Bianchi Identities and Weak Gravitational Lensing

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Abstract

Gravitational lensing is the bending of light rays due to the gravitational attraction between light and massive objects such as galaxies. Weak gravitational lensing, the distortion of the shapes of light rays, and general relativity, our modern theory of gravity, have had divergent paths. Astronomers who study weak lensing don't rely on the principles of general relativity but use approximations to understand their observations. The purpose of this paper is to study how general relativity can be used to explain weak gravitational lensing. The formal language for general relativity's explanation of weak lensing is a null tetrad of vectors associated with light rays and the spin coefficient formalism. The Bianchi identities, which come from the theory of relativity, may be the fundamental equations of weak lensing.

I. Introduction

The phenomena of gravitational lensing occurs when a large object is between an observer and light rays emitted from a distant star. According to general relativity, light is bent by significantly massive objects. Weak lensing is when a massive object (the lens) is between the source and the observer and the light from the source is bent in such a way as to make the images we would normally see more elliptical or sheared.

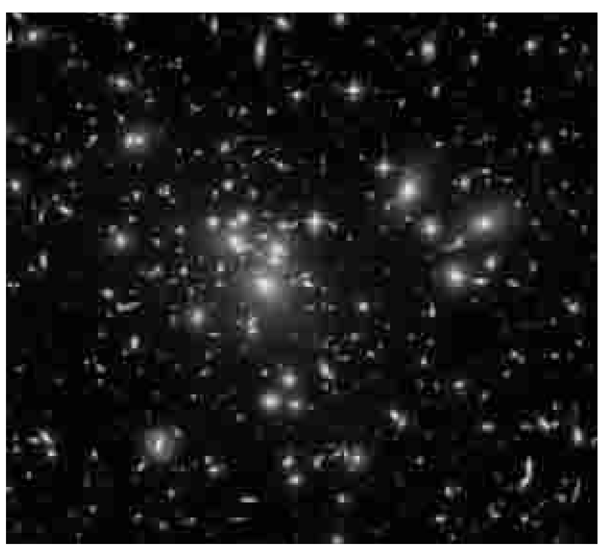


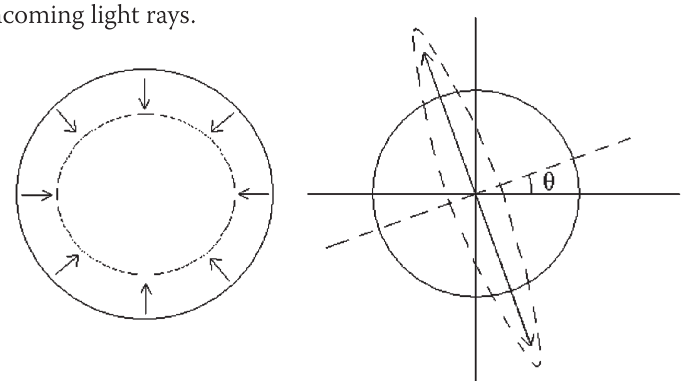
Figure 1 is an example of weak lensing. This is an HST image of Abel 1969, a cluster of galaxies (yellow blobs). Notice that the light ray rings that have formed on the edges of the edges of the galaxies. The gravity of the cluster has sheared the light rays emitted from galaxies behind it relative to us.

Currently, there is a way to find the mass of the object in the way of the source and observer. The mass of the lens can be obtained from information about the amount of shearing that the source images undergo. The approach that astrophysicists use to reach this goal does not start from a fundamental equation in general relativity. But physicists like to categorize things; finding the fundamental equations in general relativity would make the theory more complete and would give it more credibility. We believe that we may have found the fundamental equations of weak lensing in the Bianchi identities of general relativity.

We are speculating that integrating the Bianchi identities, a set of nine coupled partial differential equations which relate components of the Ricci curvature tensor to components of the Weyl curvature tensor, is the key to finding the fundamental equations of weak lensing. We will be working in the null tetrad and the spin coefficient formalism. The null tetrad is a way to track a light ray's path. This is done by identifying four vectors, one that is in the direction of the light ray, one that is perpendicular, and two complex ones that curl in opposite directions around the light ray. The spin coefficients are a set of 12 quantities that describe certain properties of the light rays (also called a pencil of rays) in question.

For example, r is the convergence of the light rays and s is the shear of the light rays as seen in Figure 2 below. The circle with the arrows on the left side of the picture represents the convergence of a pencil of rays. The radius of the cross-sectional area of the pencil of rays diminishes. The picture on the right represents the shearing or stretching of the pencil of rays. This gives the circular pencil of rays eccentricity. We can imagine looking at this picture from the front and seeing the pencil of rays as elliptical. From the side we can see that the reason is the pencil gets shifted from the plane so that light rays at the bottom of the pencil is closer than rays at the top of the pencil. This shift is what produces the elliptical shape of the pencil.

Figure 1: The figure on the left shows convergence which is when the radius of the pencil of light rays shrink. The picture on the right demonstrates shearing. When a pencil is sheared, the circle's plane gets shifted which results in the twisting of the incoming light rays.



The first Bianchi identity in spin coefficient formalism is

$$\bar{\delta}\,\Psi_0 - D\,\Psi_1 + D\,\Phi_{01} - \delta\,\Phi_{00} = (4\alpha - \pi)\,\Psi_0 - 2(2\rho + \varepsilon)\,\Psi_1 + 3\kappa\,\Psi_2$$

$$+ (\pi^{-} - 2\bar{\alpha} - 2\beta)\Phi_{00} + 2(\varepsilon + \bar{\rho})\,\Phi_{01} + 2\sigma\,\Phi_{10} - 2\kappa\,\Phi_{11} - \bar{\kappa}\,\Phi_{02}. \quad [1]$$

The lower case greek letters on the right side of Eq. 1 are spin coefficients, the Ψ_A are Weyl tensor components and the Φ_{AA} are the Ricci tensor components. On the right hand side of the equation, δ^{-}, δ, and D are derivative operators.

The Ricci and Weyl tensors are the curvature tensors that we hope to find and the topic in Section II. The null tetrad, explained in section III, is the mathematical language we have employed to express weak lensing in, as well as the Ricci and Weyl tensors. The spin coefficient formalism follows from this and is also discussed in this section. Section IV explains how the first Bianchi identity could simplify down to the fundamental equation of weak lensing by good choice of tetrad, perturbating to first order in the weak perturbation of spacetime, and restricting the light rays to the lens plane. In Section V, we discuss the relevance of our findings and future work with this project.

II. Ricci and Weyl tensors

The Ricci tensor is one of the components of the decomposition of the Riemann tensor; the other is the (trace-free) Weyl tensor. Before we look at these tensors we must first introduce the Christoffel symbols defined:

$$\Gamma^{\rho}_{\mu\upsilon} = \tfrac{1}{2}\Sigma_{\sigma} g^{\rho\sigma}\left(\frac{\partial g_{\upsilon\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\mu\sigma}}{\partial x^{\upsilon}} - \frac{\partial g_{\mu\upsilon}}{\partial x^{\sigma}}\right) \quad [2]$$

It is the idea of covariance that is pivotal to general relativity. The idea of parallel transport of vectors is related to this. If one parallel transports a vector in flat space around a small closed loop with ordinary derivative operators, the vector will return to its original position with its integrity intact. But this is not true in curved space or even for spheres in flat space. As we change coordinate systems we must use covariant derivatives to preserve the integrity of vectors. The Christoffel symbols mediate what the derivative operators are in different coordinate systems. The symbols are a map to the covariant derivatives for the particular coordinate system. The Riemann tensor is a (0,4) curvature tensor that utilizes the fact that parallel transporting is path dependent in a curved space. It is defined in terms of the Christoffel symbols, with ``, index" denoting partial derivative with respect to the index:

$$R_{\mu\upsilon\rho}{}^{\sigma} = \Gamma^{\sigma}_{\mu\rho,\upsilon} - \Gamma^{\sigma}_{\upsilon\rho,\mu} + \Sigma_{\alpha}\left(\Gamma^{\alpha}_{\mu\rho}\Gamma^{\sigma}_{\alpha\upsilon} - \Gamma^{\alpha}_{\upsilon\rho}\Gamma^{\sigma}_{\alpha\mu}\right).$$

[2] (3)

The Riemann tensor obeys four properties,

1. $R_{abc}{}^{d} = -R_{bac}{}^{d}$,
2. $R_{[abc]}{}^{d}$, where "[ ]" denotes commutation of the indices (i.e. $R_{[abc]} = R_{abc} - R_{bac}$),
3. For the derivative operator ∇_a naturally associated with the metric, $\nabla_a g_{bc} = 0$, we have $R_{abcd} = -R_{abdc}$.
4. The Bianchi identity holds $\nabla_{[a} R_{bc]d}{}^{e} = 0$.

Contracting Eq. 3 yields the Ricci tensor,

$$R_{\mu\rho} = \Sigma_{\upsilon} R^{\upsilon}_{\mu\upsilon\rho} = \Sigma_{\upsilon}\Gamma^{\upsilon}_{\mu\rho,\upsilon} - \Sigma_{\upsilon}\Gamma^{\upsilon}_{\upsilon\rho,\mu} + \Sigma_{\alpha,\upsilon}\left(\Gamma^{\alpha}_{\mu\rho}\Gamma^{\upsilon}_{\alpha\upsilon} - \Gamma^{\alpha}_{\upsilon\rho}\Gamma^{\upsilon}_{\alpha\mu}\right)$$

(4)

The Weyl tensor is the trace-free, anti-symmetric part of the decomposition of the Riemann tensor. It is a product of commuting the metric and the Ricci tensor,

$$C_{abcd} = \frac{2}{n-2}\left(g_{a[c}R_{d]b} - g_{a[c}R_{d]a}\right) - \frac{2}{(n-1)(n-2)} R g_{a[c}g_{d]b} - R_{abcd},$$

[2] (5)

where R is the scalar curvature, defined as the trace of the Ricci tensor. The four properties above that hold for the Riemann tensor also hold for the Weyl tensor.

III. Null tetrad and spin coefficient formalism

The null tetrad is a set of four vectors and is used to track a light ray's path. The four vectors identified are in the direction of the light ray, perpendicular to it, and two complex ones that curl in opposite directions around the light ray. The spin coefficients are a set of 12 quantities that describe certain properties of the light rays in question. For example, r is the convergence of the light rays and s is the shear of the light rays.

Since we are using the null tetrad and spin coefficient formalism, the Ricci and Weyl tensors must be expressed in that way. The tensor quantities in spin coefficient formalism are attained by contracting the quantities with prescribed tetrad components. We have computed the Φ_{AA} and the Ψ_{A} in a weak field metric of a gravitational potential $\varphi(x,y,z)$ satisfying $\nabla^2\varphi = 4\pi\rho$ where $\rho(x,y,z)$ is the matter density.

We align the light rays along the z-axis by choosing the tetrad to be

$$\ell^a = \frac{1}{\sqrt{2}}(1,0,0,1), \qquad n^a = \frac{1}{\sqrt{2}}(1,0,0,-1),$$
$$m^a = \frac{1}{\sqrt{2}}(0,1,i,0), \qquad \overline{m}^a = \frac{1}{\sqrt{2}}(0,1,-i,0). \tag{6}$$

This allows us to define four derivative operators as

$$\delta = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \qquad D = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right)$$
$$\overline{\delta} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \Delta = \frac{-1}{\sqrt{2}}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right) \tag{7}$$

We assume Minkowski (flat) space for the spin coefficients. This condition forces the spin coeffients to be 0. The calculation of the Ricci tensor is given in Appendix A.1. We choose to perturbate to first order in j, and as a result, the Ricci tensor simplifies to

$$\Phi_{00} = \Phi_{22} = 2\Phi_{11} = \frac{\nabla^2\varphi}{2}. \tag{8}$$

The calculation of the Weyl tensor is in Appendix A.2 and is

$$\Psi_0 = \Psi_4 = \frac{1}{2}\left(\varphi_{xx} - \varphi_{yy} + 2i\varphi_{xy}\right),$$
$$\Psi_1 = -\overline{\Psi}_3 = \frac{1}{2}\left(-\varphi_{xx} + i\varphi_{yz}\right), \tag{9}$$
$$\Psi_2 = \frac{1}{2}\left(\varphi_{zz} - \frac{\nabla^2\varphi}{3}\right).$$

IV. 1st Bianchi identity: the fundamental equation of weak lensing?

At this point we can create a differential relation between the Ricci and Weyl tensors by using the results of the previous section and using a thin lens approximation. We believe that the first Bianchi identity may simplify to the fundamental equations of weak lensing. The Bianchi identities are the result of taking covariant derivatives of the commuted Riemann tensor and contracting it.

They are defined as

$$\nabla_{[a}R_{bc]d}{}^{e} = 0 \quad [2] \qquad (10)$$

Start with the first Bianchi identity,

$$\bar{\delta}\,\Psi_0 - D\,\Psi_1 + D\,\Phi_{01} - \delta\,\Phi_{00} = (4\alpha - \pi)\,\Psi_0 - 2(2\rho + \varepsilon)\,\Psi_1 + 3\kappa\,\Psi_2$$
$$+ (\bar{\pi} - 2\bar{\alpha} - 2\beta)\Phi_{00} + 2(\varepsilon + \bar{\rho})\,\Phi_{01} + 2\sigma\,\Phi_{10} - 2\kappa\,\Phi_{11} - \bar{\kappa}\,\Phi_{02}, \; [1]$$

(11)

we identify $D, \Delta, \delta, \text{and}\,\bar{\delta}$ are derivative operators that are defined in Eq. 7 and are associated with the null tetrad. The first Bianchi identity after we assume flat space and perturbate to first order discussed in previous sections is

$$\bar{\delta}\,\Psi_0 - D\,\Psi_1 - \delta\,\Phi_{00} = 0 \qquad (12)$$

We assume that the Weyl tensor has support in the lens plane and define $\Phi_{00} = {}_L\Phi_{00}\,\delta(z\text{-}L)$ (the lens is thin) and $\Psi_i = {}_L\Psi_i\,\delta(z\text{-}L)$ where $\delta(z\text{-}L) = \begin{cases} 1 & z=L \\ 0 & z\neq L \end{cases}$ and $\int \delta(z)dz = 1$. We will proceed to integrate out in the z direction (perpendicular to the lens plane) to compress all the matter into the lens plane, yielding

$$\int_{-\infty}^{\infty} \bar{\delta}\,\Psi_0 \; dz = \int_{-\infty}^{\infty} \bar{\delta}_L \Psi_0 \; \delta(z-L)dz = \bar{\delta}_L \Psi_0 \;,$$

$$\int_{-\infty}^{\infty} \delta\,\Phi_{00} \; dz = \int_{-\infty}^{\infty} \delta_L \Phi_{00} \; \delta(z-L)dz = \delta_L \Phi_{00} \;,$$

(13)

Due to the condition of asymptotic flatness in Minkowski space, φ is zero at ∞ and the same goes for any change in φ, thus,

$$\int_{-\infty}^{\infty} D\Psi_1 \; dz = \int_{-\infty}^{\infty} -\frac{1}{2}\left(\varphi_{xz} + i\varphi_{yz}\right)dz = -\frac{1}{2}\left(\varphi_x + i\varphi_y\right)\Big|_{-\infty}^{\infty} = 0.$$

(14)

Now we can write the final result as

$$\bar{\delta}\,\Psi_0 - \delta\,\Phi_{00} = 0 \qquad (15)$$

The Weyl tensor is a measurable quantity and we can find the mass density from this equation.

V. Discussion

An integral relationship between the Ricci and the Weyl tensors has already been found where the Weyl tensor is some kernel of the Ricci tensor and the setting is the lens plane:

$$_L\Phi_{00}(\vec{r}) = \int (d\vec{r}\,')\frac{{}_L\Psi_0 \vec{r}}{\pi}\frac{e^{-2i\eta}}{\left|\vec{r} - \vec{r}\,'\right|^2}, \quad [3]$$

(16)

We have found a differential relation between the two. These two different formats for weak lensing lends to each other's relevance. The integral relation is stable, smoothes out noise and is less precise than the differential version which is at times unstable, but more exact.

A very important aspect about the differential version is that it was found by starting with first principles. If this equation is relevant, then it lends to the foundation and the strength of general relativity as a theory, as well as being another approach to weak lensing. Also, it could serve as a first principle to other physicists who would like to find an equation for something starting with first principles in general relativity.

To prove the relevance of this equation of weak lensing we must first plug in values for a simple mock case. If the numbers come out correct there, then we will try it out with real data. This involves writing code that will map the values of the spin coefficients and Weyl tensor at every point in the lens plane to the Ricci tensor that we expect.

A. Appendix

A.1 Ricci tensor

The Ricci tensor components R_{ab} in spin coefficient formalism Φ_{AA} are computed by contracting over a combination of two vectors from the null tetrad chosen,

$$\ell^a = \frac{1}{\sqrt{2}}(1,0,0,1), \qquad n^a = \frac{1}{\sqrt{2}}(1,0,0,-1),$$
$$m^a = \frac{1}{\sqrt{2}}(0,1,i,0), \qquad \bar{m} = \frac{1}{\sqrt{2}}(0,1,-i,0). \qquad (17)$$

The contractions that produce non-trivial results are defined:

$$\Phi_{00} = -\frac{1}{2}R_{00} = -\frac{1}{2}R_{ab}\ell^a\ell^b = -\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right)\nabla^2\varphi = \frac{-\nabla^2\varphi}{2},$$
$$\Phi_{11} = -\frac{1}{2}\left(R_{01}+R_{23}\right) = -\frac{1}{2}\left(R_{ab}\ell^a n^b + R_{ab}m^a\overline{m}^b\right) = -\frac{1}{2}\left(0+\left(\frac{1}{2}-\frac{i^2}{2}\right)\nabla^2\varphi\right) = \frac{-\nabla^2\varphi}{4}$$
$$\Phi_{22} = -\frac{1}{2}R_{11} = R_{ab}n^a n^b = -\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right)\nabla^2\varphi = \frac{-\nabla^2\varphi}{2}.$$

The subscripts in the component denotes which vector to contract with. For instance, $R_{01} = R_{ab}\lambda^a n^b$ while $R_{23} = R_{ab}m^a\overline{m}^b$.When the vectors are chosen only the contractions over the same part of respective vectors are eligible to be non-zero since $R_{aa} = \nabla^2\varphi$.

A.2 Weyl tensor

The Weyl tensor components (C_{abcd}) in spin coefficient formalism (Ψ_A) are computed by contracting over a combination of 4 vectors from the null tetrad. The spin coefficient version components of the Weyl tensor are defined:

$$\Psi_0 = -C_{0202} = -C_{abcd}\ell^a m^b \ell^c m^d,$$
$$\Psi_1 = -C_{0102} = -C_{abcd}\ell^a n^b \ell^c m^d,$$
$$\Psi_2 = -\frac{1}{2}\left(C_{0101} - C_{0123}\right) = -\frac{1}{2}\left(C_{abcd}\ell^a n^b \ell^c n^d - C_{abcd}\ell^a n^b m^c \overline{m}^d\right),$$
$$\Psi_3 = C_{0113} = C_{abcd}\ell^a n^b n^c \overline{m}^d,$$
$$\Psi_4 = -C_{1313} = -C_{abcd}n^a m^{b-} n^c \overline{m}^d.$$

(19)

A symbolic logic program calculated the components of the Weyl tensor to be

$$C_{0i0i} = \frac{1}{3}\left(-3\varphi_{ii} + \nabla^2\varphi\right), \qquad C_{0i0j} = -\varphi_{ij}, \quad i \neq j$$
$$C_{ijij} = \frac{1}{3}\left(3\varphi_{kk} - \nabla^2\varphi\right), \qquad C_{ijik} = -\varphi_{jk}, \quad i \neq j \neq k$$

(20)

where i, j and k can represent either x, y or z.

The non-zero calculations omitting the specific vector contractions are

$$\Psi_0 = \Psi_4 = -\left(C_{obod} + C_{3b3d}\right) = \frac{1}{2}\left(\varphi_{xx} - \varphi_{yy} + 2i\varphi_{xy}\right),$$
$$\Psi_1 = -\Psi_3^- = -\left(C_{03cd} + C_{30cd}\right) = \frac{1}{2}\left(-\varphi_{xx} + i\varphi_{yz}\right),$$
$$\Psi_2 = -2\left(C_{03cd} + C_{30cd}\right)\frac{1}{2}\left(\varphi_{zz} - \frac{\nabla^2\varphi}{3}\right).$$

(21)

References

[1] E. T. Newman, K. P. Tod. Asymptotically Flat Space-Times from General relativity and Gravitation, Vol. 2, (Picnum publishing Corporation, 1980).

[2] Robert Wald. General Relativity, (University of Chicago press, Chicago 1984).

[3] Kling, T.P. 2002, Talk at Connecticut college.

[4] Jeremy Bernstein, Paul M. Fishbane, Stephen Gasiorowicz. Modern Physics, (Prentice Hall, New Jersey, 2000).

[5] Albert Einstein. Relativity, (Crown Publishers Inc., New York 1961).

[6] James B. Hartle. Gravity: An introduction to Einstein's general relativity, (Pearson Education Inc., San Francisco, 2003).

[7] Steven R. Lay. Analysis with an Introduction to Proof, (Upper Saddle River, New Jersey: Prentice Hall Inc., 1999).

[8] H. A. Lorentz, H. Weyl, H. Minkowski. Notes by A. Sommerfield. The Principle of Relativity, (General Publishing Company, Toronto 1952).

[9] James Peacock. Cosmological Physics, (Cambridge University press, Cambridge 1999).

[10] R. Penrose, W. Rindler. Spinors and spacetime: volume I, (Cambridge University Press, Cambridge 1984).

[11] R. Penrose, W. Rindler. Spinors and spacetime: volume II, (Cambridge University Press, Cambridge 1986).