How Does Tutoring to Develop Conceptual Understanding Impact Student Understanding?

Kelsey Cummings

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How Does Tutoring to Develop Conceptual Understanding Impact Student Understanding?

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Introduction

The debate over conceptual understanding versus procedural knowledge has caught the eye of many teachers in school systems all around the world. Conceptual understanding is the comprehension of not only what to do, but also why you do it. Procedural knowledge, also known as imperative knowledge, is the knowledge exercised in the performance of some task. In both cases, students understand how to complete an assignment, but the way they think about it differs. One thing that many teachers agree on is that students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge, as stated in the *NCTM Principles and Standards for School Mathematics* (Miles 1).

Balka, Hull, and Harbin Miles believe that,

For decades, the major emphasis in school mathematics was on **procedural knowledge**, or what is now referred to as procedural fluency. Rote learning was the norm, with little attention paid to understanding of mathematical concepts. Rote learning is not the answer in mathematics, especially when students do not understand the mathematics. In recent years, major efforts have been made to focus on what is necessary for students to learn mathematics, what it means for a student to be **mathematically proficient**. (Miles 1)

In order to be mathematically proficient, a student must have conceptual understanding of the topic. This understanding awards users with the power to adapt their behavior to the environment, and to shape their environment to suit their own needs. Richard Skemp, author of *The Psychology of Mathematics*, believes that the power of concepts also comes from their ability to combine and relate many different experiences and classes of experience. The more abstract the concepts, the greater their power to do this. He concludes that, “the person who says
‘Don’t worry me with theory—just give me the facts’ is speaking foolishly. A set of facts can be used only in the circumstances to which they belong, whereas an appropriate theory enables us to explain, predict and control a great number of particular events in the classes to which it relates” (Skemp 17).

There are a number of productive beliefs dedicated to teaching and learning mathematics. On page 11, *Principles to Actions: Ensuring Mathematical Success For All*, by the National Council of Teachers of Mathematics, provides readers with a table to compare unproductive versus productive beliefs about teaching and learning mathematics.

<table>
<thead>
<tr>
<th><strong>Beliefs about teaching and learning mathematics</strong></th>
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</thead>
<tbody>
<tr>
<td><strong>Unproductive beliefs</strong></td>
</tr>
<tr>
<td>Mathematics learning should focus on practicing procedures and memorizing basic number combinations.</td>
</tr>
<tr>
<td>Students need only to learn and use the same standard computational algorithms and the same prescribed methods to solve algebraic problems.</td>
</tr>
<tr>
<td>Students can learn to apply mathematics only after they have mastered the basic skills.</td>
</tr>
<tr>
<td>The role of the teacher is to tell students exactly what definitions, formulas, and rules they should know and demonstrate how to use this information to solve mathematics problems.</td>
</tr>
<tr>
<td>The role of the student is to memorize information that is presented and then use it to solve routine problems on homework, quizzes, and tests.</td>
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</tbody>
</table>
An effective teacher makes the mathematics easy for students by guiding them step by step through problem solving to ensure that they are not frustrated or confused.

An effective teacher provides students with appropriate challenge, encourages perseverance in solving problems, and supports productive struggle in learning mathematics. (NCTM 11)

During my high school career, I believe that my teachers taught mathematics in a procedural fashion, by teaching the steps of what to do, but not necessarily why you do it. I could always give the right answers during class and receive the highest grades on tests because I memorized the material. Unfortunately, this method was unproductive in the end because the information is not retained. After becoming a tutor, I began gaining conceptual understanding of the topic using productive methods of learning. I now have a conceptual comprehension for algebra, geometry, trigonometry, and calculus, which has equipped me to become a much better mathematician.

Overall, I believe most teachers are teaching procedurally rather than conceptually. Not only do I know this from my own experiences, but also from the ten to twenty hours a week I spend tutoring my students. When I realized that my clients might know how to solve a problem, but certainly not why, I started asking them questions like: do you know why? Or, how did you know you had to do that? Most of the time if it is a new topic that we have not addressed, they will reply that they have no idea. I always jump at the opportunity to explain the reasoning in full detail. This technique has generated great success, with students increasing their test grades as much as fifty points. The purpose of this thesis is to study exactly this. Are students improving their understanding after my conceptual explanations?

After conducting all of my research, I can conclude that yes, students are improving their understanding after I exposed them to conceptual explanations. By using productive conceptual teaching techniques, I was able to improve the student scores by over fifty percent. Even though...
the problems for each question slightly changed, these students were able to use what they learned during our tutoring sessions to determine the correct answer. They understood what they were supposed to do and why they were doing it, helping them give a detailed explanation for all five questions. When students are able to combine procedures they have learned with the conceptual understanding of each, their scores improve. When students understand conceptually, the information is stored in the long-term part of their brain, and they do not have to worry about forgetting a formula the next day. They know where the information comes from and how to retrieve it; therefore, they will always be able to derive it.

**Conceptual Understanding vs. Procedural Knowledge**

Conceptual understanding is knowledge rich in relationships. Professor Metcalf includes a quote from Hiebert and Lefevre (1986) in her dissertation that has a more in depth look at the nature of conceptual understanding.

> It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network. In fact, a unit of conceptual knowledge cannot be an isolated piece of information: by definition it is a part of conceptual knowledge only if the holder recognizes its relationship to other pieces of information. (Metcalf 24)

The more conceptually a topic is taught, the more meaningful the learning becomes.

The definition of meaningful learning refers to the idea that the learned knowledge is fully understood by the individual and that the individual knows how that specific fact relates to other stored facts. When activities or ideas have no meaning in life, they easily become boring
and pointless. Richard Skemp explains in *The Psychology of Mathematics* that many schoolchildren derive little benefit and no enjoyment from mathematics, therefore not seeing its full potential. He states,

> This is almost certainly because they are not really learning mathematics at all…What is inflicted on all too many children and older students is the manipulation of symbols with little or no meaning attached, according to a number of rote-memorized rules. This is not only boring (because meaningless); it is very much harder, because unconnected rules are much harder to remember than an integrated conceptual structure. (Skemp 18)

Bringing meaning and connections to concepts is the key to success in conceptual understanding.

Richard Skemp believed entirely in conceptual understanding for a long time, until he really investigated at the pros of procedural knowledge. There are three major advantages to teaching procedurally, especially when managing a large class with varying degrees of comprehension and learning styles. First, it is much easier for the student to absorb. Various one step problems could take hours to solve, potentially confusing students. Next, the rewards for the students are immediate and more apparent. Many students are preoccupied with high grades and will memorize material to pass especially if disinterested in the subject. As long as they know the formula to get the right answer, most students don’t care about the methodology. Finally, since less knowledge is involved, students are able to reach a solution more quickly. At home or in class, students just want to breeze through their work. Procedural knowledge enables students to finish work speedily, compared to conceptual, which could take all night. Ironically, although full comprehension is not achieved, procedural instruction is usually the necessary technique for
a classroom teacher. An instructor’s time constraints do not permit conceptual understanding to be achieved.

Both conceptual understanding and procedural knowledge are needed for a student to be successful in their mathematics career. Hiebert and Lefevre (1986), as quoted in Professor Metcalf’s dissertation, link the two together, discussing their benefits.

1) Building relationships between conceptual knowledge and procedures of mathematics contributes to memory of procedures and to their effective use (p. 26).

2) Linking conceptual knowledge with symbols creates a meaningful representation system, an essential prerequisite for intelligent mathematical learning in performance (p.26).

3) Linking conceptual knowledge with rules, algorithms, or procedures that must be learned increases the likelihood that an appropriate procedure will be recalled and used effectively (p.26).

4) Procedural knowledge that is informed by conceptual knowledge results in symbols that have meaning and procedures that can be remembered better and used more effectively (p. 27).

5) Procedural knowledge provides a formal language and action that raise the level and applicability of conceptual knowledge (p.27).

Conceptual understanding and procedural knowledge are not mutually exclusive and both are integral parts of the learning process. If a student has only conceptual understanding, he or she may have a strong instinctual feel for mathematics, but may not be able to solve the problems. On the other hand, if the student is only proficient with procedural knowledge, they will be able
to produce answers but may not comprehend what they are doing. In the book, *Conceptual and Procedural Knowledge: The Case of Mathematics*, edited by James Hiebert, the reader is introduced to several reasons to believe that connecting procedures with their conceptual underpinnings is the key in producing procedures that are stored and retrieved more successfully:

First, if procedures are linked with conceptual knowledge, they become stored as part of a network of information, glued together with semantic relationships. Such a network is less likely to deteriorate than an isolated piece of information, because memory is especially good for relationships that are meaningful and highly organized. Second, retrieval is enhanced because the knowledge structure, or network, of which the procedure in part comes equipped with numerous links that enable access to the procedure. The “conceptual” links increase the chances that the procedure will be retrieved when needed, because they serve as alternate access routes for recall.

(Hiebert 11)

Linking procedural knowledge with conceptual knowledge shows an increase in the value of procedural knowledge.

**Methodology**

**Participants**

In this study, data were collected from four high school students from an urban area in Massachusetts. I currently tutor all four students in Pre-Calculus, meeting on a weekly basis. One hundred percent of these students are female and they are 17-18 years old. I was limited on the selection of students due to the topic of my thesis. I chose to work from a small pool of
individuals to make sure that they were all from the same pre-calculus class, learning the same exact properties of trigonometry.

**Materials**

The two major materials for this study were the pre-test and post-test that I had created, which are both included in the appendix. Using the aid of the pre-calculus textbook written by Roland Larson and Robert Hostetler, as well as the article, *Teaching Trigonometric Functions: Lessons Learned from Research*, by Keith Weber, I created 5 short answer questions on the topic of trigonometry. Both tests have the same question structure, but the numbers in each question have been altered.

Question 1 contains parts a-d, asking the students to convert degrees to radian measure. The angles in parts a and b will appear directly on their unit circles, while c and d are not in standard form, and require them to multiply by $\pi/180$.

Question 2 contains parts a-d, asking the students to convert radians to degrees. The angles in parts a and b will appear directly on their unit circles, while c and d are not in standard form, and require them to multiply by $180/\pi$.

Question 3 contains parts a-e, asking the students to use a given function value to evaluate five other trigonometric expressions.

Question 4 contains parts a-e, and asks the students to use the Pythagorean Identities and a given function value to evaluate five other trigonometric expressions.

Question 5 contains parts a-d, and asks the students to answer qualitative questions rather than quantitative. These questions included determining whether sine and cosine are positive or negative for certain inputs, as well as which of sine or cosine is bigger for a given input.
All five questions require the students to answer an open response question. We are using this to assess their conceptual understanding of the topic, specifically to see if they are able to explain why they did what they did.

**Procedure**

I began by administering a pre-test (see Appendix A) to each of the four participants. Each student was given a fifty-minute time slot to complete the exam. This is equivalent to the time they are given in a math classroom while taking an exam. After all pre-tests were completed, I reviewed all four exams. This gave me a better understanding of the concepts the students already had a procedural knowledge and conceptual understanding of. At the next meeting with the student, I began tutoring with the goal of helping students gain a conceptual understanding, knowing that they also needed to become proficient with procedural skills, as well. I explained why a problem develops into its answer. Practice problems were used in the tutoring process, as well as student verbal explanations of why they did what they did. After three one-hour tutoring sessions were completed with each student, the post-test was administered (see Appendix B). Finally, I reviewed the results, determining if an improvement occurred. I checked for improvement by comparing the number of correct answers from both tests, also, the work put into each question (i.e. diagrams, pictures, etc.), as well as the advancement in explanation and writing skills.

**Strategy**

Prior to beginning this research project, the participants had been introduced to Trigonometry. They were aware of the Unit Circle and how find degrees, radians, and coordinate points. Also, they understood how to form a right triangle and use the acronym SOH CAH TOA to find angle measures and side lengths. *Teaching Trigonometric Functions: Lessons Learned*
from Research, by Keith Weber, explains that “Trigonometry presents many first-time challenges for students: It requires students to relate diagrams of triangles to numerical relationships and manipulate the symbols involved in such relationships. Further, trigonometric functions are typically among the first functions that students cannot evaluate directly by performing arithmetic operations” (Weber 144).

In both the pre-test and post-test, the students are asked to answer written response questions explaining how they arrived at their answers. Writing in mathematics is a very valuable assessment tool. As Bernadette Russek states in, Writing to Learn Mathematics, “It is used to assess attitudes and beliefs, mathematics ability, and ability to express ideas clearly” (Russek 36). These are essential things that a teacher must use to assess his or her students, as well as students to evaluate themselves. Writing in math provides students with a deeper understanding for what they are learning, giving them the ability to communicate their actions and ideas, rather than just giving a numerical answer. This type of assessment will inform the teacher on how well his or her students understand the topic.

I believe that writing assessments promote conceptual understanding. If a student is able to answer a question, explain why, and give examples, then he or she has a conceptual understanding of the topic. There are problems that only require procedural knowledge to solve. If the student can restate a memorized definition, along with an example they learned in class, they often will receive full credit. Unfortunately, when the problem is slightly changed, the student would not be able to formulate an answer and example, along with an explanation because they do not have a conceptual understanding. I tested this theory by changing each question slightly from pre-test to post-test. I believe that if students are able to explain all of their
work in writing, including why they did what they did, they have a full knowledge and understanding for the topic, which is the desired for all teachers and students.

My research question was “how does tutoring, to develop conceptual understanding, impact student understanding?” To investigate my research question, I chose a method for tutoring that I have developed over the past seven years. In order to be able to help students build conceptual understanding, I believe the first step is to have a profound understanding of mathematics. Not only must I have a conceptual grasp of the subject, but also I must be able to communicate it to those at a lower conceptual level. When introducing a new subject, I always relate it back to an older topic that I know my students understand. Whether it is something they learned in elementary school, middle school, or just last week, I use it. For the topic used in this study, I related trigonometry to the Pythagorean theorem, circles, triangles, etc. Also, I explained to them how you get the Pythagorean identities and why, rather than making them memorize the three formulas, with no idea where they came from. The factual knowledge is important, but it must be placed in context, if it is to be retained. Memory of factual knowledge is enhanced by conceptual knowledge, and conceptual knowledge is clarified as it is used to help organize important details.

Another strategy that I use is to constantly keep asking my students, “What questions do you have?” I want them to be constantly thinking, rather than me talking and them listening. If I can get them talking, that means they are actively thinking. After each problem they would need to explain why they did every step they did. All of these strategies help students’ gain conceptual understanding.
Results

The purpose of this study was to discover whether pre-calculus students improve their understanding when conceptual understanding was being reinforced. After conducting tutoring sessions and collecting written work, all four Pre-Calculus students were able to complete all problems on the post-test; Whereas, they correctly completed fewer than half of the questions on the pre-test. They were able to apply the different strategies while performing the tasks to reach the correct solution. Along with their mathematical skills improving, their written responses improved as well.

Question 1: Students were asked to express four different angle measures in radian measure as a multiple of $\pi$. On the pre-test, all four students were able to answer parts a and b. Before administering the test, I assumed this would be the case. All four students knew how to use their unit circle and both of these degree measures could be found directly on it. Parts c and d introduced negative measures, so the students would have to know that they must go backwards on their unit circles. Three out of four students were able to correctly answer d, while no student was able to answer c. I believe this provides evidence that prior to my tutoring, the students were not fluent in the multiple representations of angles, i.e. degrees and radians. Twenty degrees is not directly written on the unit circle, therefore the students were unable to answer the question posed. On the post-test all four students correctly answered all four questions. Along with getting the correct answers, their written responses improved. All students showed that they could not only derive an answer, but could justify their answers as well. This was evident by their responses to the required questions: Explain how you got your answer. Can you use this method to convert
any degree measure to radians? Why or why not? A sample response to question one can be found in Appendix C.

**Question 2:** Students were asked to express four different radian measures in degree measure. On the pre-test, all four students were able to answer parts a and b. Before administering the test, I assumed this would be the case. All four students knew how to use the unit circle and both of these radian measures could be found directly on it. Three students did not attempt to complete parts c or d. One student did complete c and d, getting both correct. This student showed signs of knowing the conversion formula, but the written response shows that the student did not understand what he or she was doing, but just completed the procedural step taught. On the post-test all four students got all four questions correct. Along with getting the correct answers, their written responses improved, all giving evidence that they could not only derive an answer, but provide an explanation, as well. They were required to answer the question: Explain how you got your answer. Can you use this method to convert any radian measure to degrees? Why or why not? A sample response to question two can be found in Appendix D.

**Question 3:** Students were asked to evaluate five different trigonometric expressions given a value for $\sin \theta$. On the pre-test, students 3 and 4 did not attempt to answer this question. They expressed that they did not know where to begin. Student 1 drew a right triangle, which was the correct method, but incorrectly labeled the sides. Also, student 1 did not know how to evaluate tangent, secant, cosecant, or cotangent, so she was not able to complete those questions. Student 2 showed the most understanding of this question. Student 2 drew a right triangle and used the Pythagorean theorem to label all sides correctly. She then answered all questions correctly, but
did not rationalize the denominator on parts b and c. In the written response, Student 2 showed that she knew what to do, but not why. On the post-test all four students got all five parts correct. Along with getting the correct answers, their written responses improved, all giving evidence that they could not only derive an answer, but provide an explanation, as well. They were required to answer the question: Explain how you got your answer referring to any figures you used. Why are you able to use right triangles to evaluate trigonometric functions? A sample response to question three can be found in Appendix E.

**Question 4:** Students were asked to evaluate five different trigonometric expressions using a given function value for \( \cos \theta \) and the Pythagorean Identities, for example, \( \sin^2 \theta + \cos^2 \theta = 1 \). On the pre-test, students 3 and 4 did not attempt to answer this question. Students 1 and 2 both drew a right triangle, labeling all sides correctly. They both were able to find the correct values of \( \sin \theta \) and \( \tan \theta \), but neither used the Pythagorean Identities to solve the problem. Student 1 was unable to solve for secant, cotangent, and cosecant. Student 4 correctly solved for all five, but did not rationalize the denominator. Regarding the written responses, Student 1 was able to correctly identify the origin of the Pythagorean Identity, while Student 2 was able to tell how to create the additional Pythagorean Identities. On the post-test all four students answered all five questions correctly. Along with answering correctly, their written responses improved, giving evidence that they could not only derive an answer, but provide an explanation, as well. They were required to answer the question: Where does the original Pythagorean identity \( \sin^2 \theta + \cos^2 \theta = 1 \) come from? How do you create the additional Pythagorean identities using the original? A sample response to question four can be found in Appendix F.
Question 5: Students were asked to answer four questions without doing any computations and include an explanation of how they arrived at their answers. Of the questions given, this question assessed mostly conceptual understanding. There was no mathematical calculation required; students needed to use what they knew about sine and cosine to answer questions of positive, negative, bigger, or smaller. On the pre-test, all students attempted parts a and b. All students correctly answered part a, while three out of four students correctly answered b. Only one student was able to correctly explain why the value of the expression was negative or positive. For parts c and d, three out of the four students took a guess, but because of their “how do you know responses”, I determined they did not know if their answer was correct. On the post-test all four students got all five questions correct. Along with getting the correct answers, all students written responses improved, giving evidence that they could not only derive an answer, but provide an explanation, as well. A sample response to question five can be found in Appendix G.

Conclusion

H. Lynn Erickson once said, “Trying to teach in the 21st century without conceptual schema for knowledge is like trying to build a house without a blueprint”(Erickson 8). Procedures such as worksheets, drills, and memorization of facts have proven to work as a tool to help students earn good grades. The truth is, most of these procedures will be forgotten once they put the pencil down having completed the test. The instructional method of procedural knowledge cannot stand alone, if the goal is for students to carry concepts throughout their entire mathematical career. Providing students with definitions or relations of the concepts, rather than just the facts, will help them become mathematically proficient.

Some limitations may have had an impact on the results of this study. Due to the small sample size of only four students, results may not be generalized to the entire population of high
school students. Also, I did not have a comparison group. Regardless of these limitations, the results can still shed light on how conceptual understanding impacts test performance. Nevertheless, future research could help address these limitations. The same methodology could be used with a much larger number of participants to see if similar results can be achieved. Also, since this research only used 17-18 year old students from pre-calculus on the topic of trigonometry, a wider age range and a multitude of concepts could be examined.

The results of this research will be useful to educators, who want to learn more on the topic of conceptual understanding, and how much it helps a student’s long-term learning. Specifically, this information can be used when planning lessons for teachers. Also, it can help guide tutors to a successful way to provide their students with the tools to build infinite knowledge.

Conceptual understanding is an extremely important skill to have in not only mathematics, but also all subjects in school. This research was designed to test whether focusing on conceptual understanding when providing instruction impacts student understanding. The results show that tutoring students in trigonometry with a goal of improving students’ conceptual understanding, using the aid of procedural knowledge, will produce a considerably higher quality of work. Building students’ conceptual understanding throughout their education will ensure that they retain their understanding throughout their lifetime.
Appendix A

Pre-Test

Test Number: ____________________

1. Express the following angles in radian measure as a multiple of π. (Do not use a calculator.)

   a) 30° = ____________
   b) 150°=___________
   c) -20°=____________
   d) -240°=___________

   Explain how you got your answer. Can you use this method to convert any degree measure to radians? Why or why not?

   ————————————————————————————————————————————————————
   ————————————————————————————————————————————————————
   ————————————————————————————————————————————————————
   ————————————————————————————————————————————————————
   ————————————————————————————————————————————————————
   ————————————————————————————————————————————————————

2) Express the angle in degree measure. (Do not use a calculator.)

   a) 3π/2=______________
   b) 7π/6=______________
   c) 7π/3=______________
   d) -11π/3=_____________

   Explain how you got your answer. Can you use this method to convert any radian measure to degrees? Why or why not?

   ————————————————————————————————————————————————————
   ————————————————————————————————————————————————————
   ————————————————————————————————————————————————————
   ————————————————————————————————————————————————————
   ————————————————————————————————————————————————————
   ————————————————————————————————————————————————————
3) Evaluate the following knowing that \(\sin \theta = \frac{2}{3}\)

a) \(\cos \theta = \underline{\quad}\)

b) \(\tan \theta = \underline{\quad}\)

c) \(\sec \theta = \underline{\quad}\)

d) \(\csc \theta = \underline{\quad}\)

e) \(\cot \theta = \underline{\quad}\)

Explain how you got your answer referring to any figures you used. Why are you able to use right triangles to evaluate trigonometric functions?

4) Use the given function value and Pythagorean Identities to find the indicated trigonometric functions.

Given: \(\cos \theta = \frac{1}{2}\)

a) \(\sin \theta\)

b) \(\tan \theta\)

c) \(\sec \theta\)

d) \(\cot \theta\)

e) \(\csc \theta\)
Where does the original Pythagorean identity \( \sin^2 \theta + \cos^2 \theta = 1 \) come from? How do you create the additional Pythagorean identities using the original?

5) Without doing the computations, answer the following questions. How do you know?

a) Is \( \sin 140^\circ \) a positive number or a negative number?

b) Is \( \cos 200^\circ \) a positive number or negative number?

c) Which is bigger- \( \sin 23^\circ \) or \( \sin 37^\circ \)?

d) Which is bigger- \( \cos 300^\circ \) or \( \cos 330^\circ \)?
Appendix B

Post-Test

Test Number: ________________

2. Express the following angles in radian measure as a multiple of \( \pi \). (Do not use a calculator.)

\[
\begin{align*}
\text{c)} \quad & 60^\circ = \text{ } \quad \text{c)} \quad -40^\circ = \text{ } \\
\text{d)} \quad & 135^\circ = \text{ } \quad \text{d)} \quad -230^\circ = \text{ }
\end{align*}
\]

Explain how you got your answer. Can you use this method to convert any degree measure to radians? Why or why not?

_________________________________________________________________________________________________________

_________________________________________________________________________________________________________

_________________________________________________________________________________________________________

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_________________________________________________________________________________________________________

2) Express the angle in degree measure. (Do not use a calculator.)

\[
\begin{align*}
\text{c)} \quad & \pi/2 = \text{ } \quad \text{c)} \quad 7\pi/4 = \text{ } \\
\text{d)} \quad & 5\pi/6 = \text{ } \quad \text{d)} \quad -13\pi/3 = \text{ }
\end{align*}
\]

Explain how you got your answer. Can you use this method to convert any radian measure to degrees? Why or why not?
3) Evaluate the following knowing that \( \sin \theta = 1/2 \)

f) \( \cos \theta = \) 

g) \( \tan \theta = \) 

h) \( \sec \theta = \) 

i) \( \csc \theta = \) 

j) \( \cot \theta = \) 

Explain how you got your answer referring to any figures you used. Why are you able to use right triangles to evaluate trigonometric functions?

_________________________________________________________________________________________________________
_________________________________________________________________________________________________________
_________________________________________________________________________________________________________
_________________________________________________________________________________________________________

4) Use the given function value and Pythagorean Identities to find the indicated trigonometric functions.

Given: \( \cos \theta = \frac{\sqrt{3}}{2} \)

f) \( \sin \theta \)

g) \( \tan \theta \)

h) \( \sec \theta \)

i) \( \cot \theta \)

j) \( \csc \theta \)
Where does the original Pythagorean identity \( \sin^2 \theta + \cos^2 \theta = 1 \) come from? How do you create the additional Pythagorean identities using the original?


5) Without doing the computations, answer the following questions. How do you know?

   e) Is \( \cos 140^\circ \) a positive number or a negative number?

   f) Is \( \sin 200^\circ \) a positive number or negative number?

   g) Which is bigger - \( \cos 23^\circ \) or \( \cos 37^\circ \)?

   h) Which is bigger - \( \sin 300^\circ \) or \( \sin 330^\circ \)?
Appendix C

Sample from Question 1

Pre-Test

1. Express the following angles in radian measure as a multiple of $\pi$. (Do not use a calculator.)
   
   a) $30^\circ = \frac{\pi}{6}$
   
   b) $150^\circ = \frac{5\pi}{6}$
   
   c) $-20^\circ = \ ?$
   
   d) $-240^\circ = -\frac{4\pi}{3}$

   Explain how you got your answer. Can you use this method to convert any degree measure to radians? Why or why not?

   I used my unit circle to find the radians. This method only works for degrees on the unit circle.

Post-Test

1. Express the following angles in radian measure as a multiple of $\pi$. (Do not use a calculator.)
   
   a) $60^\circ = \frac{\pi}{3}$
   
   b) $135^\circ = \frac{3\pi}{4}$
   
   c) $-40^\circ = \frac{-40\pi}{180} = \frac{-20\pi}{90}$
   
   d) $-230^\circ = \frac{-23\pi}{180} = \frac{-23\pi}{2\pi}$

   Explain how you got your answer. Can you use this method to convert any degree measure to radians? Why or why not?

   I got my answer by multiplying the degrees by $\pi/180$. This method works to convert any degrees to radians because it is similar to multiplying by $\pi/1$ since $\pi = 180^\circ$. 
Appendix D

Sample from Question 2

Pre-Test

2) Express the angle in degree measure. (Do not use a calculator.)
   a) \( \frac{3\pi}{2} = 270^\circ \)
   c) \( \frac{7\pi}{3} = \ ? \)
   b) \( \frac{7\pi}{6} = 210^\circ \)
   d) \( \frac{-11\pi}{3} = \ ? \)

Explain how you got your answer. Can you use this method to convert any radian measure to degrees? Why or why not?

I used my unit circle to find the measurements, which does not help me convert any radian measure to degrees.

Post-Test

2) Express the angle in degree measure. (Do not use a calculator.)
   a) \( \frac{\pi}{2} = 90^\circ \)
   c) \( \frac{3\pi}{4} = 315^\circ \)
   b) \( \frac{5\pi}{6} = 150^\circ \)
   d) \( \frac{-13\pi}{3} = \ ? \)

Explain how you got your answer. Can you use this method to convert any radian measure to degrees? Why or why not?

I got my answer by multiplying by \( \frac{180^\circ}{\pi} \) which works to convert any radians to degrees because \( \pi = 180^\circ \) so it is the same as multiplying \( \pi \).
Appendix E

Sample from Question 3

Pre-Test

3) Evaluate the following knowing that \( \sin \theta = \frac{2}{3} \)
   
   a) \( \cos \theta = \) ?

   b) \( \tan \theta = \) ?

   c) \( \sec \theta = \) ?

   d) \( \csc \theta = \) ?

   e) \( \cot \theta = \) ?

Explain how you got your answer referring to any figures you used. Why are you able to use right triangles to evaluate trigonometric functions?

Post-Test

3) Evaluate the following knowing that \( \sin \theta = \frac{1}{2} \)
   
   a) \( \cos \theta = \) \( \frac{\sqrt{3}}{2} \)

   b) \( \tan \theta = \) \( \frac{\sqrt{3}}{3} \)

   c) \( \sec \theta = \) \( \frac{2\sqrt{3}}{3} \)

   d) \( \csc \theta = \) ?

   e) \( \cot \theta = \) \( \frac{\sqrt{3}}{2} \)

\[ a^2 + b^2 = c^2 \]
\[ 1^2 + b^2 = 2^2 \]
\[ b^2 = 3 \]
\[ b = \sqrt{3} \]

Explain how you got your answer referring to any figures you used. Why are you able to use right triangles to evaluate trigonometric functions?

Right triangles can be made by the unit circle using the \( \cot \) and \( \tan \) rules. All sides of the triangle are used.
Appendix F

Sample from Question 4

Pre-Test

4) Use the given function value and Pythagorean Identities to find the indicated trigonometric functions.

Given: \( \cos \theta = \frac{1}{2} \)

a) \( \sin \theta = \frac{\sqrt{3}}{2} \)

b) \( \tan \theta = \frac{\sqrt{3}}{1} \)

c) \( \sec \theta = \frac{2}{1} \)

d) \( \cot \theta = \frac{1}{\sqrt{3}} \)

e) \( \csc \theta = \frac{2}{\sqrt{3}} \)

\[ 1 + B^2 = 4 \]

Where does the original Pythagorean identity \( \sin^2 \theta + \cos^2 \theta = 1 \) come from? How do you create the additional Pythagorean identities using the original?

To get the other ones, you divide by \( \sin^2 \) on both sides to get one. To get the other, you divide both sides by \( \cos^2 \).
4) Use the given function value and Pythagorean Identities to find the indicated trigonometric functions.

Given: \( \cos \theta = \frac{\sqrt{3}}{2} \)

a) \( \sin \theta = \frac{1}{2} \)

b) \( \tan \theta = \frac{\sqrt{3}}{3} \)

c) \( \sec \theta = \frac{2}{\sqrt{3}} \)

d) \( \cot \theta = \frac{1}{\sqrt{3}} \)

e) \( \csc \theta = \frac{2}{\sqrt{3}} \)

\[ \sin^2 \theta + \cos^2 \theta = 1 \]
\[ \sin^2 \theta + \frac{3}{4} = 1 \]
\[ \sin^2 \theta = \frac{1}{4} \]
\[ \sin \theta = \frac{1}{2} \]

Where does the original Pythagorean identity \( \sin^2 \theta + \cos^2 \theta = 1 \) come from? How do you create the additional Pythagorean identities using the original?

Since the legs of the right triangle in the unit circle have the values of \( \sin \theta \) and \( \cos \theta \), the Pythagorean theorem can be used to obtain \( \sin^2 \theta + \cos^2 \theta = 1 \). To obtain the other 2, you can divide by \( \cos^2 \theta \) to get \( \tan^2 \theta + 1 = \sec^2 \theta \). Or you can divide by \( \sin^2 \theta \) to get \( \cot^2 \theta + 1 = \csc^2 \theta \).
Appendix G

Sample from Question 5

Pre-Test

5) Without doing the computations, answer the following questions. How do you know?

a) Is sin140° a positive number or a negative number?
   positive
   because the coordinate point is negative

b) Is cos200° a positive number or negative number?
   negative
   because both coordinate points are negative

c) Which is bigger- sin23° or sin37°?
   sin 37°
   because it is not negative

d) Which is bigger- cos300° or cos330°?
   cos 330°
   because it is not negative
5) Without doing the computations, answer the following questions. How do you know
\( \cos 140^\circ \) a positive number or a negative number?

\( \cos 140^\circ \) is a negative number because it is in the 2nd quadrant, where all \( \cos \) numbers are negative.

b) Is \( \sin 200^\circ \) a positive number or negative number?

\( \sin 200^\circ \) is a negative number because it is in the 3rd quadrant, where all \( \sin \) numbers are negative.

c) Which is bigger- \( \cos 23^\circ \) or \( \cos 37^\circ \)?

\( \cos 23^\circ \) is bigger because \( \cos \) deals with the length of the base.

d) Which is bigger- \( \sin 300^\circ \) or \( \sin 330^\circ \)?

\( \sin 330^\circ \) is bigger because it is higher up on the y-axis.
References


