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BRIDGEWATER STATE UNIVERSITY

MATHEMATICS

**A Mathematical Foundation of the
Quantum-Classical Correspondence**

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Abstract

In this thesis we explore the mathematical foundations that unite physics at a quantum scale, quantum mechanics, with a macroscopic scale, classical mechanics. We seek to understand the mathematical motivation behind the quantum-classical correspondence and how it unites two seemingly different theories of the physical world. We show how this correspondence binds the Hamiltonian theory of classical physics to the Hilbert space theory in quantum mechanics, and establish a way to translate between classical observables and quantum operators, using the Fourier transform. These approaches to “quantizing” a physical state can be applied generally to a wide variety of observable quantities in classical mechanics.

1 Introduction

How does mathematics unite the quantum-scale theory of quantum mechanics with macroscopic-scale observations in classical mechanics? On one hand, quantities in classical physics may be seen and measured with simple tools; on the other hand, in quantum physics measurements are taken at the level of a single particle of matter. Whether we wish to measure a single particles such as an electron, or a massive object such as a falling apple, the mathematics behind the two must be connected in order to make a consistent physical theory, as objects are made of particles. Classical mechanics has been studied for centuries: understanding the behavior of massive objects around us was almost necessary to survival. Measurements of so-called “observable” quantities, properties of a state defined by a real number, were for the most part easy to take. Math comes into the picture in the forms of differential equations and calculus, to examine how these state properties change continuously over time.

On the other hand, the physical properties of particles at the quantum scale, such as for a single subatomic particle, are not as easily measured. Contrary to classical assumptions, early experiments showed that these properties may not be deterministic at the quantum scale, but rather probabilistic.

Young's "Double Slit Experiment" was one of the first to suggest the need for revolutionizing notions of quantum mechanics. In this experiment, Young used a pair of parallel slits to project a beam of light through in order to measure the intensity of the photons traveling to a screen. The result was not as expected: the particles of light behaved like waves, giving a distribution of intensity bands rather than a single value. Finding out that the particles moved in a wave-like behavior rather than a direct path, as predicted, Young was able to determine that particles behave like waves. Therefore it is possible to treat the state of a particle as a function that mathematically describes a wave.

This perspective later came to be called the Copenhagen formalism, and from it the probabilistic approach to quantum mechanics came about [4]. In this formalism, the state of a system is characterized by a wave function that defines a probability density for a particle's properties. In contrast to the deterministic nature of classical physics, with the introduction of waves, quantum mechanics becomes much more complex, focusing instead on probabilities, most likely and average values, and uncertainty in measured values.

Later, Bohr and Heisenberg studying quantum mechanics came up with the "postulate of the quantum", written into Copenhagen's formalism as well. Both also came up with the idea that while wave functions may not be real-valued, once an operator is applied to the wave function real-valued properties may be measured [10].

Heisenberg is also credited with the famous Uncertainty Principle. The Principle relates joint un-

certainty in the measurement of two observable quantities of a quantum state to the commutativity (or lack thereof) of their quantum operators, stating that if the operators do not commute then it is impossible to measure both corresponding physical properties at once to an arbitrary degree of accuracy. An example of this is position and momentum. While one can be determined to any degree of accuracy you wish, both cannot be so determined at the same time. With this probabilistic approach, it became necessary to qualify the measurements of particles with probabilities. This required mathematical methods to approaching quantum physics that were not needed in the classical approach.

While classical and quantum mechanics seem very different there is a mathematical foundation that pulls them together. This relation is known as the quantum-classical correspondence.

Definition 1. *The **correspondence principle** is the axiom that the behavior of a system predicted for each of its particles by quantum mechanics should agree with the behavior of the whole system predicted by classical mechanics, in the limit as the number of particles becomes large.*

This principle is essential to physics, as we should be able to translate between classical predictions made for an object consisting of many particles and quantum predictions of the states of the particles themselves. For this reason, mathematics becomes very important to the quantum-classical correspondence.

As we read through this paper, it will lead through the mathematical foundations of the theories of classical and quantum physics, and then to the connections between them. In the second section we will discuss the some of the mathematics behind the classical formalism, focusing on the Hamiltonian in particular. For the third section, we will look into the mathematical aspects of quantum mechanics, focusing on Hilbert and Banach spaces as well as the applications of these

ideas into physical operators. Lastly, for the fourth section we will explore the mathematics behind the quantum-classical correspondence, using the Fourier transform and the Weyl quantization to explore the reasons mathematics connects the differences between classical mechanics and quantum mechanics.

2 Classical Formalism of Physics

Classical Mechanics is the study of observables and how they react in space. These observables can be measured in many different ways and at the foundation of these measurements is mathematics. Whether employing a differential equation or a basic calculus problem much of the work done in physics corresponds with mathematics. One approach to classical mechanics is through the Hamiltonian formalism.

In a Hamiltonian system, particles are acted upon by a conservative force, that is, a force that transfers energy in a path-independent fashion into and out of the system. (For example, gravity is conservative force: it adds the same amount of energy to a falling apple regardless of the path the apple takes to reach the ground.) In a conservative system, forces F arise as the gradient of a potential energy function V according to the equation $F = -\frac{dV}{dx}$.

Due to this requirement, the motion of particles in a Hamiltonian system can be completely determined by the properties of a total energy function, known as the Hamiltonian function.

2.1 Phase Space

Total energy in a Hamiltonian system is divided into two components: potential energy, which is energy that gives rise to a force acting on the system and depends on the particle's position, and

kinetic energy, which is the energy of motion and depends on the particle's momentum (defined by $p = mv = m \frac{dx}{dt}$). For this reason, the Hamiltonian total energy function has a domain in which both the particle's position and momentum are known. This is known as phase space.

Definition 2. *The phase space of a one-dimensional, physical system refers to the plane \mathbb{R}^2 equipped with coordinates (x, p) where x is the position and p the momentum of the system.*

Phase space can be used to measure momentum and position as well as other physical quantities that depend on momentum and position. Functions on phase space make this possible, and the Hamiltonian function is the most important example.

Definition 3. *The Hamiltonian function of a one-dimensional physical system is the function $H : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by*

$$H(x, p) := K(p) + V(x) = \frac{p^2}{2m} + V(x),$$

where $K(p) = \frac{p^2}{2m}$ is called the kinetic energy of the system and $V(x)$ is called the potential energy.

The potential energy is determined by the physical system itself, and differs based on the circumstances of the environment. For example:

- A “free particle” is defined by $V(x) = 0$.
- An “infinite square well” potential is defined by $V(x) = \begin{cases} 0 & 0 \leq x \leq a \\ \infty & \text{else.} \end{cases}$
- A harmonic oscillator, such as a mass on a spring, is defined by $V(x) = kx^2$ where $k > 0$ is the spring constant.

2.1.1 Hamilton's Equations of Motion

In the Hamiltonian approach, dynamics are created in phase space by equations of motion arising from Newton's second law relating force to acceleration, i.e. $F = ma$. Recalling that acceleration is the second derivative of position, we may rewrite $ma = mx''$ and note that this is also equivalent to the derivative of momentum.

Thus, Newton's second law is equivalent to

$$p' = F = ma = mx''.$$

Since kinetic energy, $K(p) = \frac{p^2}{2m}$, is dependent only on the momentum, p , and potential energy $V(x)$ depends only on position, x , we can gather from the definitions $H(x, p) = K(p) + V(x)$ and $F(x) = -\frac{dV}{dx}$ that

$$\frac{dH}{dx} = V'(x) = \frac{dV}{dx} = -F(x) = -mx'' = -p'$$

and

$$\frac{dH}{dp} = K'(p) = \frac{dK}{dx} = \frac{p}{m} = v = x'$$

where m is the mass of the particle and v is the particle's velocity. These two equations together are called Hamilton's equations of motion:

Theorem 1. *The motion of a Hamiltonian system on phase space with Hamiltonian function*

$H(x, p)$ is determined by Hamilton's equations of motion

$$\begin{aligned}x' &= \frac{dH}{dp} \\ p' &= -\frac{dH}{dx}.\end{aligned}$$

Example: Harmonic oscillator. For the harmonic oscillator we have the potential energy $V(x) = kx^2$, where $k > 0$ is a spring constant. Since $dH/dp = \frac{1}{m}p$ and $dH/dx = 2kx$, Hamilton's equations of motion are a linear system of differential equations:

$$\begin{pmatrix} x'(t) \\ p'(t) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ -2k & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ p(t) \end{pmatrix}.$$

We solve this system by determining the eigenvalues and eigenvectors of its matrix:

$$\begin{aligned} & \begin{pmatrix} 0 & \frac{1}{m} \\ -2k & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ p \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ p \end{pmatrix} \\ & \left[\begin{pmatrix} 0 & \frac{1}{m} \\ -2k & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right] \cdot \begin{pmatrix} x \\ p \end{pmatrix} = 0 \\ & \begin{pmatrix} 0 & \frac{1}{m} - \lambda \\ -2k - \lambda & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ p \end{pmatrix} = 0 \end{aligned}$$

To find λ we can set the determinant of the matrix equal to zero. Thus,

$$\begin{aligned} \det \begin{pmatrix} 0 & \frac{1}{m} - \lambda \\ -2k - \lambda & 0 \end{pmatrix} &= 0 \\ 0 &= \lambda^2 + \frac{2k}{m} \\ \lambda &= \pm i \sqrt{\frac{2k}{m}} \end{aligned}$$

Now that we have found our eigenvalues we next need to find the eigenvectors. To do this we use the equation:

$$\begin{pmatrix} 0 & \frac{1}{m} \\ -2k & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ p \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ p \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{m} \cdot p \\ -2k \cdot x \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda p \end{pmatrix}$$

It then follows that,

$$\frac{1}{m}p = i\sqrt{\frac{2k}{m}}x$$

or

$$\frac{1}{m}p = -i\sqrt{\frac{2k}{m}}x$$

Thus $p = \pm i\sqrt{\frac{2k}{m}}x$ So our eigenvectors will be equal to:

$$\begin{pmatrix} 1 \\ i\sqrt{\frac{2k}{m}} \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ -i\sqrt{\frac{2k}{m}} \end{pmatrix}$$

Using this representation we can represent

$$H = C_1 \cdot e^{i\sqrt{\frac{2m}{k}}t} \cdot \begin{pmatrix} 1 \\ i\sqrt{\frac{2k}{m}} \end{pmatrix} + C_2 \cdot e^{-i\sqrt{\frac{2m}{k}}t} \cdot \begin{pmatrix} 1 \\ -i\sqrt{\frac{2k}{m}} \end{pmatrix}$$

Then using Euler's formula, this is equivalent to

$$x(t) = \cos \sqrt{\frac{2k}{m}}t$$

and

$$p(t) = -\sqrt{2km} \cdot \sin \sqrt{\frac{2k}{m}}t.$$

These solutions lie along ellipses in phase space: the value of the expression $2kmx^2 + p^2$ is con-

stant in time. This is seen by computing

$$\begin{aligned}
 2km \cdot x(t)^2 + p(t)^2 &= 2km(\cos \sqrt{\frac{2k}{m}}t)^2 + (-\sqrt{2km} \sin \sqrt{\frac{2k}{m}}t)^2 \\
 &= 2km(\cos^2 \sqrt{\frac{2k}{m}}t) + (-\sqrt{2km}^2 \sin^2 \sqrt{\frac{2k}{m}}t) \\
 &= 2km \cos^2 \sqrt{\frac{2k}{m}}t + 2km \sin^2 \sqrt{\frac{2k}{m}}t \\
 &= 2km(\cos^2 \sqrt{\frac{2k}{m}}t + \sin^2 \sqrt{\frac{2k}{m}}t) \\
 &= 2km.
 \end{aligned}$$

Therefore, solutions $(x(t), p(t))$ in phase space lie along the equation

$$2kmx^2 + p^2 = 2km$$

which when graphed on a phase space becomes an ellipse. This shows that motion in phase space occurs along ellipses.

However, dividing this equation by $2m$ gives another perspective:

$$kx^2 + \frac{p^2}{2m} = k$$

$$H(x, p) = k$$

In other words, the motion occurs along ellipses because the value of the Hamiltonian total energy function is constant along ellipses. This illustrates a general theorem in Hamiltonian mechanics.

Theorem 2. *If $H(x, p)$ is a Hamiltonian function and $(x(t), p(t))$ is a solution of Hamilton's equations of motion for H , then the value of H is constant along the solution:*

$$H(x(t), p(t)) = c.$$

This is a version of the law of conservation of energy in a closed system.

2.1.2 Observable Functions and the Poisson Bracket

Measurable properties of Hamiltonian systems, such as the position, momentum, and energy of a particle, are defined by functions on phase space known as observables.

Definition 4. An *observable* is a real-valued function $f(x, p)$ whose domain is the phase space of a Hamiltonian system.

Position and momentum themselves are observables: they are the coordinate function positions on phase space can be represented by

$$x(x, p) = x$$

and

$$p(x, p) = p$$

respectively.

Likewise, the kinetic energy function $K(x, p) = \frac{p^2}{2m}$, potential energy functions $V(x, p) = V(x)$, and the Hamiltonian function itself $H(x, p) = K + V$ of systems may be seen as observables.

Other observables include physical quantities such as angular momentum.

The values of observables may change over time as particles move. This change is determined by a quantity known as the Poisson bracket.

Definition 5. The *Poisson bracket* $\{f, g\}$ of two functions f, g on phase space is defined to be the rate of change of the function f along the direction of a trajectory of g . That is,

$$\{f, g\} = \left. \frac{d}{dt} \right|_{t=0} f(x_g(t), p_g(t))$$

where $(x_g(t), p_g(t))$ is a solution of Hamilton's equations for the Hamiltonian function g .

Proposition 3. *The Poisson bracket of two functions $\{f, g\}$ is:*

1. *Anti-symmetric, i.e. $\{f, g\} = -\{g, f\}$,*

2. *Bi-linear, i.e. if c_1, c_2 are scalars then*

$$\{c_1f + c_2h, g\} = c_1\{f, g\} + c_2\{h, g\} \quad \text{and} \quad \{f, c_1g + c_2h\} = c_1\{f, g\} + c_2\{f, h\}.$$

Proof. This proof may be found in [2], and uses Hamilton's equations of motion and the linearity of derivatives. □

Corollary 4. *If f is an observable function such that $\{f, H\} = 0$, then the value of f is constant along trajectories of the Hamiltonian system defined by H . In this case, f is called an **integral of motion** of the system.*

This provides a proof of the energy conservation seen in the harmonic oscillator example in the previous section.

Corollary 5. *The value of the total energy H itself is constant along trajectories of the Hamiltonian system defined by H .*

Proof. By anti-symmetry we have $\{H, H\} = 0$, therefore H is an integral of motion. □

Hamilton's equations of motion also give a convenient formula for the computation of the Poisson bracket $\{f, H\}$ if H is the Hamiltonian function:

Proposition 6. *For a one-dimensional system, the value of the Poisson bracket $\{f, H\}$ is given by*

$$\{f, H\} = \frac{\partial H}{\partial p} \frac{\partial f}{\partial x} - \frac{\partial H}{\partial x} \frac{\partial f}{\partial p}.$$

Proof. Refer to [2], Section 40, pp. 214-215. □

In the Hamiltonian formalism, we have a way of determining the motion of a system using a single observable function — the Hamiltonian function — and a way of comparing the simultaneous evolution of two quantities along solutions using an operation — the Poisson bracket. We will see how each of these ideas has a parallel in quantum mechanics.

3 Quantum Formalism of Physics

In quantum mechanics, particles are modeled as waves and described by “wavefunctions.” Young’s famous double slit interference experiment using beams of small particles provided evidence that those particles have wave-like behavior [5]. However, this wave-like behavior was not seen at larger scales, suggesting that wave-like effects depend on the scale of the system.

Max Planck demonstrated that the energy of particles comes in discrete packets (“quanta”), whose energy is given by Planck’s constant $h = 6.62606957 \times 10^{-34}$ Js. The energy of very small particles is comparable in magnitude to Planck’s constant; however, as particles get larger, their energies become sufficiently large that Planck’s constant is difficult to distinguish from zero. In order to approach this, the deBroglie wavelength defines a scale in which quantum mechanics dominates and Planck’s constant becomes significant.

Definition 6. *The deBroglie wavelength for a particle is defined as $\lambda_D = \frac{h}{p}$ where p is the particle’s momentum and h is Planck’s constant.*

A “quantum particle” exists at the scale defined by deBroglie’s wavelength, so its properties are best modeled by quantum mechanics. The state of a quantum particle is modeled by a wavefunc-

tion.

Definition 7. A wavefunction for a particle in one dimension encodes the particle's state into a complex valued function, $\psi(x,t)$.

From a wavefunction describing its state, we will have the opportunity to calculate angular momentum, momentum, position, energy, and other information about the particle. The Copenhagen interpretation says that the wavefunction of a particle should determine information about the probabilities and uncertainties of measurement in these values. Therefore, wavefunctions are probability density functions in a specific sense:

Definition 8. The probability density of a particle with wavefunction $\psi(x,t)$ is given by $|\psi(x,t)|^2$.

This function is called a probability density due to the requirement that its total integral is equal to 1. In physics, this is said to “normalize” the wavefunction.

Definition 9. A wavefunction is said to be normalized if the following integral is equal to 1:

$$\int_{-\infty}^{\infty} |\psi(x,t)|^2 dx = 1$$

Mathematically, we will see that normalization requires quantum wavefunctions to exist in a Lebesgue space, known as L^2 .

Now that the wavefunction is normalized and the probability of the particle has been found, the wavefunction evolves in position and time according to the Schrödinger's equation. We will later see how this equation resembles Hamilton's equations of motion in classical mechanics.

Definition 10. Schrödinger's equation for a particle with wavefunction $\psi(x,t)$ is defined by the equation:

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x) \psi$$

where $\hbar = \frac{h}{2\pi} = 1.054572 \times 10^{-34} \text{ Js}$.

While this equation has a time dependence, for the sake of our argument we are often interested in a time-independent Schrödinger's equation. This is because we may use separation of variables to assume $\psi(x,t) = \psi(x) \cdot f(t)$. The Schrödinger equation becomes

$$\begin{aligned} i\hbar\psi(x) f'(t) &= -\frac{\hbar^2}{2m}\psi''(x)f(t) + V(x)\psi(x)f(t) \\ &= \left(-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x)\right) f(t) \end{aligned}$$

which can be decoupled into two ordinary differential equations

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = \psi(x)$$

and

$$-i\hbar f'(t) = f(t).$$

The differential equation defining the time dependence $f(t)$ is the same for every physical system (i.e. for every choice of potential energy function $V(x)$) and has the same solution after correcting for units:

$$f(t) = ce^{-iEt/\hbar}$$

Therefore, the spatial dependence of the wavefunction, otherwise known as the stationary-state wavefunction $\psi(x)$, better characterizes the physical state of the system and its general solution may be found by solving the eigenvalue/eigenfunction equation

$$\hat{H}\psi(x) := -\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E \cdot \psi(x)$$

known otherwise as the “time-independent Schrödinger equation.”

3.1 p^{th} Lebesgue Space

One crucial aspect of quantum mechanics are Lebesgue Spaces. The p^{th} Lebesgue space is defined, for $a > 0$, to be the set of all complex valued, p^{th} -power-integrable functions on $[0, a]$:

Definition 11. The p -th Lebesgue space on $[0, a]$ is the space of p -th power summable complex-valued functions on $[0, a]$, i.e.

$$L^p([0, a]) : \left\{ f : [0, a] \rightarrow \mathbb{C} : \int_0^a |f(x)|^p dx < \infty \right\}.$$

3.1.1 L^p is a Norm Space

Equipped with the theorem that under the operations of point wise addition $(f + g)(x) = f(x) + g(x)$ and complex scalar multiplication (where $c \in \mathbb{C}$ $(c \cdot f)(x) = c \cdot f(x)$), it can be shown that $L^p([0, a])$ is a vector space [7]. Not only this, L^p may be equipped with a norm function that can measure the “sizes of” and “distances between” functions in this space.

Definition 12. A *norm space* is a pair $(V, \|\cdot\|)$ where V is a vector space and $\|\cdot\| : V \rightarrow \mathbb{R}$ is a function which satisfies the following axioms:

- **Homogeneity:** For all $c \in \mathbb{C}$, $\|c \cdot f\|_p = |c| \cdot \|f\|_p$.
- **Positive Definiteness:** For all $f \in L^p([0, a])$, we have $\|f\|_p \geq 0$. Furthermore, if $\|f\|_p = 0$, then f is the zero function “almost everywhere.”
- **Triangle Inequality:** For all $f, g \in L^p([0, a])$, $\|f + g\|_p \leq \|f\|_p + \|g\|_p$

It is important to note that in general, functions in $L^p([0, a])$ do not have to be continuous or differentiable. However, to avoid considerations of measure theory in this thesis, it will be necessary to assume at various points in the following proofs that the functions in question are at least continuous.

Theorem 7. *The p -th Lebesgue space, $L^p([0, a])$, equipped with the p -norm function $\|f\|_p$ defined by*

$$\|f\|_p = \left(\int_0^a |f(x)|^p dx \right)^{\frac{1}{p}},$$

is a norm space.

Proof. Homogeneity: By the given function, it can be assumed that

$$\|c \cdot f\|_p = \left(\int_0^a |c \cdot f(x)|^p dx \right)^{1/p}.$$

It follows,

$$\begin{aligned} \left(\int_0^a |c \cdot f(x)|^p dx \right)^{1/p} &= \left(\int_0^a (|c| \cdot |f(x)|)^p dx \right)^{1/p} \\ &= \left(\int_0^a |c|^p \cdot |f(x)|^p dx \right)^{1/p} \\ &= (|c|^p \int_0^a |f(x)|^p dx)^{1/p} \\ &= (|c|^p)^{1/p} \left(\int_0^a |f(x)|^p dx \right)^{1/p} \\ &= |c|^{p/p} \left(\int_0^a |f(x)|^p dx \right)^{1/p} \\ &= |c| \left(\int_0^a |f(x)|^p dx \right)^{1/p} \end{aligned}$$

Thus, we can conclude from the definition of the p -norm function,

$$\|c \cdot f\|_p = |c| \cdot \|f\|_p$$

Positive Definiteness: Given that $f = 0$ is the zero function, it follows that $\|f\|_p = (\int_0^a |f(x)|^p dx)^{1/p} = 0$.

Now we must show that if $\|f\|_p = 0$, then $f = 0$ is the zero function.

By way of contradiction, assume $\exists c \in [0, a]$ such that $f(c) \neq 0$.

Assuming that f is continuous on $[0, a]$, the

$$\lim_{x \rightarrow c} f(x) = f(c)$$

Thus for any $\varepsilon > 0$, there exists $\delta > 0$ such that where $|x - c| < \delta$, $|f(x) - f(c)| < \varepsilon$.

In particular, choose $\varepsilon = \frac{f(c)}{2}$ and let δ be as given above. Because $|f(x)|^p$ is a positive function, it follows that

$$\left(\int_0^a |f(x)|^p dx \right)^{1/p} \geq \left(\int_{-\delta+c}^{\delta+c} |f(x)|^p dx \right)^{1/p} \quad (1)$$

We wish to show that the previous integral is positive. For $c - \delta < x < c + \delta$ we have

$$|f(x) - f(c)| < \varepsilon$$

$$-\varepsilon < f(x) - f(c) < \varepsilon$$

$$-\varepsilon + f(c) < |f(x)| < \varepsilon + f(c)$$

$$-\frac{f(c)}{2} + f(c) < f(x) < \frac{f(c)}{2} + f(c)$$

$$\frac{f(c)}{2} < f(x) < \frac{3f(c)}{2}$$

Since $f(x) > \frac{f(c)}{2}$ and $\frac{f(c)}{2} = \varepsilon > 0$, $f(x)$ is greater than 0 for $c - \delta < x < c + \delta$.

If $f(x) > 0$, then

$$\begin{aligned} \left(\int_0^a |f(x)|^p dx \right)^{1/p} &\geq \left(\int_{-\delta+c}^{\delta+c} |f(c)|^p dx \right)^{1/p} > 0 \\ &= (f(c)^p \cdot x)^{\frac{1}{p}} \Big|_{-\delta+c}^{\delta+c} \\ &= (f(c)^p \cdot (\delta + c + \delta - c))^{\frac{1}{p}} \\ &= (f(c)^p \cdot 2\delta)^{\frac{1}{p}} > 0 \end{aligned}$$

Therefore $\|f\|_p > 0$ which is a contradiction.

Thus, in conclusion if $\|f\|_p = 0$, then $f = 0$.

Triangle Inequality:

For this proof we will use the Cauchy-Schwartz Inequality defined as $|f \cdot g| \leq \|f\|_p \cdot \|g\|_p$

Let $\|f + g\|_p^2 = \left(\int_0^a |(f + g)(f + g)|^p dx \right)^{1/p}$

$$\begin{aligned} &= \left(\int_0^a |f^2 + (2f \cdot g) + g^2|^p dx \right)^{1/p} \\ \left(\int_0^a |f^2 + (2f \cdot g) + g^2|^p dx \right)^{1/p} &\leq \left(\int_0^a |f|^p dx \right)^{1/p} + \left(\int_0^a |g|^p dx \right)^{1/p} + 2 \left(\int_0^a |fg|^p dx \right)^{1/p} \end{aligned}$$

It then follows,

$$\|f + g\|^2 \leq \|f\|^2 + \|g\|^2 + 2\|f\| \cdot \|g\|$$

$$\|f + g\|^2 \leq \|f + g\| \cdot \|f + g\|$$

$$\|f + g\|^2 \leq (\|f\| + \|g\|)^2$$

□

In addition to a norm function, which all Lebesgue spaces have, there is also an inner product function which only one Lebesgue space has, as we will see next.

3.1.2 L^2 is an Inner Product Space

An inner product is a function on a norm space that generalizes the idea of measuring “angles” between vectors.

Definition 13. A vector space V , equipped with a function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{C}$ is called an **inner product space** if the following axioms are satisfied:

- **Sesquilinearity:** For all $f, g \in V$, and all $a, b \in \mathbb{C}$, we have

$$\langle a \cdot f, g \rangle = \bar{a} \langle f, g \rangle \quad \text{and} \quad \langle f, b \cdot g \rangle = b \langle f, g \rangle.$$

- **Conjugate symmetry:** For all $f, g \in V$, we have

$$\langle g, f \rangle = \overline{\langle f, g \rangle}.$$

- **Positive definiteness:** The function $\|f\| = \langle f, f \rangle^{1/2}$ satisfies the axioms of a norm space.

It is not true that L^p is an inner product space in general, but there is one case in which it is.

Theorem 8. The Lebesgue space $L^2([0, a])$, equipped with the inner product defined by

$$\langle f, g \rangle = \int_0^a \overline{f(x)} g(x) dx,$$

is an inner product space.

Proof. Sesquilinearity:

$$\begin{aligned} \langle a \cdot f, g \rangle &= \int_0^a \overline{a \cdot f(x)} g(x) dx, \\ &= \int_0^a \bar{a} \cdot \overline{f(x)} \cdot g(x) dx, \\ &= \bar{a} \int_0^a \overline{f(x)} \cdot g(x) dx, \end{aligned}$$

$$\begin{aligned}
&= \bar{a} \langle f, g \rangle \\
\langle f, b \cdot g \rangle &= \int_0^a \overline{f(x)} (b \cdot g(x)) dx, \\
&= \int_0^a b \cdot \overline{f(x)} g(x) dx, \\
&= b \int_0^a \overline{f(x)} \cdot g(x) dx, \\
&= b \langle f, g \rangle
\end{aligned}$$

Conjugate Symmetry:

$$\begin{aligned}
\langle g, f \rangle &= \int_0^a \overline{g(x)} f(x) dx, \\
\langle g, f \rangle^\dagger &= \left(\int_0^a \overline{g(x)} f(x) dx \right)^\dagger \\
\langle g, f \rangle^\dagger &= \int_0^a \overline{f(x)} g(x) dx \\
&= \overline{\langle f, g \rangle}
\end{aligned}$$

Positive Definiteness: This inner product satisfies $\langle f, f \rangle^{1/2} = \|f\|_2$ as defined in Definition 11.

Theorem 7 then shows that the axioms of norm are satisfied. □

By meeting these criteria, the space L^2 space is an inner product space.

The idea of completeness lends a final property to the Lebesgue spaces.

Definition 14. A norm space $(V, \|\cdot\|)$ is called **complete** if every Cauchy sequence of points $x_i \in V$, $i = 1, 2, 3, \dots$ is convergent in V , i.e. there exists $x \in V$ such that $x = \lim_{i \rightarrow \infty} x_i$. A complete norm space is called a **Banach space**. A complete inner product space is called a **Hilbert space**.

Theorem 9. L^p is a Banach space for all $p \geq 1$. In particular, L^2 is a Hilbert space.

In order to prove that L^2 is a Hilbert space it is sufficient to prove that L^p is a norm space and complete. As shown in the latter proof, L^2 is a norm space. Rudin's *Principles of Mathematical Analysis* states and proves the following more general result from measure theory, where μ is a measure on position space:

Theorem 10. ([7], p. 329) *If f_n is a Cauchy sequence in $L^2(\mu)$, then there exists a function $f \in L^2(\mu)$ such that f_n converges to f in $L^2(\mu)$. This says in other words, that $L^2(\mu)$ is a **complete metric space**.*

Therefore, by Rudin's proof for completeness and the proof for an inner product space, we can conclude that L^2 is a Hilbert space.

3.2 Operators on L^2

Because of the probabilistic nature of their values, physical properties of quantum particles are defined differently than in classical mechanics. In quantum mechanics, they are measured using operators.

Definition 15. *An operator on L^2 is a linear function $\hat{a} : L^2([0, a]) \rightarrow \mathbb{C}$.*

We will see that these operators are differential operators which generally take the derivative of wavefunctions. An example of these operators is $\hat{p} = i\hbar \frac{\partial}{\partial x}$ which takes the derivative of whatever is following the operator while multiplying it by an $i\hbar$.

The eigenvalues of an operator determine the possible values of its physical property that may be measured. While in general there is a probability associated with this measurement, "pure states" of an operator offer certainty.

Definition 16. A pure state of an operator \hat{a} is an eigenfunction $\psi(x,t)$, in other words, a wavefunction that satisfies

$$\hat{a}\psi = \lambda \psi$$

for a scalar λ .

When the value of the physical property corresponding to \hat{a} is measured on the pure state ψ , its eigenvalue λ is returned with probability 1.

For example, when measuring the momentum of a pure momentum state, the same value will be measured every time, which is the relevant eigenvalue. From there the eigenfunctions of an operator can be found to obtain the physical values of that operator as eigenvalues.

In quantum mechanics, probabilistic effects can introduce uncertainty into the values of two physical properties measured simultaneously. This uncertainty between two values is determined by the commutator of their operators:

Definition 17. The commutator of two operators A, B is written as

$$[A, B] \equiv AB - BA.$$

The commutator of two operators is related to the statistical variance in their simultaneous measurement by the generalized uncertainty principle, as follows.

Theorem 11 (Generalized Uncertainty Principle). Let A, B be two operators, and σ_A^2, σ_B^2 be their variances. Then

$$\sigma_A^2 \sigma_B^2 \geq \frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle.$$

[6]

Corollary 12 (Heisenberg Uncertainty Principle). *The position and momentum of a quantum particle cannot be measured to simultaneous accuracy. In particular,*

$$\sigma_x^2 \sigma_p^2 \geq \frac{\hbar^2}{4}.$$

Proof. If $\psi(x,t)$ is a wavefunction, then

$$\hat{x}\hat{p}\psi = \hat{x}(-i\hbar\psi') = -i\hbar x\psi'$$

$$\hat{p}\hat{x}\psi = -i\hbar(x\psi)' = -i\hbar(x\psi' + \psi)$$

$$[\hat{x}, \hat{p}]\psi = \hat{x}\hat{p}\psi - \hat{p}\hat{x}\psi = i\hbar\psi.$$

Thus $[\hat{x}, \hat{p}] = i\hbar$ and by the generalized uncertainty principle

$$\sigma_x^2 \sigma_p^2 \geq \frac{1}{4} i\hbar^2 = \frac{\hbar^2}{4}.$$

□

3.2.1 Eigenvalues in Quantum and Classical Mechanics

Because eigenvalues are the link between operators and the values of the physical properties they measure, the properties of an operator's eigenvalues must be known. One crucial property is self-adjointness.

Definition 18. *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. The **adjoint** of an operator $\hat{a} : V \rightarrow \mathbb{C}$ is an operator $\hat{a}^\dagger : V \rightarrow \mathbb{C}$ which, for all $x, y \in V$, satisfies:*

$$\langle x, \hat{a}y \rangle = \langle \hat{a}^\dagger x, y \rangle.$$

Definition 19. *An operator \hat{a} is called self-adjoint if $\hat{a}^\dagger = \hat{a}$. A self-adjoint operator on L^2 is called a Hermitian operator.*

We will show that self-adjoint have one crucial property that allows them to measure physical properties of a state: their eigenvalues are real numbers.

Theorem 13. *Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and $\hat{a} : V \rightarrow \mathbb{C}$ be a self-adjoint operator.*

Then the following holds.

(A) *If $x \in V$ is an eigenvector of \hat{a} (i.e., if there exists a $\lambda \in \mathbb{C}$ such that $\hat{a}x = \lambda x$), then $\lambda \in \mathbb{R}$ is a real number.*

(B) *There exists a countable collection, $x_i \in V$, $i \in \{1, 2, 3, \dots\}$ of vectors that spans all of V . In other words, for any vector $x \in V$ there exists a collection of scalars $c_i \in \mathbb{C}$, $i \in \{1, 2, 3, \dots\}$ such that:*

$$x = c_1x_1 + c_2x_2 + c_3x_3 + \dots .$$

Proof. The proof for if V is finite-dimensional can be found in Arfken, section 9.4 [1]. □

If we accept that eigenvalues of quantum operators are physically measurable values of pure states, then part (A) of the above theorem shows those eigenvalues are real numbers when the operators are self-adjoint.

Furthermore, part (B) of the above theorem assures us that knowing the eigenstates of a quantum operator, such as knowing the states and measurable values, then permits us to know any other state in L^2 and write the state as a superposition of all the "pure states" of the operator.

The operators on L^2 of interest to quantum mechanics frequently involve taking derivatives of wavefunctions. However, the derivative itself is not a self-adjoint operator/

Proposition 14. *The derivative operator $\frac{d}{dx} : L^2([0, a]) \rightarrow \mathbb{C}$ is not self-adjoint, but $\hat{D} := i\frac{d}{dx}$ is self-adjoint.*

Proof. The adjoint of d/dx can be found by moving the derivative in the following expression from the function g to the function f . This uses integration by parts and the fact that if $f \in L^2([0, a])$, then $f(0) = 0$ and $f(a) = 0$.

$$\begin{aligned}\langle f, \frac{d}{dx}g \rangle &= \int_0^a \overline{f(x)}g'(x) dx \\ &= \overline{f(x)}g(x) \Big|_0^a - \int_0^a \overline{f'(x)}g(x) dx \\ &= -\langle \frac{d}{dx}f, g \rangle\end{aligned}$$

However, multiplying the derivative operator by i makes it self-adjoint:

$$\begin{aligned}\langle f, Dg \rangle &= \int_0^a \overline{f(x)}ig'(x) dx \\ &= \overline{f(x)}ig(x) \Big|_0^a - \int_0^a \overline{f'(x)}ig(x) dx \\ &= \int_0^a \overline{if'(x)}g(x) dx \\ &= \langle Df, g \rangle.\end{aligned}$$

□

This example shows how operators can be self-adjoint or may not be depending on the operator acting on the function. The operator, $i\frac{d}{dx}$ is better known, up to a factor of \hbar , as the operator for momentum, \hat{p} , which is essential to quantum mechanics.

4 Mathematics of the Quantum Classical Correspondence

While classical formalism and quantum formalism seem like very different things, they should, in general, give similar results. Classical mechanics is the study of observable quantities, and a

classical system is nothing more than a (large) collection of quantum particles. For this reason the mathematics of these two theories should not be too far apart from each other. The operators we apply in quantum mechanics should reflect in some way the classical observables, and similarly our classical observables should be able to be “scaled down” to become quantum operators.

One way this correspondence had been viewed in the past was through Bohr’s correspondence principle. Bohr drew an analogy between vibrational modes of classical mechanics and the stationary states (eigenfunctions) in quantum mechanics, postulating that each allowed quantum transition between stationary states corresponds to one harmonic component of the classical motion. [3] The deBroglie wavelength then differentiates between effects seen at the quantum scale from those seen at a classical scale.

This correspondence must also make sense of the probabilistic nature of measurement in quantum physics. One important result that accomplishes this is the theorem that shows the classical Poisson bracket corresponds to the quantum commutator.

Theorem 15 (Ehrenfest). *If \hat{F} is a quantum operator, then the expected value of \hat{F} evolves in time along a solution $\psi(x,t)$ of Schrödinger’s equation according to the equation*

$$\frac{d}{dt}\langle\hat{F}\rangle = \frac{1}{i\hbar}\langle[\hat{F},\hat{H}]\rangle.$$

Proof. See [6], Ch. 4. □

Comparing this theorem with Definition 5 which defined the Poisson bracket by the formula

$$\left.\frac{d}{dt}\right|_{t=0} f(x(t),p(t)) = \{f,H\},$$

we can see how the commutator in quantum mechanics has the same purpose as the Poisson

bracket did in classical mechanics: measuring the rate of change of physical quantities along solutions of the equations of motion.

In order to make this connection more precise, we want to know how to construct a quantum operator out of a classical observable.

4.1 Quantization of Physical Observables

To define a quantum operator whose expectation value corresponds to a physical observable, we will first look at a simple example using the Fourier transform and then at a more general approach using the Weyl quantization formula.

4.1.1 Quantization with Fourier Transforms

In physics, Fourier transforms are used to take a signal (a function whose domain is a time variable) and break it down into a collection of waves (a function whose domain is a frequency variable).

Definition 20. Let $\phi(x,t) \in L^2([0,a])$. The Fourier transform of ϕ is denoted $\Phi(p,t)$ and is defined by the formula

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \phi(x,t) dx.$$

Similarly, if $\Psi(p,t) \in L^2([0,a])$, the inverse Fourier transform of Ψ is denoted $\psi(x,t)$ and is defined by

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar} \Psi(p,t) dp$$

The Fourier transform will be used to give us a way to determine the effect of the momentum operator on a wavefunction whose domain is a position variable.

The position operator \hat{x} acts on a position-space wavefunction by multiplying $\psi(x)$ by x :

$$\hat{x}\psi(x) = x \cdot \psi(x)$$

Similarly, \hat{p} operates on a momentum-space wavefunction $\Psi(p)$ by multiplying it by p :

$$\hat{p}\Psi(p) = p \cdot \Psi(p)$$

Knowing these both separately is beneficial, but in order to understand the physical relation, the Fourier inverse transform has to be employed of $p \cdot \Psi(p)$ in order to determine the effect in position space. Using the definition that

$$\Psi(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} \psi(x) dx$$

It then follows,

$$\begin{aligned} \mathcal{F}^{-1}(p \cdot \Psi(p))(x) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx} p \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} \psi(x) dx dp \\ &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{ipx} p \left(\frac{i}{p} e^{-ipx} \psi(x) \Big|_{-\infty}^{\infty} - \frac{i}{p} \int_{-\infty}^{\infty} e^{-ipx} \psi'(x) dx \right) dp \\ &= -\frac{i}{2\pi} \int_{-\infty}^{\infty} e^{ipx} \int_{-\infty}^{\infty} e^{-ipx} \psi'(x) dx dp \\ &= -i \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx} \psi'(x) dx dp \\ &= -i \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx} \mathcal{F}(\psi')(p) dp \\ &= -i\hbar \psi'(x) \end{aligned}$$

Thus quantizing momentum, p , such that $p = -i\hbar \frac{\partial}{\partial x}$ which is the definition of momentum. Therefore, the Fourier transform is a satisfactory way to quantize things in this case, though it will not always work easily.

Now that x and p have been quantized, it is important to note that this is an algebraic homomorphism so that any observable represented by a polynomial in x and p can be quantized in this fashion.

4.1.2 Weyl Quantization

While Fourier Transforms and integrals are useful for converting observables from classical mechanics to quantum mechanics, we wish to find a more general process that will work for any observable function $a(x, p)$ on phase space. The Weyl quantization formula provides that process.

Definition 21. *The Weyl quantization is an mapping from functions on phase space to operators on Hilbert space. If $a : (\mathbb{R}^n)^2 \rightarrow \mathbb{R}$ is an observable on an n -dimensional phase space, then its Weyl quantization is an operator $\hat{a} : L^2(\mathbb{R}^n) \rightarrow \mathbb{C}$ defined by the formula:*

$$\hat{a}(x, hD)u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi.$$

For this thesis we will only be concerned with the case where $n = 1$, since the examples we looked into are one-dimensional.

If we want to use this formula to quantize physical observables, we need to know under what circumstances the operators it produces are self-adjoint. The following proposition accomplishes this. Here, we consider functions on $L^2([0, L])$.

Proposition 16. *Let $a : \mathbb{R}^2 \rightarrow \mathbb{C}$ be a function. Then the adjoint of its Weyl quantization is the Weyl quantization of its complex conjugate:*

$$Op(a)^\dagger = Op(\bar{a}).$$

Proof.

$$\hat{a}(\hat{x}, \hat{p})u(x) = \frac{1}{2\pi h} \int_0^L \int_0^L e^{\frac{-i}{h}(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) dy d\xi dx$$

In our case we will take $Op(a)$, thus we will take the complex conjugate of $Op(a)$, such that

$$\langle u, Op(a)^\dagger v \rangle = \int_0^L \overline{u(x)} \int_0^L \int_0^L e^{\frac{-i}{h}(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) v(y) dy d\xi dx$$

$$\begin{aligned}
&= \int_0^L \int_0^L \int_0^L \overline{u(x)} e^{\frac{-i}{\hbar}(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) v(y) dy d\xi dx \\
&= \int_0^L \int_0^L \int_0^L \overline{u(x)} e^{\frac{-i}{\hbar}(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) d\xi dx v(y) dy \\
&= \int_0^L \int_0^L \overline{u(x)} e^{\frac{-i}{\hbar}(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) d\xi dx \int_0^a v(y) dy \\
&= \underbrace{\int_0^L \int_0^L \overline{u(x)} e^{\frac{-i}{\hbar}(x-y)\xi} a\left(\frac{x+y}{2}, \xi\right) d\xi dx}_{\bar{f}} \int_0^a v(y) dy \\
&= \langle Op(\bar{a})u, v \rangle
\end{aligned}$$

Therefore,

$$Op(a)^\dagger = Op(\bar{a}).$$

□

Corollary 17. *If $a : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a real-valued physical observable, then its Weyl quantization $Op(a)$ is a self-adjoint operator.*

This assures us that the operator produced by the Weyl quantization from a real-valued physical observable will be self-adjoint as required in the quantum theory.

Example: To quantize the position observable $x(x, p) = x$ using the Weyl formula, let $a(x, p) = x$ in the definition. Then

$$\begin{aligned}
Op(x)u(x) &= \int_0^L \int_0^L e^{\frac{-i}{\hbar}(x-y)p} \cdot \frac{x+y}{2} u(y) dy dp \\
&= \int_0^L \left(\frac{x+y}{2}(y) \int_0^L e^{\frac{-i}{\hbar}(x+y)p} dp dy \right) \\
&= \int \frac{x+y}{2} u(y) \cdot \frac{i\hbar e^{\frac{-i}{\hbar}(x-y)p}}{(x-y)} dy \\
&= \frac{x+y}{2} \delta(x-y) dy \\
&= \frac{x+x}{2} \cdot u(x) \\
&= xu(x)
\end{aligned}$$

(Note, the Dirac δ function satisfies the identity:

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-ikx} dx \\
&= \delta(k)
\end{aligned}$$

Therefore when $k = 0$ the integral is equal to 1. When $k \neq 0$, $ie^{-ix}|_{-\infty}^{\infty} = 0$.)

Therefore, the operator on L^2 which corresponds to the observable function of position is an operator which multiplies its wavefunction by x .

The Weyl quantization is a special case of a more general formula: for a parameter $0 \leq t \leq 1$, the general formula produces an operator $Op_t(a)$ according to

$$Op_t(a)(u(x)) = \frac{1}{2\pi} \int_0^L \int_0^L e^{i/\hbar(x-y)\xi} a(tx + (1-t)y, \xi) u(y) dy d\xi.$$

This more general quantization satisfies the adjoint formula

$$Op_t(a)^\dagger = Op_{1-t}(\bar{a}).$$

The case where $t = \frac{1}{2}$ and a is real-valued gives the self-adjoint Weyl quantization. [11] It follows that the Weyl quantization still meets the properties of producing real valued numbers, thus

applicable to classical formalism.

5 Conclusion

Starting out by looking at the mathematical applications in classical physics we looked in depth into how observables measure physical properties of Hamiltonian systems. Continuing on we followed a similar pattern for quantum mechanics, finding the mathematical characterization of wavefunctions in the Hilbert space L^2 and how self-adjoint operators measure physical properties through their real-valued eigenvalues. Finally we concluded by looking into the connection between quantum mechanics and classical mechanics. In doing this we saw that there is a wide variety of links connecting classical observables to quantum operators. One example of this was our example on the position and momentum observables which when quantized arrived at a quantum operator. This solution can be generalized for a variety of classical observables. While we used this specific example, the Weyl quantization is capable of quantizing any classical observable, producing an operator that is self-adjoint. This connects the (complex-valued) wavefunctions of quantum mechanics to (real-valued) systems of classical physical observables.

A further connection between quantum and classical mechanics is found in semiclassical analysis [11]. This bridge uses the connection of Planck's constant \hbar , treating it as a small parameter in the system. In the quantum realm, \hbar is a (significant) positive real number while in the "semiclassical" limit as $\hbar \rightarrow 0$, the theory produces classical mechanics. While we looked into a couple ways, we could have further solidified our connection with more methods to quantize a physical observable. Another thing to notice is that while there is a correspondence between classical and quantum mechanics, general relativity and quantum mechanics are incompatible. A large part of

this incompatibility is the geometry of the underlying space: quantum mechanics, due to its small scale, is thought of as a theory on a flat space, while the scale of general relativity makes the curvature of large volumes of spacetime impossible to ignore. To understand more of this, refer to Sach's text [8].

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