonon that stimulated it. By enclosing this active region where there is inversion between mirrors, a single spontaneously emitted photon can be reflected from the mirror to stimulate the production of a 2nd coherent photon and then they in turn encounter the mirror to get

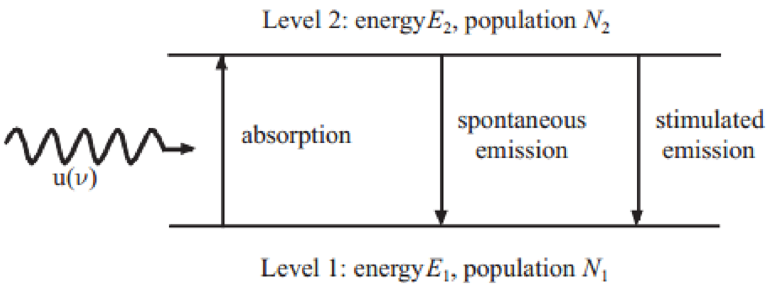

Fig. 1.1: Quantum mechanical view of a laser. From "Atomic and Laser Physics," A.M. Fox, 2014.

reflected back once again this time stimulating 2 more coherent photons and so on. In this process with the photons at the speed of light an exponential buildup of coherent photons is produced and this is what produces a laser beam. The process is described through suggestions provided from Dr. Edward Deveney.

1.2 History and Importance

The laser idea was a fundamental one that was first talked about by Einstein in 1917 through one of his papers(Milonni & Eberly, 1988). Even though he had these ideas about what would become a laser he didn't know what he had created using stimulated emission. Theodore H. Maiman, a physicist at Hughes Research Laboratories made the laser first in 1960(Milonni & Eberly, 1988).In 1961 the first continuous Helium Neon Laser was successfully used and recorded.

We have since moved on from what was the first idea behind what a laser could do. Now we have more advanced lasers that are able to perform tasks that advance our scientific knowledge. From the initial advancement of producing lasers we have incorporated great advancements to our everyday lives, for example in medical devices and compact discs (Milonni & Eberly, 1988).

1.3 Mathematical Analysis

In section 1.1, we discussed that three or four level lasers are easier to build and maintain than simple two level lasers. Even so, the underlying mathematical representation of the laser action is essentially the same regardless of the number of laser levels (Fox, 2014). The critical part is that some system excites the electrons in the medium to a higher energy level, and then photons in the cavity are emitted from those excited atoms through stimulated emission(Fox, 2014). Regardless, the underlying mathematics and analysis is basically the same, so in this paper we will examine a two state laser.

The quantum mechanical analysis of these systems can be very complicated, but an approach that uses the techniques of non-linear dynamics can simplify the process of understanding how a laser functions. In this thesis, I first discuss what a dynamical system is and then we introduce a simplified version of a laser system. Section 4 concludes with a discussion of a more realistic laser model.

Chapter 2

Dynamical Systems

When solving and analyzing ordinary differential equations, the solution which you compute can be difficult to understand. For example, consider the differential equation:

$$\dot{x} = \sin x \text{ where } \dot{x} = \frac{dx}{dt} \tag{2.1}$$

with the solution:

$$t = -\ln|\csc(x) + \cot(x)| + c. \tag{2.2}$$

Analyzing equation 2.2 is very difficult. For this reason we can analyze flows of first order equations as in Figure 2.1 which represents the differential equation of $\dot{x} = \sin x$. We were able to easily plot x-$\dot{x}$ because the graph of $\sin(x)$ is a trivial graph. Using the trivial graph we analyze flows of first order differential equations to analyze the dynamics.

2.1 Flows

We first define a general first order system as:

$$\dot{x_n} = f_n(x_1, \cdots, x_n). \tag{2.3}$$

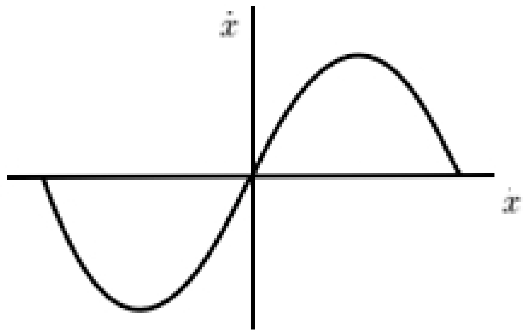

Fig. 2.1: Graph of $\dot{x} = \sin x$.

We use this to visualize the flow through n-dimensional phase space. A simple case to look at is $n = 1$, where

$$\dot{x} = \frac{dx}{dt} = f(x). \tag{2.4}$$

The $f(x)$ is a real-valued function of x. Equation 2.4 is called a one-dimensional first order equation.

2.2 Fixed Points

When we set the derivative term of equation 2.4 to zero, we find values of x that are unchanging in time as follows:

$$\dot{x} = f(x) = 0. \tag{2.5}$$

These equilibrium solutions are called fixed points x^*. We will show the fixed points as x-intercepts in the x-$\dot{x}$ plane in the discussion below.

Stable Fixed Point: If $\dot{x}$ decreases from positive to negative as we increase x across the fixed point x^*, then the fixed point is stable. Such a point on the graph in the x-$\dot{x}$ plane is denoted by a solid black circle. See Figure 2.2. The flow directed towards to the fixed point x^* as shown in Figure 2.2.

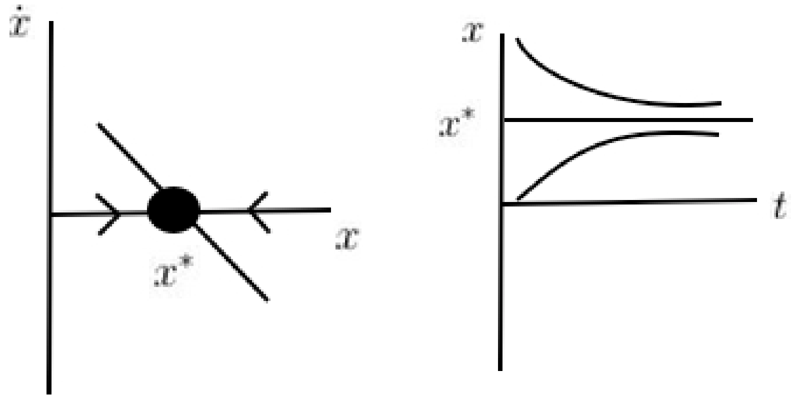

Fig. 2.2: Stable fixed point at $x = x^*$ in the x-$\dot{x}$ plane (left) and t-x plane (right).

We also see the attraction to the stable fixed point in the $t - x$ plane on the right side of Figure 2.2. Note that the trajectory with $x(0)$ greater than x^* has negative slope $\dot{x}$. The trajectory below the line $x(t) = x^*$ (the trajectory with $x(0)$ less than x^*) has positive slope $\dot{x}$, consistent with the left side of Figure 2.2.

Unstable Fixed Point: If $\dot{x}$ increases from negative to positive as we increase x across the fixed point x^*, then the fixed point is unstable. Such a point on the graph in the x-$\dot{x}$ plane is denoted by a empty black circle. See left side of Figure 2.3. The flow on the sides of the point is repelled from the fixed point x^* as shown in left side of Figure 2.3.

We also see the repulsion from the fixed point in the right side of Figure 2.3. Note that the trajectory with $x(0)$ greater than x^* has positive slope $\dot{x}$. The trajectory below the line $x(t) = x^*$ (the trajectory with $x(0)$ less than x^*) has negative slope $\dot{x}$, consistant with the left side of Figure 2.3.

Metastable: If $\dot{x}$ has the same sign for x to the left and right of the fixed point x^*, then the fixed point is metastable. The flow on one side acts like the fixed point is

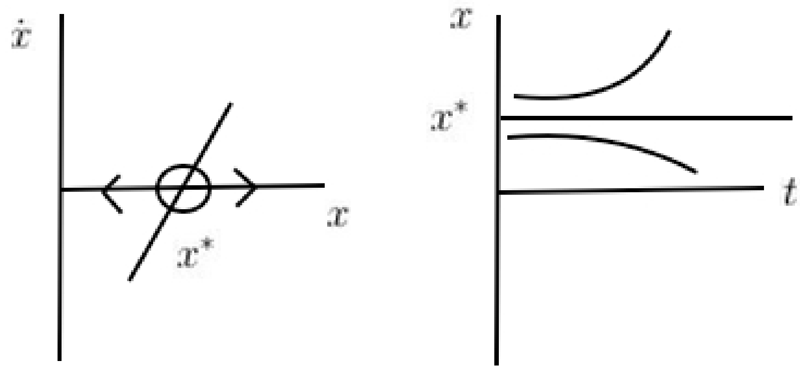

Fig. 2.3: Unstable fixed point at $x = x^*$ in the x-$\dot{x}$ plane (left) and t-x plane (right).

stable and on the other side the flow acts like the fixed point is unstable. Such a point on the graph in the x-$\dot{x}$ plane is denoted by a circle that is half empty, half solid. See, for example, the left side of Figure 2.4. The flow on the sides of the point is repelled from the fixed point x^* on one side and attracted in the other. As shown in Figure 2.4.

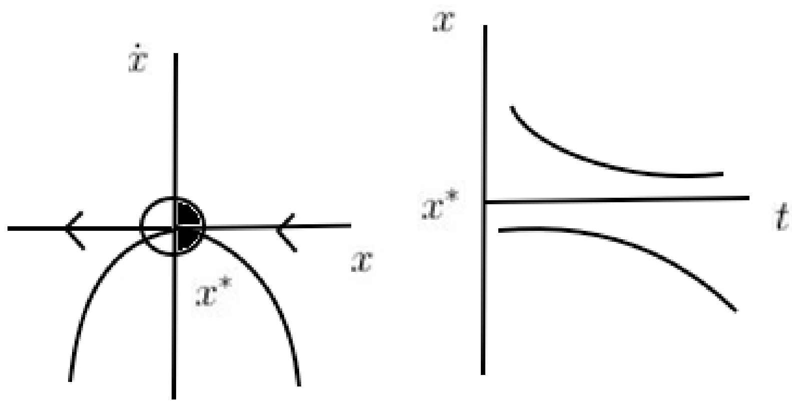

Fig. 2.4: Metastable fixed point at $x = x^*$ in the x-$\dot{x}$ plane (left) and t-x plane (right).

We also see the repulsion/attraction from the fixed point in the right side of Figure 2.4. Note that the trajectory with $x(0)$ greater than x^* has negative slope $\dot{x}$. The trajectory below the line $x(t) = x^*$ (the trajectory with $x(0)$ less than x^*) also has negative slope $\dot{x}$.

2.3 Example

Consider a particular example of the first-order system in equation 2.4 as follows:

$$\dot{x} = rx - x^2. \tag{2.6}$$

We find the fixed points as in section 2.2. In particular, we solve

$$rx - x^2 = x(r - x) = 0$$

to find the fixed points

$$x = 0, \quad x = r.$$

As we will see below, these are the x-intercepts of the graph of equation 2.6 in the x-$\dot{x}$ plane. Consider these three cases:

Case r > 0: Figure 2.5 shows the graph of equation 2.6 for $r > 0$. Note that the figure shows $\dot{x}$ decreasing from positive to negative as we increase x across the fixed point $x = r$. As such the fixed point is stable. The point $(r, 0)$ on the graph in the x-$\dot{x}$ plane is denoted by a solid black circle.

Figure 2.5 also shows $\dot{x}$ increasing from negative to positive as we increase x across the fixed point $x = 0$. As such the fixed point is unstable. The point $(0, 0)$ on the graph in the x-$\dot{x}$ plane is denoted by an empty circle.

Case r = 0: If $r = 0$, then equation 2.6 is

$$\dot{x} = -x^2,$$

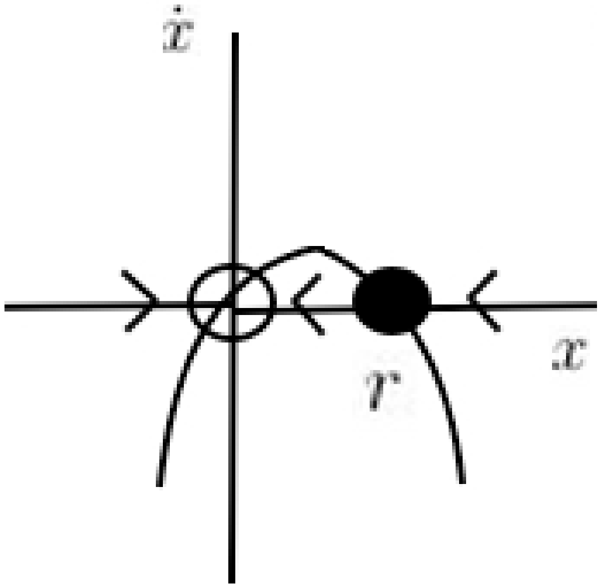

Fig. 2.5: Stable fixed point at $x = r$, unstable fixed point at $x = 0$.

and the fixed point $x = 0$ is metastable as shown in Figure 2.4 and discussed in the previous section.

Case r < 0: Figure 2.6 shows the graph of equation 2.6 for $r < 0$. Note that the figure shows $\dot{x}$ increasing from negative to positive as we increase x across the fixed point $x = r$. As such the fixed point is unstable. The point $(r, 0)$ on the graph in the x-$\dot{x}$ plane is denoted by an empty circle.

Figure 2.6 also shows $\dot{x}$ decreasing from positive to negative as we increase x across the fixed point $x = 0$. As such the fixed point is stable. The point $(0, 0)$ on the graph in the x-$\dot{x}$ plane is denoted by a solid black circle. Figure 2.6 also shows $\dot{x}$ increasing from negative to positive as we increase x across the fixed point is unstable. The point $(r, 0)$ on the graph in the x-$\dot{x}$ plane is denoted by an empty circle.

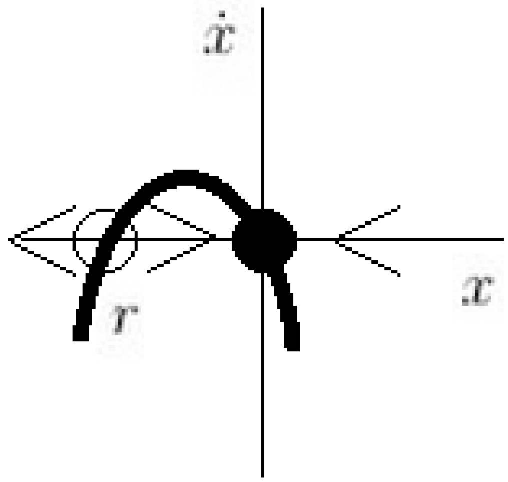

Fig. 2.6: Unstable fixed point at $x = r$, stable fixed point at $x = 0$.

2.4 Bifurcations

Figure 2.7 captures the flow for all values of r. When r is negative you have an unstable fixed point at $x = r$ and a stable fixed point at $x = 0$. When r is zero you have a metastable fixed point at $x = 0$. When r is positive you have a stable fixed point at $x = r$ and an unstable fixed point at $x = 0$.

The equilibrium at $x = 0$ changes from stable to unstable ("loses stability") as r increases from negative to positive. As such, $r = 0$ is called a bifurcation point. Similarly, the equilibrium $x = r$ changes from unstable to stable at $r = 0$. The solution $x = r$ experiences a bifurcation at $r = 0$. In general, when a fixed point is created, is destroyed, or changes stability at a value of the parameter of the dynamical system, the value is called a bifurcation point.

The bifurcation points is shown at the origin in figure 2.7. In this figure we see the r parameter as the independent variable and x as the dependent variable. This graph is called a bifurcation diagram. The fixed points $x = 0$ and $x = r$ appear as lines called "branches of equilibria." They lose and gain stability respectively as you increase r across

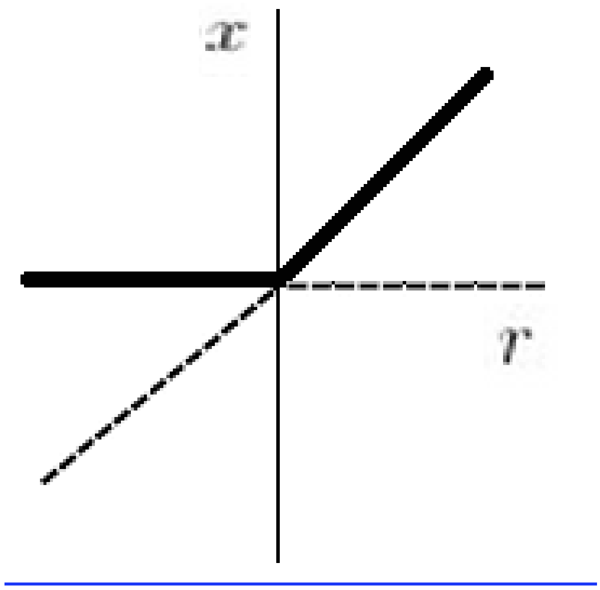

Fig. 2.7: Bifurcation diagram for $\dot{x} = rx - x^2$.

the bifurcation point at the origin. When two branches of equilibria cross as shown in Figure 2.7, the bifurcation point is called transcritical.

Chapter 3
Basic Model

A simple model of a laser (Strogatz, 1994) is given by

$$\dot{n} = gain - loss = GnN - kn. \tag{3.1}$$

Here $n(t)$ is the number of photons in a laser field at time t. The equation gives the total rate of change of the number of photons as gain minus loss. The variable $N(t)$ is the number of excited atoms at time t, which we assume to be

$$N(t) = N_0 - \alpha n \tag{3.2}$$

(Strogatz, 1994), where N_0 is the number of excited atoms maintained by the pump in the absence of the laser effect, and the parameter $\alpha > 0$ governs the rate of loss of excited atoms to the laser process. Gain occurs through stimulated emission, where the photons stimulate the excited atoms to emit additional photons. This process is a random occurrence between photons and excited atoms as described in section 1. The gain term in equation 3.1 is proportional to the number of times nN that photons encounter excited atoms in the chamber. The constant of proportionality $G > 0$ is called the gain coefficient. It measures how readily photons are produced. Photons are lost proportionally to the number present. The loss term includes the constant of proportionality $k > 0$, which is the decay rate.

A mathematical goal is to determine the number of photons n in the laser cavity as time evolves. The behavior of the solution depends on the values of the parameters. Different choices of parameter values produce qualitatively different dynamics.

Our equation for the number of excited atoms at time t is used in our simple model equation. We are able to have an equation in terms of only the variable n. This is where we the next step is to find the bifurcation diagram, but we must first find the fixed points through non-dimensioning the equation first.

3.1 Nondimensionalization

We seek to reexpress the model given by equations 3.1 and 3.2 in terms of dimensionless quantities. We will reduce the number of parameters from four (G, k, N_0, α) to one (r).

The variable t has the units of time. The variables $n(t)$ and $N(t)$ are dimensionless counts. The units on the rate constants G and k are per time. The parameters N_0 and α are dimensionless.

We introduce the dimensionless variables:

$$\tilde{t} = Gt \tag{3.3}$$

$$\tilde{n} = \alpha n. \tag{3.4}$$

Making this change into the dimensionless variables $\tilde{t}$ and $\tilde{n}$, equations 3.1 and 3.2 become

$$\frac{d\tilde{n}}{d\tilde{t}} = \tilde{n}N - \frac{k}{G}\tilde{n} \tag{3.5}$$

and

$$N = N_0 - \tilde{n}. \tag{3.6}$$

Substituting equation 3.6 into the ordinary differential equation 3.5, we have the dimensionless model

$$\frac{d\tilde{n}}{d\tilde{t}} = \tilde{n}\left(N_0 - \frac{k}{G} - \tilde{n}\right). \tag{3.7}$$

Introducing the dimensionless parameter

$$N_0 - \frac{k}{G} = r, \tag{3.8}$$

and dropping the tildes, we can write the basic model as

$$\dot{n} = n(r - n). \tag{3.9}$$

Thus we have expressed the model by equations 3.1 and 3.2 in terms of dimensionless quantities. This is the example we analyzed in section 2.3.

3.2 Fixed Points of Basic Model

The analysis of the fixed points of the nondimensionalized basic model is analogous to the discussion in section 2.3. We find the fixed points of the dynamical system in Equation 3.9 by solving:

$$\dot{n} = n(r - n) = 0. \tag{3.10}$$

The fixed points are:

$$n = 0, \quad n = r. \tag{3.11}$$

These are the n-intercepts of the graph in the n-$\dot{n}$ plane.

We will consider the parameter n to be non-negative due to the fact that we can't have a negative number of photons in the laser cavity. Figures 3.1, 3.2, and 3.3 show the cases $r > 0$, $r = 0$, and $r < 0$, respectively. See Section 2.3 for a discussion of flow and stability.

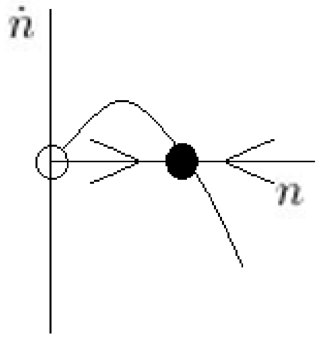

Fig. 3.1: Case $r > 0$.

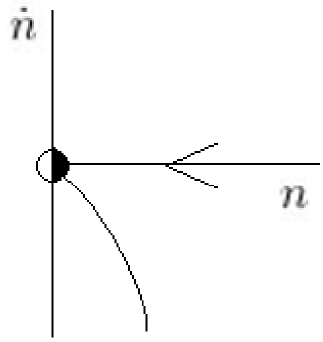

Fig. 3.2: Case $r = 0$.

3.3 Dynamics of the Basic Model

In the previous section we showed a bifurcation at $r = 0$. As the parameter r drops below zero, the fixed point $x = r$ becomes unstable. From equation 3.8, we can write

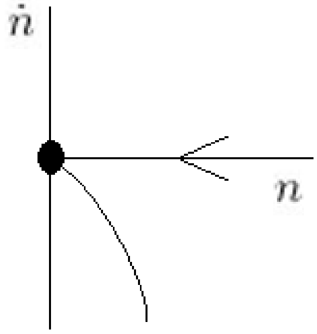

Fig. 3.3: Case $r < 0$.

$r = 0$ in terms of the parameters of a laser as

$$N_0 - \frac{k}{G} = 0.$$

So for $r \leq 0$ we can conclude

$$N_0 \leq \frac{k}{G}.$$

The number of photons tends to zero. (See Figures 3.2 and 3.3.)

In the regime $r > 0$, the laser action is turned on. From Equation 3.8, this case corresponds to

$$N_0 > \frac{k}{G}.$$

(See Figure 3.1.)

Depending on the ratio of the rate of escape k to the gain coefficient G we need a particular pump strength to create a laser. In particular, the pump strength must exceed the ratio of $\frac{k}{G}$. Equivalently, if the ratio of rate of escape k to the gain coefficient G is less than the number of excited atoms N_0 in the absence of the laser, the laser action persists. We see this case for sufficiently small losses k or sufficiently large gain G. But if the ratio of rate of escape k to the gain coefficient G is greater than or equal to the number of atoms in the absence of the laser, then we have a simple lamp instead of a laser. This physical situation corresponds to stability for the equilibrium $n = 0$.

Chapter 4
System Model

In Section 3, our basic model included equation 3.2. In that equation, we set the number of excited atoms $N(t)$ at time t to be the number of excited atoms N_0 maintained by the pump in the absence of the laser effect, with excited atoms lost proportionally to the number $n(t)$ of photons present. We present a more realistic model

$$\dot{n} = GnN - kn, \tag{4.1}$$

$$\dot{N} = -GnN - fN + p \tag{4.2}$$

(Strogatz, 1994), where equation 4.1 is the same as equation 3.1 in the previous chapter. In this model, the algebraic equation 3.2 has been replaced by the differential equation 4.2. Equation 4.2 gives the net rate of change $\dot{N}$ of excited atoms. In particular, excited atoms are lost proportionally to the number of times nN that photons encounter excited atoms in the chamber. Excited atoms decay proportionally to the number $N(t)$ present at a rate of f. Excited atoms are introduced by the pump at a rate p.

Recall that G is the gain coefficient, and k is the loss coefficient for photons. Here f is the loss coefficient for excited atoms, and p is the pump strength. All parameters are positive, except for p, which can be nonpositive.

4.1 Nondimensionalization

We nondimensionalize the system via

$$\tilde{t} = kt,$$

$$\tilde{n} = \frac{G}{k}n,$$

and

$$\tilde{N} = \frac{G}{k}N.$$

When we substitute into the system 4.1–4.2 and drop the tildes, the outcome is

$$\dot{n} = nN - n, \tag{4.3}$$

$$\dot{N} = -nN - r_1N + r_2, \tag{4.4}$$

where

$$r_1 = \frac{f}{k}, \tag{4.5}$$

and

$$r_2 = \frac{pG}{k^2}. \tag{4.6}$$

4.2 Fixed Points of the System Model

To find the fixed points of the system 4.3–4.4, we will set the derivative terms to zero:

$$0 = n(N-1), \tag{4.7}$$

$$0 = -nN - r_1N + r_2. \tag{4.8}$$

Equation 4.7 implies $n = 0$ or $N = 1$.

If $n = 0$, equation 4.8 implies that $N = \frac{r_2}{r_1}$. One fixed point is

$$(n, N) = \left(0, \frac{r_2}{r_1}\right). \tag{4.9}$$

If $N = 1$, then equation 4.8 implies that $n = -r_1 + r_2$. The fixed point is

$$(n, N) = (r_2 - r_1, 1). \tag{4.10}$$

So equations 4.9 and 4.10 give the fixed points.

4.3 Dynamics of the System Model

Equation 4.9 corresponds to the physical case of no photons, where we have a simple lamp instead of a laser. If $r_1 \neq r_2$, then the equilibrium solution in equation 4.10 has the number of photons nonzero. This case corresponds to a laser.

From equations 4.5–4.6, we can write $r_1 = r_2$ in terms of the parameters of a laser as

$$\frac{k}{G} = \frac{p}{f},$$

which corresponds of the ratio of loss to gain being in perfect balance with the ratio of pump strength to loss of excited atoms. In any physically realistic situation $r_1 \neq r_2$.

In further work we could identify parameter regimes that correspond to stability and instability of the fixed points given by equations 4.9 and 4.10. If we have parameter values that correspond to stability for the fixed point in equation 4.10 and instability for the fixed point in equation 4.9, then we have a laser.

The stability can be analyzed by looking at the eigenvalues of the system 4.3–4.4 linearized near each fixed point. Conditions that produce two negative real eigenvalues or two eigenvalues with negative real part would correspond to stability for the fixed point in question.

Chapter 5

Conclusions

Lasers produce a strong focused ray of light that is useful in many industries. In this thesis I characterized a global view of the laser at the macroscopic level. We analyzed the collective behavior of the photons through the dynamics of a basic model and a system model. Different choices of parameter values produce qualitatively different behaviors of the system. This shows that lasers work not because there is an exact number of photons, but because of the stability of an equilibrium solution under the right parameter conditions. This is important physically because lasers would not be useful if they required specific numerical values for physical quantities represented in the models.

I performed a bifurcation analysis in order to capture the different physical situations, identifying equilibrium solutions and, in the case of the basic model, their stability properties. I sketched representative solutions using the bifurcation points as a guide. In some parameter regimes the laser action persists. In other situations the number of photons tends to zero.

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