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Abstract

In this paper we examine the geodesic deviation equation using the Newman-Penrose (N-P) formalism for a flat Friedmann-Lemaitre-Robertson-Walker(FLRW) metric (Carroll,S. (2004); Ryden, B. (2003); Newman & Penrose(1962)). We solved the geodesic deviation equation for angular diameter distance, using the relevant N-P components, and the resulting expression was the Dyer-Roeder equation of cosmology (Ryden, B. (2003)) (Schneider et al.(1992)). This leads us to believe that we can apply the N-P formalism to a perturbed FLRW metric and find a solvable equation for angular diameter *distance* (Kling & Campbell(2008)). The perturbed FLRW metric incorporates clumps of matter into a metric that is on average homogeneous and isotropic. Deriving a solvable equation for angular diameter distance, in a perturbed FLRW metric, could prove useful to astronomers. By including clumps of matter along the line-of-sight into the math, we can calculate distances to light emitting objects that are obstructed by weak gravitational fields. This thesis should serve as a test of both our tetrad and our methodology, by showing that they work in the flat unperturbed metric.

Contents

Section 1

Purpose and Introduction

1.1 Purpose Statement

The purpose of this thesis is to derive the Dyer-Roeder equation from the geodesic deviation equation for a flat Friedmann-Lamaitre-Robertson-Walker (FLRW) metric using the Newman-Penrose null-tetrad.

1.2 Introduction

The Dyer-Roeder equation is an equation used for calculating distances to astronomical objects. The Dyer-Roeder equation approximates the universe to have a uniform density in all directions, i.e. that the universe is homogeneous and isotropic (Foster et al (1995)). The metric used to derive the equation assumes there are no clumps of matter in the space-time, which makes the equation for distance simple enough to derive. The assumption that the universe has a uniform density limits the scope of objects to which we can calculate the distance. If light from an astronomical object on its way to earth, passes through the gravitational field of a clump of matter then we cannot calculate the distance to that object using Dyer-Roeder equation. Attempts to find a solvable expression for angular diameter distance using a metric that allows for clumps of matter have historically been unfruitful. We think this failure is a direct result of using basis vectors that are best suited to deal with flat space-times rather than curved space-times. In this thesis we attempt to show that the N-P null-tetrad of basis vectors is better suited for the curvature of space-time associated with clumpy cosmologies.

The N-P formalism for General Relativity is useful in dealing with motion of light-bundles, or a propagating pencil of light rays, in a curved space-time. It allows us to deal with problems arising from the curvature of space-time due to local variations in matter density by introducing the null tetrad of basis vectors. In the phrase “null-tetrad”, the word “null” means light-like and “tetrad” means a set of four. The reason we need four vectors is due to the fact the we are working in a four dimensional space-time which requires four independent bases vector to span the whole space. Since astronomers study astronomical objects by observing light emitted from these objects, it makes sense that we would use the null-tetrad of basis vectors when deriving an equation for the distance to these objects. Before this can be accomplished we should be able to show that the N-P null-tetrad can produce the equation for angular diameter distance for a flat FLRW. Once this is shown to be the case we can calculate angular diameter distance for the perturbed FLRW.

This derivation of the Dyer-Roeder equation is a first step in obtaining an equation for angular diameter distance in a perturbed FLRW metric using the N-P terad. We start our derivation with a discussion about the null tetrad. We then use the flat FLRW metric to calculate the N-P components needed to solve the geodesic deviation equation for angular diameter distance. Finally we make the appropriate substitutions to get the Dyer-Roeder equation.

Section 2

Background

2.1 Historical Background

The principle of relativity, as attributed to Galileo, roughly states that the laws of classical mechanics hold true for all observers in inertial reference frames (McEvoy, J. P. (2010), p. 110). The principle did not come to include gravity and the laws of electromagnetism until the twentieth century. When the laws of electricity and magnetism were being developed by physicists in the eighteenth century it became clear that light was an electromagnetic wave. The aether was introduced to explain how the wave propagates through space. The aether is a medium that fills all space and was famously proposed by Descartes to explain the propagation of heat (Whittaker, E. T. (1951)). The idea is to explain action at a distance using mechanical models. The aether is then a medium made of small objects that transmit the heat or light. The idea that something can, without the aid of a medium or an agent, affect other objects had been rejected by many great scientists at the time of the discovery of laws of electromagnetism. When writing about the effects of gravity, Sir Isaac Newton said:

> *"That gravity should be innate, inherent and essential to matter, so that one body should act upon another at a distance, through a vacuum, without the mediation of anything else, by and through which their force may be conveyed from one to another, is to me so great an absurdity that I believe no man who has, in philosophical matters, a competent faculty of thinking, can ever fall into it. Gravity must be caused by an agent, acting constantly according to certain laws, but whether this agent be material or immaterial, I leave to the consideration of my reader."*

Newton was not alone in this opinion. When attempts to preserve the law of electromagnetism for all reference frames using the idea of the aether failed, along with the failure to detect the effects of the aether experimentally, modern theories of relativity were proposed. What Einstein, and others, realized was that the assumptions made about the nature of space-time itself were the key to the failures of earlier attempts to preserve the laws of electricity and magnetism for all inertial observers. Einstein recognized that to preserve the laws of physics he did not need to postulate the aether, for which no evidence could be produced. Instead, Einstein questioned other assumptions for which he had no experimental motivation. He knew that the laws of electromagnetism came directly from experiment and that there was no evidence for the space-time of Newton. Building on the work of Lorentz and others Einstein proposed the theory of special relativity (McEvoy, J. P. (2010)).

The theory of special relativity preserves the laws of electromagnetism by only assuming the speed of light is preserved in all reference frames at rest or in constant motion relative to an inertial observer. This assumption was motivated by the fact that the speed of light comes directly from Maxwell's equations of electricity and magnetism which are experimentally derived. These equations were formulated using the experimental values for strength of the electric and magnetic fields. Since the laws of electricity and magnetism come from experiment and the idea that space and time remains static and constant for all inertial observers does not, he chose to discard the latter. The classical laws of mechanics, including the implied relativity, were formulated assuming a static Cartesian space-time and that was a mistake. The success of Newton's theories led physicists to believe that this seemingly natural assumption was true. Newton's laws, while incomplete, are still good enough to get humans to the moon and back and are still taught in all introductory physics classes today.

When Einstein omitted this assumption from Newton's theory, and only assumed the laws of physics are the same for all inertial observers, he solved the problem of a constant speed of light, as predicted by Maxwell. A constant speed of light appears to be a violation of classical relativity in a static-Newtonian space-time. This is not be confused with static, or steady-state, cosmologies which will be referred to later in this thesis. When Einstein postulates that space is contracted and time is dilated for objects in relative constant motion to an inertial observer, he is able to make Maxwell's equations universal by preserving them in all inertial reference frames. Einstein's theory of special relativity uses a coordinate transformation that predicts time and space are not static upending Newtonian space-time. For observers in relative motion,to an observer in an inertial frame, measurements of time and space are not the same but the result is that the laws of physics are preserved. Predictions of Einstein's theory have been confirmed and are used so widely that any theory is considered incomplete unless it is fully relativistic in the Einstein sense.

Einstein then turned his attention to gravity. Eventually he noticed that if he wanted to only preserve the laws of classical mechanics, he needed to abandon the Cartesian space-time of the eighteenth century and allow for curved space-time. In classical mechanics, objects at rest or moving with a constant velocity will remain in that state unless acted upon by an external force. Physicists observed that small objects near massive objects appear to accelerate (change direction and/or velocity) toward the massive objects. This change in direction and velocity was attributed to the force of gravity. Sometime in the 1600's the aether was proposed as a means of mediating this force, making it a mechanical force. The failure of the aether theories and the emergence of special relativity meant that hopes of explaining gravity as a force transmitted by the aether were dashed as well.

Einstein showed that gravity is not a real force. Instead he explained the apparent bending of trajectories near massive objects to be a consequence of the curvature of space-time. From his work on special relativity, Einstein knew that space-time was not the static Cartesian space-time assumed by Newton. Space-time is a relative phenomenon having differing size and pace for different observers in motion relative to each other. By allowing space-time to be curved, e.g., not Euclidean, around massive objects he was able to explain gravity as motion along the shortest path in curved space-time. This revelation is very important since it turns gravity into an artifact of geometry, rather than an inherent force of nature. This also means that electromagnetic waves, e.g., light, will be affected by this curvature and no interaction with particles or mediators is necessary. This very useful if one is near a gravitational well and needs to communicate using electromagnetic waves. The gravitational well created by the mass of the earth needs to be accounted for in order to get a Global Positioning System (GPS) which operates accurately using electromagnetic waves. The clocks on satellites used for GPS run faster than those measuring time on the surface of earth. This is because the distance from the center of the gravitational well created by the mass of the earth is larger than at the surface. The curvature of space-time only causes small discrepancies in the form of time dilation at the surface but these discrepancies are enough to make GPS useless (Xu, G. (2003), p. 64).

Einstein proposed his theory as a set of equations combined into a tensor equation called the Einstein Field Equation (EFE) (Carroll,S. (2004)). Exact solutions to the EFE were not initially offered by Einstein himself; instead, he offered an approximate solution. Einstein thought that it might not be possible to find exact solutions to the EFE due to mathematical complications. The first exact solution to the EFE was offered in the same year the equations were introduced by Einstein. In 1915 a German Astronomer, Karl Schwarzschild, came upon an exact solution to the EFE by using a different coordinate system than that used by Einstein (Brown, L. M. et al(1995)). While other solutions have been found, we will only be interested in the exact solution found by Alexander Friedmann and Georges Lemaître independently in the 1920's. This is the solution now used by cosmologists to model the expanding space-time in which we appear to reside.

Initially the solution was arrived at by Friedmann and he recognized that the solution supported an expanding universe. The solution was published in 1924 but never gained any attention. It was dismissed by those who did notice it as a purely mathematical solution and one that was not a description of the physical universe. Friedmann died soon after and his solution was forgotten. A few years later, in 1927, Georges Lemaître independently came up the same solution. The publication also went mostly unnoticed, until he published again in 1931 proposing the "big-bang" theory. Lemaître explained the apparent redshift of galaxies observed by astronomers in terms of his expanding universe. He then explained that this implies a universe that has expanded from a primordial atom. He was able to site Edwin Hubble's confirmation of the predicted expansion rate, calculated using his solution, as evidence for his expanding cosmos. Later in the 1930's Howard P. Robertson and Arthur Geoffrey Walker proved that the Friedmann-Lemaitre metric is the only exact solution for homogeneous and isotropic universe (Earman, J. (1993), pp. 360-408).

The FLRW metric, which is a solution to the EFE that models an expanding,homogeneous and isotropic universe, allows for three geometries: open, flat, or closed. As of 2013 scientists have been able to determine that our universe is flat with a 0.4% margin of error. This determination is based on evidence from the Wilkinson Microwave Anisotropy Probe (WMAP) (NASA 2014). For this reason I will focus on the flat metric in this thesis, but it is worth mentioning that a more general derivation is necessary and could be easily produced using the techniques outlined here.

2.2 Mathematical Background

2.2.1 The General FLRW Metric

The most general form of the FLRW, as discussed in section 2.1, allows for global curvature. The metric can be expressed to account for all three possible global curvatures in a homogeneous and isotropic universe.

$$ds^2 = dt^2 - a^2(t) \begin{cases} Closed: & d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta d\phi^2\right) \\ Flat \ : & dx^2 + dy^2 + dz^2 \\ Open \ : & d\psi^2 + \sinh^2\psi\left(d\theta^2 + \sin^2\theta d\phi^2\right) \end{cases} \tag{2.1}$$

The metric is expressed for the three possible geometries. The possible geometries arise from the EFE equation(2.2 ,(Carroll,S. (2004), p. 332).

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab} \tag{2.2}$$

In the EFE the metric is expressed as a second rank tensor. T_{ab} is the stress tensor which codes the distribution of pressure, matter, and energy. R_{ab} and R are the Ricci tensor and scalar respectively (see section 3.1). Using

$$T_{ab} = \rho u_a u_b + P(g_{ab} + u_a u_b) \tag{2.3}$$

,where u is a velocity four vector and ρ is the mass density of our dust model. In the flat case, the case in which we are interested, the metric can be be expressed in terms of the scale factor $(a(t))$, as:

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2). \tag{2.4}$$

2.2.2 Null-Tetrad

For our derivation we will use the N-P formalism. This formalism makes use of a tetrad, or a set of four, basis vectors associated with a light ray. The null-tetrad is:

$$\lambda_i^a = \{l^a, n^a, m^a, \overline{m}^a\}, \tag{2.5}$$

Where $l^a, n^a, m^a, \overline{m}^a$ is a set of four basis vectors. In our tetrad, l^a is tangent to the light ray, n^a is perpendicular to l^a in the metric space/plane. m^a and $\overline{m}^a$ are complex axial vectors for a cross section which slices an ellipsoid shaped bundle of light-rays (see figure 1). The proposed null-tetrad in terms of the parameter ξ and the cosmological scale factor $a(t)$, are:

$$l^a = \frac{1}{a(t)\sqrt{2}(1+\xi\overline{\xi})}\langle(-1-\xi\overline{\xi}), \frac{1}{a(t)}(\overline{\xi}+\xi), \frac{i}{a(t)}(\overline{\xi}-\xi), \frac{1}{a(t)}(-1+\xi\overline{\xi})\rangle, \tag{2.6a}$$

$$n^a = \frac{a(t)}{\sqrt{2}(1+\xi\overline{\xi})}\langle(-1-\xi\overline{\xi}), \frac{-1}{a(t)}(\overline{\xi}+\xi), \frac{i}{a(t)}(\xi-\overline{\xi}), \frac{1}{a(t)}(-1+\xi\overline{\xi})\rangle, \tag{2.6b}$$

$$m^a = \frac{1}{a(t)\sqrt{2}(1+\xi\overline{\xi})}\langle 0, (1-\overline{\xi}^2), -i(\overline{\xi}^2+1), 2\overline{\xi}\rangle, \tag{2.6c}$$

$$\overline{m}^a = \frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})}\langle 0, (1-\xi^2), i(\xi^2+1), 2\xi \rangle. \tag{2.6d}$$

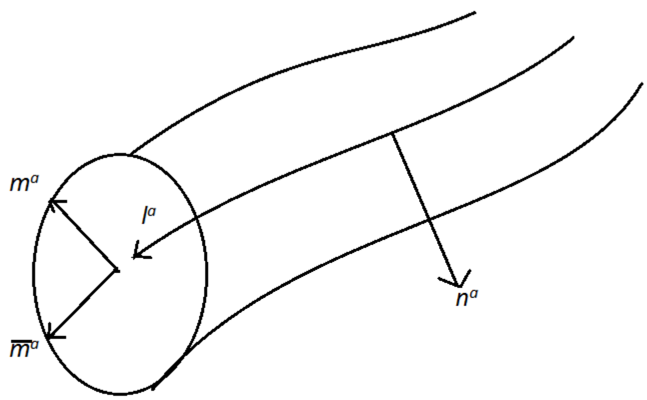

Figure 1 *:Shows an ellipsoid light-bundle traveling along the l^a direction. The tetrad are represented by the four arrows in the figure*

This is the general expression for the N-P null-tetrad in Cartesian, x, y, z, coordinates. In equations (2.6), ξ & $\bar{\xi}$ are stereographic projections onto a complex plane. They map all points on a semi-sphere onto a flat complex plane (see figure 2).

$$\xi = \cot\left(\frac{\theta}{2}\right) e^{i\phi}, \tag{2.7}$$

where $0 < \theta < \pi$ *and* $0 \leq \phi \leq 2\pi$.

In this derivation we can make some assumptions about the space-time and the tetrad to make them simpler. Since Dyer-Roeder assumes a flat homogeneous and isotropic universe, in our derivation, we assume the same thing. If we are looking at an object that is directly overhead, the light from that object will not have components in the x-y plane (this amounts to choosing an origin for a coordinate system). The only direction along which the light ray should progress is the z-direction, using standard Cartesian coordinates

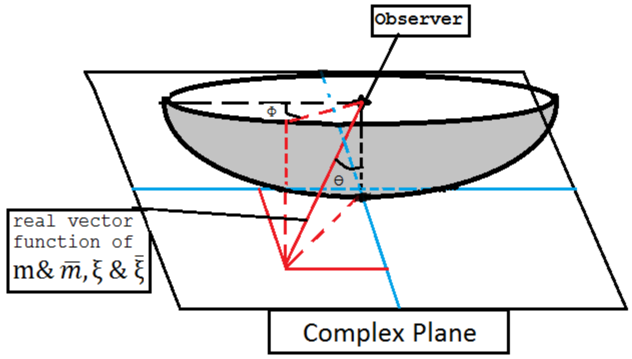

Figure 2: *Shows an observer at the center of a semi-sphere of light. The semi-sphere of light ray-vectors* $v = v(\xi, \bar{\xi}, m, \bar{m})$ *is mapped onto a complex plane.*

(see figure 3).

The vector l^a is tangent to the light-ray, which means that $l^a = \frac{1}{\sqrt{2}}\langle \dot{t}, \dot{x}, \dot{y}, \dot{z}\rangle$ and since we have no movement in the x-y plane, $\dot{x} = 0$ and $\dot{y} = 0$. In terms of our stereographic coordinates, ξ and $\bar{\xi}$, the angle θ is measured from the center of a sphere with respect to the point at which the semi-sphere touches the plane (see figure 2). This strategic choice of coordinates makes the task of calculating distance easy by making $\xi = \bar{\xi} = 0$. Our null tetrad now has spatial components in only the z direction, and the time component remains unchanged.

$$l^a = \frac{1}{a(t)\sqrt{2}} \quad \langle -1, 0, 0, \frac{-1}{a(t)} \rangle \quad , \tag{2.8}$$

And since $\xi = \bar{\xi} = 0$, m^a becomes:

$$m^a = \frac{1}{a(t)\sqrt{2}} \langle 0, 1, -i, 0 \rangle. \tag{2.9}$$

Now that we have l^a and m^a we solve for the N-P components needed to solve the geodesic deviation equation. To find an expression for angular diameter "distance", or the equivalent of Dyer-Roeder equation, for a perturbed FLRW metric, we will need to use the tetrad in equations (2.6).

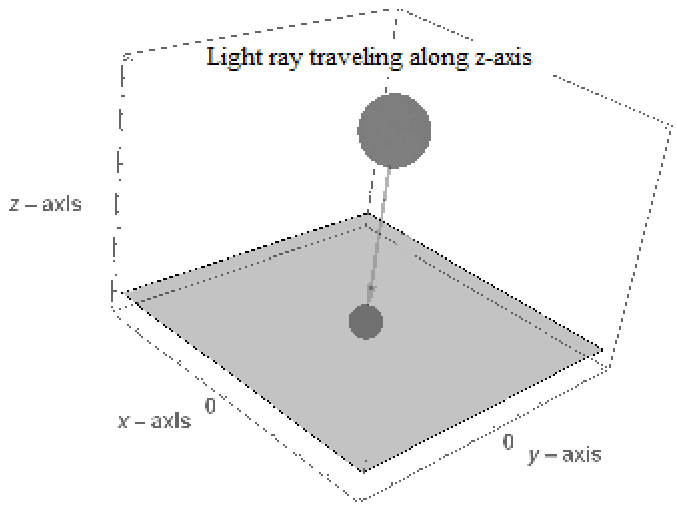

Figure 3: *Shows a light emitting object directly over head. The light ray only travels in the z-direction.*

Note: $n^a, m^a,$ *and* $\overline{m}^a$ *all have components that are NOT tangent to the light-ray. We only make that argument for* l^a *and use* $\xi = \overline{\xi} = 0$ *to get the rest of the tetrad in the derivation that follows.*

2.2.3 Angular Diameter Distance in a Flat, Homogeneous, and Isotropic Universe

Angular Diameter Distance is a way to talk about the distance to faraway objects of known size. Suppose there is a sphere of radius (l). When viewed by an observer from some distance (d), it subtends an angle θ as measured by the observer(see figure 4). If the angle is small enough, then the diameter is approximately equal to the arc length. The arc length is a product of the radius and the angle of the arc. Here the radius is the Angular Diameter-Distance (D_A).

$$D_A = \frac{l}{\theta} \tag{2.10}$$

Astronomers use the angular diameter distance to estimate the distance to an object. This is done by measuring the angle subtended by an object of known size. If the angular diameter distance of the object is known, then its size can be estimated using the same relation. The angular diameter distance is related to the Luminosity Distance (d_L) by redshift (z_R) and comoving transverse distance (d_M) (Ryden, B. (2003)):

$$d_L = d_M\,(1 + z_R) \tag{2.11a}$$

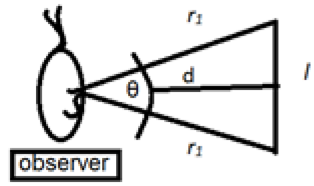

Figure 4

$$D_A = \frac{d_M}{1+z_R}. \tag{2.11b}$$

The comoving transverse distance (d_M) is defined in terms of the proper transverse velocity (u) as measured using redshift. The velocity is calculated using $u = \frac{c}{H_0}\frac{(1-z_R)^2}{(1+z_R)^2}$ and the angular velocity ($\dot{\theta}$) as measured relative to an observer (Carroll,S. (2004), pp. 344-349).

$$d_M = \frac{u}{\dot{\theta}} \tag{2.12}$$

Due to the expansion of the universe with time, astronomers must consider the effects the expansion has on measurement and calculation. One of these effects is that in an expanding universe, a fixed, non-expanding, coordinate system will give different coordinates for objects which are at rest otherwise.

To deal with this difficulty we make use of a comoving coordinate system. This is a system of coordinates that expands at the same rate as that of the universe, allowing objects that move due to expansion only to keep the same coordinates. The actual distance is then obtained via a coordinate transformation. The comoving distance is then the separation distance between the source of the light and the observer in this expanding coordinate system. The comoving distance (transverse) between any two objects in this system is the separation distance between the two points. This comoving distance is not the actual distance one would travel if one wanted to get to the object in question. To get the actual distance we need to include the cosmic scaling factor which is a parameter with a magnitude that varies with time.

Note: this is only true for a Flat, homogenous, and Isotropic universe.

Section 3

Derivation

3.1 Calculating the N-P Components for the Flat FLRW Metric Using the Tetrad

We start our derivation of the Dyer-Roeder equation from the geodesic deviation equation by choosing a cosmology or a metric. The metric we use is the FLRW metric for a flat cosmology as discussed in section (2.2.1). This is an expanding cosmology that has a uniform matter density in every direction, and it is flat everywhere. We can express the flat FLRW metric in two important ways:

$$ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2) \tag{3.1a}$$

$$g_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ 0 & 0 & 0 & -a^2(t) \end{bmatrix}, \tag{3.1b}$$

The N-P components, needed for the geodesics deviation equation, are:

$$\Phi_{00} = -\frac{1}{2}R_{ab}l^a l^b \tag{3.2a}$$

$$\Psi_0 = -C_{abcd}l^a m^b l^c m^d \tag{3.2b}$$

Where R_{ab} is the Ricci tensor and C_{abcd} is the Weyl tensor. These tensors encode the curvature of the space in question and they are calculated by contracting the Riemann tensor R^a_{bcd}. The Ricci tensor is the symmetric part of the Riemann tensor and it is expressed as:

$$R_{ab} = R^{c}{}_{acb}, \tag{3.3}$$

where the Riemann tensor is contracted along the repeated index. The Weyl tensor is the curvature tensor "with all of its contractions removed", it is the anti-symmetric part of the Riemann tensor (Carroll,S. (2004)). For a four dimensional manifold the Weyl tensor is:

$$C_{abcd} = R_{abcd} - g_{a[c}R_{d]b} - g_{b[c}R_{d]a} + \frac{1}{3}g_{a[c}R_{d]b}. \tag{3.4}$$

The Riemann tensor $R_{abcd} = g_{ae}R^{e}_{bcd}$ is defined in terms of the Levi-Civita connection Γ^{a}_{bc} as:

$$R^{a}{}_{bcd} = \partial_c\Gamma^{a}{}_{db} - \partial_d\Gamma^{a}{}_{cb} + \Gamma^{a}{}_{ce}\Gamma^{e}{}_{db} - \Gamma^{a}{}_{cb}. \tag{3.5}$$

We can calculate the Levi-Civita Connection $\Gamma^{a}{}_{bc}$ from the metric tensor, equation (3.1). The connection is defined to be:

$$\Gamma^{a}{}_{bc} = \frac{1}{2}g^{ad}\left(\partial_b g_{cd} + \partial_c g_{db} - \partial_d g_{bc}\right). \tag{3.6}$$

We used the xAct package in Mathematica to find the components of the connection, Ricci, and Weyl tensors. The surviving connection terms are (Wald, R. M. (1984), p. 97):

$$\Gamma^{0}{}_{ii} = a\dot{a} \tag{3.7a}$$

$$\Gamma^{i}{}_{0i} = \Gamma^{i}{}_{i0} = a\dot{a} \tag{3.7b}$$

for *i=1,2,3.* The Weyl tensor vanishes in a flat FLRW, or $C_{abcd} = 0$ in equation (3.2b), and we are only left with the Ricci terms. The Ricci tensor has only the following components:

$$R_{00} = \frac{-3\ddot{a}}{a} \tag{3.8a}$$

and

$$R_{ii} = a\ddot{a} + 2\dot{a}^2\ , \tag{3.8b}$$

for *i=1, 2, 3.* Now that we have these components we can calculate Φ_{00} by substituting equations (3.8) into equation (3.2a), or:

$$\Phi_{00} = \tfrac{-1}{2} R_{ab} l^a l^b = \tfrac{-1}{2} \left[R_{00} l^0 l^0 + R_{11} l^1 l^1 + R_{22} l^2 l^2 + R_{33} l^3 l^3 \right]$$

$$\Phi_{00} = \tfrac{-1}{2} \left[\tfrac{-3\ddot{a}}{a} (l^0)^2 + (a\ddot{a} + 2\dot{a}^2)(l^1)^2 + (a\ddot{a} + 2\dot{a}^2)(l^2)^2 + (a\ddot{a} + 2\dot{a}^2)(l^3)^2 \right] ,$$

Substituting in our tetrad:

$$\Phi_{00} = \tfrac{-1}{2} \left[\tfrac{-3\ddot{a}}{a} (\tfrac{1}{a^2 2}) + (a\ddot{a} + 2\dot{a}^2)(0)^2 + (a\ddot{a} + 2\dot{a}^2)(0)^2 + (a\ddot{a} + 2\dot{a}^2)(\tfrac{1}{a^4 2}) \right] ,$$

Then by removing the zero terms, distribution, and expansion we get,

$$\Phi_{00} = \tfrac{-1}{2} \left[(\tfrac{-3\ddot{a}}{2a^3}) + \tfrac{a\ddot{a}}{2a^4} + \tfrac{2\dot{a}^2}{2a^4} \right] = \tfrac{-1}{2a^2} \left[(\tfrac{-3\ddot{a}}{2a}) + \tfrac{\ddot{a}}{2a} + \tfrac{\dot{a}^2}{a^2} \right] = \tfrac{-1}{2a^2} \left[(\tfrac{\dot{a}}{a})^2 - \tfrac{\ddot{a}}{a} \right] .$$

We can write our final expression so that it is ready for use in the derivation of Dyer-Roeder as:

$$\Phi_{00} = \frac{1}{2a^2} \left[\left(\frac{\dot{a}}{a} \right)^2 - \frac{\ddot{a}}{a} \right] . \tag{3.9}$$

Now that we have Φ_{00} we are ready for the final derivation of the Dyer-Roeder equation using the N-P null-tetrad.

3.2 Geodesic Deviation and Angular Diameter Distance

The Dyer-Roeder equation is an equation for angular diameter distance. Angular diameter distance, as discussed in section 2.2.3, is the length of an object divided by the angle subtended by the object according to some observer in a flat FLRW cosmology. For our derivation of angular diameter distance we rely on the geodesic deviation equation. The geodesic deviation equation will supply the diameter of the object, or (l), from section 2.2.3. Geodesic deviation refers to the behavior of rays of light as they travel through some space and how that behavior deviates from a linear behavior. Two light rays traveling in a flat space will diverge linearly or not at all (figure 5). In order for the light rays to diverge non-linearly they need to be accelerated. This acceleration, in a clumpy cosmology, is provided by the curvature of the space-time. We measure this acceleration by taking the second derivative of the displacement vector between the rays of a light bundle with respect to time. If this derivative is a constant the space is flat, if it is not a constant the space is curved(see figure 5).

In the N-P formalism the second derivative is replaced by a second order differential operator (D^2). We apply this operator to a set of two complex vectors ζ & η and their complex conjugates. Together with the complex vectors m & $\bar{m}$ from our null tetrad, we can calculate the real vector q for the displacement of light rays in a light bundle.

$$q = \zeta \bar{m}^a + \bar{\eta} m^a \tag{3.10}$$

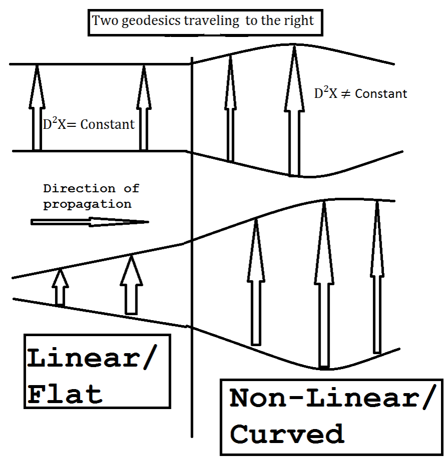

Figure 5: *Shows two sets of two light rays(geodesics) traveling through a flat space on the left and a curved space on the right. The geodesic deviation vector X is represented by the arrows and changed linearly in the flat space and non-linearly in the curved space.The second order deferential operator D^2 measures the extent to which the geodesics are accelerating.*

In the N-P formalism the geodesic deviation vectors are collected in the matrix X.

$$X = \begin{bmatrix} \zeta & \eta \\ \overline{\eta} & \overline{\zeta} \end{bmatrix} \tag{3.11}$$

The differential operator (D) is the operator used in the N-P formalism (Kling & Campbell(2008)). By applying (D)two times to the deviation vectors we are in effect measuring the distortion of an image as viewed by an observer due to some acceleration caused by the curvature of the space-time. The differential operator (D) is given by a the change along the light ray, (l^a):

$$D = l^a \frac{\partial}{\partial x^a}. \tag{3.12}$$

We will show that dividing the deviation vectors X by the angle subtended in an observer's sphere of light will yield the equation for angular diameter distance as described in section 2.2.3,

$$D_A = \frac{l}{\theta} \rightarrow \frac{X}{\theta} \tag{3.13}$$

which is the Dyer-Roeder equation. We can then solve the geodesic deviation equation for the deviation vectors X. In the N-P formalism is

$$D^2X = QX \tag{3.14}$$

where,(Q) is the matrix which codes for the effects of the space time on the deviation vectors,

$$Q = \begin{bmatrix} \Phi_{00} & \Psi_0 \\ \overline{\Psi}_0 & \Phi_{00} \end{bmatrix}. \tag{3.15}$$

As discussed in section (3.1)the Weyl tensor vanishes and the Ricci component is given by equation (3.2a).

3.3 Deriving the Dyer-Roeder Equation from the Geodesic Deviation Equation

Our goal is to start with the geodesic deviation equation and to derive the Dyer-Roeder equation. The geodesic deviation equation can be expressed in the N-P formalism as,

$$D^2 X = QX. \tag{3.16}$$

In this equation Q, X, are given by equations (3.15) and (3.11). The matrix product QX in equation (3.14) is calculated as:

$$QX = \begin{bmatrix} \Phi_{00}\zeta + \Psi_0\overline{\eta} & \Phi_{00}\eta + \Psi_0\overline{\zeta} \\ \overline{\Psi}_0\zeta + \Phi_{00}\overline{\eta} & \overline{\Psi}_0\eta + \Phi_{00}\overline{\zeta} \end{bmatrix}. \tag{3.17}$$

Since D^2X can be represented in matrix form as $D^2X = \begin{bmatrix} D^2\zeta & D^2\eta \\ D^2\overline{\eta} & D^2\overline{\zeta} \end{bmatrix}$, we can write the following four equations:

$$D^2\zeta = \Phi_{00}\zeta + \Psi_0\overline{\eta} \tag{3.18a}$$

$$D^2\eta = \Phi_{00}\eta + \Psi_0 \tag{3.18b}$$

$$D^2\overline{\eta} = \overline{\Psi}_0\zeta + \Phi_{00}\overline{\eta} \tag{3.18c}$$

$$D^2\overline{\zeta} = \overline{\Psi}_0\eta + \Phi_{00}\overline{\zeta} \tag{3.18d}$$

In this case D is an operator which acts on the components of the matrix X as follows:[2]

$$D = l^a \frac{\partial}{\partial x^a} = l^0 \frac{\partial}{\partial t} + l^3 \frac{\partial}{\partial z}. \tag{3.19}$$

Which means $D^2 = l^a \frac{\partial}{\partial x^a} l^b \frac{\partial}{\partial x^b}$. The Dyer-Roeder equation calculates distances in a flat-homogeneous cosmology, or a cosmology described by the FLRW metric, where Φ_{00} is equation (3.2a) and $\Psi_0 = 0$. Since the Dyer-Roeder equation is written in terms of red-shift distance (z_R) and matter density (Ω_m) we need a change of variable from time (t)in equation (3.19) to the red-shift distance and matter density. We don't need to worry about the z derivative in equation (3.19) because neither l_a nor Φ_{00} depend on z and the second terms in equations (3.18a)-(3.18d) vanish. To accomplish a change of variables we utilized the Hubble parameter (Ryden 2003).

$$H \equiv \frac{\dot{a}}{a}, \tag{3.20}$$

The null vector l^a:

$$l^a = \frac{1}{a\sqrt{2}}\langle -1, 0, 0, \frac{-1}{a}\rangle, \tag{3.21}$$

And the relationship between red-shift distance and the cosmological scale factor $a(t)$. This is generally $\frac{a(t_0)}{a(t)} = 1 + z_R$, where ($t_0$) is the time now, so by letting $a(t_0) = 1$ we can write,

$$\frac{1}{a(t)} = 1 + z_R. \tag{3.22}$$

By solving equation (3.22)for z_R and taking the derivative of both sides with respect to time we get,

$$\frac{\partial}{\partial t} = \frac{-\dot{a}}{a^2}\frac{\partial}{\partial z_R}. \tag{3.23}$$

As discussed, since nothing in equation(17) depends on z, and by substituting equations (3.20 - 3.23) into equation (3.19)we get:

$$D = l^0 \frac{\partial}{\partial t} = -l^0 \frac{\dot{a}}{a^2}\frac{\partial}{\partial z_R} = -\left(-\frac{1}{a\sqrt{2}}\right)\left(\frac{\dot{a}}{a^2}\frac{\partial}{\partial z_R}\right) = \left(\frac{1}{a^2\sqrt{2}}\right)\left(H\frac{\partial}{\partial z_R}\right),$$

or,

$$D = \frac{\sqrt{2}}{2}(1+z)^2 H \frac{\partial}{\partial z_R}. \tag{3.24}$$

Operating with this differential operator two times can be expressed in terms of equation (3.20) and equation (3.22) by taking the appropriate derivatives and simplifying.

$$D^2 = \frac{1}{2} H(1+z_R)^2 \frac{\partial}{\partial z_R} H(1+z_R)^2 \frac{\partial}{\partial z_R} \tag{3.25}$$

First we deal with the derivatives,

$$\tfrac{\partial}{\partial z_R} H(1+z_R)^2 \tfrac{\partial}{\partial z_R} = H(1+z_R)^2 \tfrac{\partial^2}{\partial^2 z_R} + \left(\tfrac{\partial}{\partial z_R} H(1+z_R)^2\right) \tfrac{\partial}{\partial z_R}$$

and,

$$\left(\tfrac{\partial}{\partial z_R} H(1+z_R)^2\right) \tfrac{\partial}{\partial z_R} = (1+z_R)^2 \tfrac{\partial H}{\partial z_R} \tfrac{\partial}{\partial z_R} + 2(1+z_R) H \tfrac{\partial}{\partial z_R}.$$

Then by combining the last two equations and simplifying,

$$H(1+z_R)^2 \tfrac{\partial^2}{\partial^2 z_R} + (1+z_R)^2 \tfrac{\partial H}{\partial z_R} \tfrac{\partial}{\partial z_R} + 2(1+z_R) H \tfrac{\partial}{\partial z_R} = H(1+z_R)^2 \tfrac{\partial^2}{\partial^2 z_R} + \left((1+z_R)^2 \tfrac{\partial H}{\partial z_R} + 2(1+z_R)H\right) \tfrac{\partial}{\partial z_R}.$$

Substituting this result into equation (3.25) gives,

$$D^2 = \left[\tfrac{1}{2} H(1+z_R)^2\right] \left[H(1+z_R)^2 \tfrac{\partial^2}{\partial^2 z_R} + \left((1+z_R)^2 \tfrac{\partial H}{\partial z_R} + 2(1+z_R)H\right) \tfrac{\partial}{\partial z_R}\right].$$

By distributing $H(1+z_R)^2$ we can write,

$$D^2 = \tfrac{1}{2} \left[H^2(1+z_R)^4 \tfrac{\partial^2}{\partial^2 z_R} + \left((1+z_R)^4 H \tfrac{\partial H}{\partial z_R} + 2(1+z_R)^3 H^2\right) \tfrac{\partial}{\partial z_R}\right].$$

We now factor out a $\frac{1}{2}$ and we have D^2 in terms of redshift distance and the Hubble parameter in a form that will become useful later in our derivation.

$$D^2 = \frac{1}{4} \left[\left[(1+z_R) 2H \frac{\partial H}{\partial z_R} + 4H^2\right] (1+z_R)^3 \frac{\partial}{\partial z_R} + 2H^2 (1+z_R)^4 \frac{\partial^2}{(\partial z_R)^2}\right]. \tag{3.26}$$

Since in a flat FLRW cosmology $\Psi_0 = 0$, equation (3.18a) becomes $D^2\zeta = \Phi_{00}\zeta$. Dividing both sides by the angle α (see section 2.2.3), where $D_A = \frac{\zeta}{\alpha}$ and D_A is the angular diameter distance, we get (Ryden, B. (2003)):

$$D^2 D_A = \Phi_{00} D_A. \tag{3.27}$$

Then by substituting equations (3.26) and (3.9) into equation (3.27) our expression becomes:

$$\frac{1}{4} \left[\left((1+z_R) 2H \frac{\partial H}{\partial z_R} + 4H^2\right) (1+z_R)^3 \frac{\partial D_A}{\partial z_R} + 2H^2 (1+z_R)^4 \frac{\partial^2 D_A}{(\partial z_R)^2}\right] = -\frac{1}{2a^2} \left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{\ddot{a}}{a}\right] D_A, \tag{3.28}$$

From equation (3.20) we have:

$$\frac{\partial H}{\partial t} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2. \tag{3.29}$$

Where we substitute equations (3.23) and (3.29) into equation (3.28) we get,

$$\frac{1}{4}\left(\left[(1+z_R)2H\frac{\partial H}{\partial z_R}+4H^2\right](1+z_R)^3\frac{\partial D_A}{\partial z_R}+2H^2(1+z_R)^4\frac{\partial^2 D_A}{(\partial z_R)^2}\right)=\frac{-1}{2a^3}\left[\frac{\dot{a}}{a}\frac{\partial H}{\partial z_R}\right]D_A. \tag{3.30}$$

By combining equations (3.20) (3.22) and (3.30), then setting the expression equal to zero and multiplying both sides by 4 we get:

$$\left[2H^2(1+z_R)^4\right]\frac{\partial^2 D_A}{(\partial z_R)^2}+\left[(1+z_R)^4 2H\frac{\partial H}{\partial z_R}+4(1+z_R)^3H^2\right]\frac{\partial D_A}{\partial z_R}+\left[(1+z_R)^3 2H\frac{\partial H}{\partial z_R}\right]D_A=0. \tag{3.31}$$

The only variable that does not depend on red-shift distance in equation (3.31) is the Hubble parameter and its derivatives which depend on t indirectly through $a(t)$. The Hubble parameter can be expressed in terms of redshift distance by the Hubble constant H_0. For the most general case, not flat and uniform, can be expressed as:

$$H^2=H_0^2\left[\frac{\Omega_m}{a^3}+\frac{1-\Omega_m-\Omega_\Lambda}{a^2}+\Omega_\Lambda\right]=H_0^2\left[\Omega_m(1+z_R)^3+(1-\Omega_m-\Omega_\Lambda)(1+z_R)^2+\Omega_\Lambda\right]. \tag{3.32}$$

We are trying to find the Dyer-Roeder equation as it appears in Ehlers(1992). In this version the author assumes that $\Omega_\Lambda=0$. In our metric we assume that the cosmology is flat, or that $1-\Omega_m-\Omega_\Lambda=0$ [(Carroll,S. (2004)). This reduces equation (3.32) to:

$$H^2=H_0^2\Omega_m(1+z_R)^3. \tag{3.33}$$

Taking the derivative of both sides with respect to the red-shift distance we get,

$$\frac{\partial H^2}{\partial z_R}=2H\frac{\partial H}{\partial z_R}=3H_0^2\Omega_m(1+z_R)^2. \tag{3.34}$$

By substituting equation (3.33) and (3.34) into equation (3.31) we get,

$$\begin{aligned}&\left[2(1+z_R)^4)\left[\Omega_m H_0^2(1+z_R)^3\right]\right]\frac{\partial^2 D_A}{(\partial z_R)^2}\\&\quad+\left[\left((1+z_R)^4(3H_0^2\Omega_m(1+z_R)^2)\right)+\left(4(1+z_R)^3(\Omega_m H_0^2(1+z_R)^3)\right)\right]\frac{\partial D_A}{\partial z_R}\cdot\\&\quad+\left[3H_0^2\Omega_m(1+z_R)^3(1+z_R)^2\right]D_A=0\end{aligned} \tag{3.35}$$

Reorganizing we get,

$$\begin{aligned}\left[2H_0^2(1+z_R)^7\Omega_m\right]&\frac{\partial^2 D_A}{(\partial z_R)^2}\\&+\left[\left(3H_0^2(1+z_R)^6\Omega_m)\right)+\left(4H_0^2(1+z_R)^6\Omega_m)\right)\right]\frac{\partial D_A}{\partial z_R}\cdot\\&+3H_0^2\Omega_m(1+z_R)^5 D_A=0\end{aligned}\tag{3.36}$$

Dividing both sides by $H_0^2(1+z_R)^5$ and combining the coefficients of the first derivative in the differential equation we get,

$$2(1+z_R)^2\Omega_m\frac{\partial^2 D_A}{(\partial z_R)^2}+7(1+z_R)\Omega_m\frac{\partial D_A}{\partial z_R}+3\Omega_m D_A=0.\tag{3.37}$$

I did not divide by Ω_m in the last simplification in order to make it clear that we have forced $\Omega_m = 1$ because we assumed $0 = 1 - \Omega_m - \Omega_\Lambda$ and Ehlers assumes $\Omega_\Lambda = 0$. When we divide both sides of equation (3.37) by 2 and setting $\Omega_m = 1$ we get:

$$(1+z_R)^2\frac{\partial^2 D_A}{(\partial z_R)^2}+\frac{7}{2}(1+z_R)\frac{\partial D_A}{\partial z_R}+\frac{3}{2}D_A=0.\tag{3.38}$$

This is our final expression for angular diameter distance in a flat FLRW cosmology. This is the same expression that is found in Ehlers', after setting $\Omega = 1 + \frac{kc^2}{(R_0H_0)^2} = 1$. For flat FLRW $k = 0$. The Actual expression for Dyer-Roeder on pg. 137 of Ehlers' book is:

$$(z+1)(\Omega\ z+1)\frac{\partial^2 D}{dz^2}+(\frac{7}{2}\Omega\ z+\frac{\Omega}{2}+3)\frac{\partial D}{\partial z}+\frac{3}{2}\Omega D=0.\tag{3.39}$$

By setting $\Omega = 1$ Ehler's expression reduces to:

$$(z+1)(z+1)\tfrac{\partial^2 D}{dz^2}+(\tfrac{7}{2}z+\tfrac{1}{2}+3)\tfrac{\partial D}{\partial z}+\tfrac{3}{2}D=0.$$

Finally, by combining like terms and factoring the second term, Ehler's equation matches our expression from equation (3.38) , where $z_R = z$, or:

$$(z+1)^2\frac{\partial^2 D}{dz^2}+\frac{7}{2}(1+z)\frac{\partial D}{\partial z}+\frac{3}{2}D=0,\tag{3.40}$$

This means that by using the N-P formalism and starting from the geodesic deviation equation in a flat FLRW cosmology, we have reproduced the Dyer-Roeder equation.

Section 4

Conclusion and Next Steps

4.1 Conclusion

We have shown that the N-P formalism can give us an equation for angular diameter distance that matches those obtained using traditional coordinate basis. We used the flat FLRW metric and the null tetrad to derive the Dyer-Roeder equation from the geodesic deviation equation. Our derivation is an expression for angular diameter distance in terms of red-shift distance. We were able to confirm that we have the right expression for angular diameter diatance by comparing it to the Dyer-Roeder equation in Ehlers(1992). Now that we have shown the N-P tetrad capable of producing the Dyer-Roeder equation for a flat FLRW metric, we think it is possible derive the Dyer-Roeder equation for angular diameter distance in a perturbed FLRW cosmology from the geodesic deviation equation. This derivation also serves as proof that the Dyer-Roeder equation is the geodesic deviation equation.

4.2 Next Steps

As discussed in the introduction, our final expression is only valid for objects to which we have a clear line of sight. This restriction is a result of starting with the the unperturbed flat FLRW metric. For objects to which we do not have a clear line-of-sight, we must account for the curvature produced by the presence of clumps of matter along the line-of-sight. This can be done by perturbing the FLRW metric with gravitational potential and allowing for global curvature. By using the null tetrad we think, it is possible to find useful expressions for angular diameter distance in a perturbed FLRW cosmology. We believe that using the null-tetrad will simplify the mathematics and allow us to solve the the geodesic deviation equation for angular diameter distance. If so, the resulting equation could prove to be a useful tool for astronomers looking at objects through gravitational lenses.

Section 5

Acknowledgments

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Section 6

Appendix A

6.1 Checking the normalization conditions for our complex null tetrad

6.1.1 The Null Tetrad

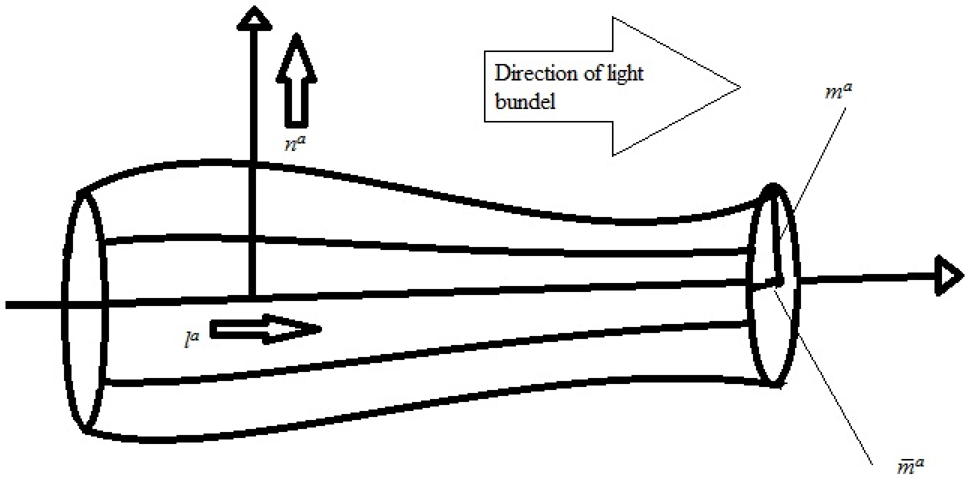
The proposed null tetrad is the set of basis vectors $\{l^a, n^a, m^a, \overline{m}^a\}$, and they are:

$l^a = \frac{1}{a(t)\sqrt{2}(1+\xi\overline{\xi})}\left\langle(-1-\xi\overline{\xi}), \frac{1}{a(t)}(\overline{\xi}+\xi), \frac{i}{a(t)}(\overline{\xi}-\xi), \frac{1}{a(t)}(-1+\xi\overline{\xi})\right\rangle$,

$n^a = \frac{a(t)}{\sqrt{2}(1+\xi\overline{\xi})}\left\langle(-1-\xi\overline{\xi}), \frac{-1}{a(t)}(\overline{\xi}+\xi), \frac{i}{a(t)}(\xi-\overline{\xi}), \frac{-1}{a(t)}(-1+\xi\overline{\xi})\right\rangle$,

$m^a = \frac{1}{a(t)\sqrt{2}(1+\xi\overline{\xi})}\left\langle 0, (1-\overline{\xi}^2), -i(\overline{\xi}^2+1), 2\overline{\xi}\right\rangle$, and

$\overline{m}^a = \frac{1}{a(t)\sqrt{2}(1+\xi\overline{\xi})}\left\langle 0, (1-\xi^2), i(\xi^2+1), 2\xi\right\rangle$.



We need to check the following normalization conditions for the flat unperturbed Robertson-Walker Metric in order to make sure that this is a valid null-tetrad.

6.1.2 1^{st} Condition

The first normalization condition is:

$$l^a \nabla_a l^b = l^a \frac{\partial}{\partial x^a} l^b + \Gamma^b_{ac} l^a l^c = 0. \tag{6.1}$$

Where the Christoffel Connection for the flat unperturbed FLRW metric has the following values:

$\Gamma^0_{ii} = \dot{a}a$, $\Gamma^0_{ii} = \frac{\dot{a}}{a}$, and zero everywhere else.

First we check the case where $\underline{b=0}$. In this case equation (6.1) becomes:

$l^a \nabla_a l^0 = l^a \frac{\partial}{\partial x^a} l^0 + \Gamma^0_{ac} l^a l^c = l^0 \frac{\partial}{\partial t} l^0 + \Gamma^0_{11} l^1 l^1 + \Gamma^0_{22} l^2 l^2 + \Gamma^0_{33} l^3 l^3.$

Then by substituting in our proposed complex null tetrad we get:

$$l^a \nabla_a l^0 = \frac{(-1-\xi\bar{\xi})}{a\sqrt{2}(1+\xi\bar{\xi})} \frac{\partial}{\partial t} \frac{(-1-\xi\bar{\xi})}{a\sqrt{2}(1+\xi\bar{\xi})} + \dot{a}a\left(\frac{(\bar{\xi}+\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}\right)\left(\frac{(\bar{\xi}+\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}\right) + \dot{a}a\left(\frac{i(\bar{\xi}-\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}\right)\left(\frac{i(\bar{\xi}-\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}\right) + \dot{a}a\left(\frac{(-1+\bar{\xi}\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}\right)\left(\frac{(-1+\bar{\xi}\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}\right),$$

$$= \frac{-(1+\xi\bar{\xi})}{a\sqrt{2}(1+\xi\bar{\xi})} \frac{\partial}{\partial t} \frac{-(1+\xi\bar{\xi})}{a\sqrt{2}(1+\xi\bar{\xi})} + \dot{a}a\left(\frac{(\bar{\xi}+\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}\right)\left(\frac{(\bar{\xi}+\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}\right) + \dot{a}a\left(\frac{i(\bar{\xi}-\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}\right)\left(\frac{i(\bar{\xi}-\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}\right) + \dot{a}a\left(\frac{(-1+\bar{\xi}\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}\right)\left(\frac{(-1+\bar{\xi}\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}\right),$$

$$= \frac{1}{2a} \frac{\partial}{\partial t} \frac{1}{a} + \dot{a}\left(\frac{(\bar{\xi}+\xi)^2}{2a^3(1+\xi\bar{\xi})^2}\right) + \dot{a}\left(\frac{ii(\bar{\xi}-\xi)^2}{2a^3(1+\xi\bar{\xi})^2}\right) + \dot{a}\left(\frac{(-1+\bar{\xi}\xi)^2}{2a^3(1+\xi\bar{\xi})^2}\right),$$

$= \frac{1}{2a}\frac{\partial}{\partial t}a^{-1} + \frac{\dot{a}}{2a^3(1+\xi\bar{\xi})^2}\left[(\bar{\xi}+\xi)^2) + -(\bar{\xi}-\xi)^2 + (-1+\bar{\xi}\xi)^2\right],$

$= \frac{1}{2a}(-\frac{\dot{a}}{a^2}) + \frac{\dot{a}}{2a^3(1+\xi\bar{\xi})^2}\left[\bar{\xi}^2 + 2\xi\bar{\xi} + \xi^2 - \bar{\xi}^2 + 2\xi\bar{\xi} - \xi^2 + 1 - 2\xi\bar{\xi} + (\xi\bar{\xi})^2\right],$

$= (\frac{-\dot{a}}{2a^3}) + \frac{\dot{a}}{2a^3(1+\xi\bar{\xi})^2}\left[1 + 2\xi\bar{\xi} + (\xi\bar{\xi})^2\right],$

$= (\frac{-\dot{a}}{2a^3}) + \frac{\dot{a}}{2a^3} = 0,$

$$\therefore l^a\nabla_a l^0 = l^a\frac{\partial}{\partial x^a}l^0 + \Gamma^0_{ac}l^a l^c = 0, \tag{6.2}$$

For the case where $\underline{b=1}$ equation (6.1) becomes:
$l^a\nabla_a l^1 = l^a\frac{\partial}{\partial x^a}l^1 + \Gamma^1_{ac}l^a l^c = l^0\frac{\partial}{\partial t}l^1 + \Gamma^1_{01}l^0 l^1 + \Gamma^1_{10}l^1 l^0,$
and since $\Gamma^i_{0i} = \Gamma^i_{i0}$, we can write equation (6.1) as:
$l^a\nabla_a l^1 = l^0\frac{\partial}{\partial t}l^1 + 2\Gamma^i_{0i}l^0 l^1,$
where by substituting in our null tetrad we get:

$= (\frac{(-1-\xi\bar{\xi})}{a\sqrt{2}(1+\xi\bar{\xi})})\frac{\partial}{\partial t}(\frac{(\bar{\xi}+\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}) + 2\frac{\dot{a}}{a}(\frac{(-1-\xi\bar{\xi})}{a\sqrt{2}(1+\xi\bar{\xi})})(\frac{(\bar{\xi}+\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}),$

$= (\frac{-(1+\xi\bar{\xi})}{a\sqrt{2}(1+\xi\bar{\xi})})\frac{\partial}{\partial t}(\frac{(\bar{\xi}+\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}) + 2\frac{\dot{a}}{a}(\frac{-(1+\xi\bar{\xi})}{a\sqrt{2}(1+\xi\bar{\xi})})(\frac{(\bar{\xi}+\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}),$

$= (\frac{-(\bar{\xi}+\xi)}{2a(1+\xi\bar{\xi})})\frac{\partial}{\partial t}(\frac{1}{a^2}) + (\frac{-2\dot{a}(\bar{\xi}+\xi)}{2a^4(1+\xi\bar{\xi})}),$

$= (\frac{-(\bar{\xi}+\xi)}{2a(1+\xi\bar{\xi})})\frac{\partial}{\partial t}a^{-2}) + (\frac{-2\dot{a}(\bar{\xi}+\xi)}{2a^4(1+\xi\bar{\xi})}) = (\frac{-(\bar{\xi}+\xi)}{2a(1+\xi\bar{\xi})})(-2)(a^{-3})\dot{a} + (\frac{-2\dot{a}(\bar{\xi}+\xi)}{2a^4(1+\xi\bar{\xi})}),$

$= (\frac{\dot{a}(\bar{\xi}+\xi)}{a^4(1+\xi\bar{\xi})}) - (\frac{\dot{a}(\bar{\xi}+\xi)}{a^4(1+\xi\bar{\xi})}) = 0,$

$$\therefore l^a\nabla_a l^1 = l^a\frac{\partial}{\partial x^a}l^1 + \Gamma^1_{ac}l^a l^c = 0. \tag{6.3}$$

For the case where $\underline{b=2}$ equation (6.1) becomes:

$l^a\nabla_a l^2 = l^a\frac{\partial}{\partial x^a}l^2 + \Gamma^2_{ac}l^a l^c = l^0\frac{\partial}{\partial t}l^2 + \Gamma^2_{02}l^0 l^2 + \Gamma^2_{20}l^2 l^0,$

as with the case where $b=1$ we can write:

$l^a\nabla_a l^2 = l^0\frac{\partial}{\partial t}l^2 + 2\Gamma^i_{0i}l^0 l^2,$

$$= (\tfrac{(-1-\xi\bar{\xi})}{a\sqrt{2}(1+\xi\bar{\xi})})\tfrac{\partial}{\partial t}(\tfrac{i(\bar{\xi}-\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}) + 2\tfrac{\dot{a}}{a}(\tfrac{(-1-\xi\bar{\xi})}{a\sqrt{2}(1+\xi\bar{\xi})})(\tfrac{i(\bar{\xi}-\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}),$$

$$= (\tfrac{-(1+\xi\bar{\xi})}{a\sqrt{2}(1+\xi\bar{\xi})})\tfrac{\partial}{\partial t}(\tfrac{i(\bar{\xi}-\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}) + 2\tfrac{\dot{a}}{a}(\tfrac{-(1+\xi\bar{\xi})}{a\sqrt{2}(1+\xi\bar{\xi})})(\tfrac{i(\bar{\xi}-\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})}),$$

$$= (\tfrac{-i(\bar{\xi}-\xi)}{2a(1+\xi\bar{\xi})})\tfrac{\partial}{\partial t}(\tfrac{1}{a^2}) + (\tfrac{-2\dot{a}i(\bar{\xi}-\xi)}{2a^4(1+\xi\bar{\xi})}),$$

$$= (\tfrac{-i(\bar{\xi}-\xi)}{2a(1+\xi\bar{\xi})})\tfrac{\partial}{\partial t}a^{-2}) + (\tfrac{-2\dot{a}i(\bar{\xi}-\xi)}{2a^4(1+\xi\bar{\xi})}) = (\tfrac{-i(\bar{\xi}-\xi)}{2a(1+\xi\bar{\xi})})(-2)(a^{-3})\dot{a} + (\tfrac{-2\dot{a}i(\bar{\xi}-\xi)}{2a^4(1+\xi\bar{\xi})}),$$

$$= (\tfrac{\dot{a}i(\bar{\xi}-\xi)}{a^4(1+\xi\bar{\xi})}) - (\tfrac{\dot{a}i(\bar{\xi}-\xi)}{a^4(1+\xi\bar{\xi})}) = 0,$$

$$\therefore l^a\nabla_a l^2 = l^a\frac{\partial}{\partial x^a}l^2 + \Gamma^2_{ac}l^a l^c = 0. \tag{6.4}$$

For the case where $\underline{b=3}$ equation (6.1) becomes:

$l^a\nabla_a l^3 = l^a\frac{\partial}{\partial x^a}l^3 + \Gamma^3_{ac}l^a l^c = l^0\frac{\partial}{\partial t}l^3 + \Gamma^3_{03}l^0 l^3 + \Gamma^3_{30}l^3 l^0,$

as before we can write:

$l^a\nabla_a l^3 = l^0\frac{\partial}{\partial t}l^3 + 2\Gamma^i_{0i}l^0 l^3,$

using our null tetrad we get:

$$= (\tfrac{(-1-\xi\bar{\xi})}{a\sqrt{2}(1+\xi\bar{\xi})})\tfrac{\partial}{\partial t}(\tfrac{(-1+\xi\bar{\xi})}{a^2\sqrt{2}(1+\xi\bar{\xi})}) + 2\tfrac{\dot{a}}{a}(\tfrac{(-1-\xi\bar{\xi})}{a\sqrt{2}(1+\xi\bar{\xi})})(\tfrac{(-1+\xi\bar{\xi})}{a^2\sqrt{2}(1+\xi\bar{\xi})}),$$

$$= (\tfrac{-(1+\xi\bar{\xi})}{a\sqrt{2}(1+\xi\bar{\xi})})\tfrac{\partial}{\partial t}(\tfrac{(-1+\xi\bar{\xi})}{a^2\sqrt{2}(1+\xi\bar{\xi})}) + 2\tfrac{\dot{a}}{a}(\tfrac{-(1+\xi\bar{\xi})}{a\sqrt{2}(1+\xi\bar{\xi})})(\tfrac{(-1+\xi\bar{\xi})}{a^2\sqrt{2}(1+\xi\bar{\xi})}),$$

$$= (\tfrac{-(-1+\xi\bar{\xi})}{2a(1+\xi\bar{\xi})})\tfrac{\partial}{\partial t}(\tfrac{1}{a^2}) + (\tfrac{-2\dot{a}(-1+\xi\bar{\xi})}{2a^4(1+\xi\bar{\xi})}),$$

$$= (\tfrac{-(-1+\xi\bar{\xi})}{2a(1+\xi\bar{\xi})})\tfrac{\partial}{\partial t}a^{-2}) + (\tfrac{-2\dot{a}(-1+\xi\bar{\xi})}{2a^4(1+\xi\bar{\xi})}) = (\tfrac{-(-1+\xi\bar{\xi})}{2a(1+\xi\bar{\xi})})(-2)(a^{-3})\dot{a} + (\tfrac{-2\dot{a}(-1+\xi\bar{\xi})}{2a^4(1+\xi\bar{\xi})}),$$

$= (\frac{\dot{a}(-1+\xi\bar{\xi})}{a^4(1+\xi\bar{\xi})}) - (\frac{\dot{a}(-1+\xi\bar{\xi})}{a^4(1+\xi\bar{\xi})}) = 0,$

$$\therefore l^a \nabla_a l^3 = l^a \frac{\partial}{\partial x^a} l^3 + \Gamma^3_{ac} l^a l^c = 0. \tag{6.5}$$

According the to equations (6.2 -6.5), equation (6.1) is true for the case where $b = 0, 1, 2, 3$, which means that our complex null tetrad meets this condition for the flat unperturbed FLRW metric.

6.1.3 2^{nd} Condition

The second normalization condition is:

$$l^a \nabla_a m^b = l^a \frac{\partial}{\partial x^a} m^b + \Gamma^b_{ac} l^a m^c = 0. \tag{6.6}$$

For the case where $\underline{b = 0}$ equation (6.6) becomes:

$$l^a \nabla_a m^0 = l^a \tfrac{\partial}{\partial x^a} m^0 + \Gamma^0_{ac} l^a m^c = l^0 \tfrac{\partial}{\partial x^0} m^0 + \Gamma^0_{11} l^1 m^1 + \Gamma^0_{22} l^2 m^2 + \Gamma^0_{33} l^3 m^3 \ .$$

Where by substituting in our null tetrad into the first term we have:
$l^a \frac{\partial}{\partial x^a} m^0 = \frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})} \langle(-1-\xi\bar{\xi})\rangle \frac{\partial}{\partial t} \frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})}(0) = 0,$
and equation (6.6) can be re-written as:

$$l^a \nabla_a m^0 = \Gamma^0_{11} l^1 m^1 + \Gamma^0_{22} l^2 m^2 + \Gamma^0_{33} l^3 m^3.$$

We can write this using Eisenstein summation notation as:
$\Gamma^0_{ii} l^i m^i = \Gamma^0_{11} l^1 m^1 + \Gamma^0_{22} l^2 m^2 + \Gamma^0_{33} l^3 m^3,$
then by substituting in our null tetrad we get:
$\Gamma^0_{ii} l^i m^i = \dot{a} a \langle \frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})} \rangle^2 \left\langle \frac{1}{a(t)}(\bar{\xi}+\xi)(1-\bar{\xi}^2) + \frac{i}{a(t)}(\bar{\xi}-\xi) - i(\bar{\xi}^2+1) + \frac{1}{a(t)}(-1+\xi\bar{\xi})2\bar{\xi} \right\rangle ,$

$= \frac{\dot{a}}{2a^2(1+\xi\bar{\xi})^2} \left\langle \bar{\xi} + \xi - \bar{\xi}^3 - \xi\bar{\xi}^2 - i^2(\bar{\xi}^3 - \xi\bar{\xi}^2 + \bar{\xi} - \xi) - 2\bar{\xi} + 2\bar{\xi}^2\xi \right\rangle,$

$= \frac{\dot{a}}{2a^2(1+\xi\bar{\xi})^2} \left\langle \bar{\xi} + \xi - \bar{\xi}^3 - \xi\bar{\xi}^2 + \bar{\xi}^3 - \xi\bar{\xi}^2 + \bar{\xi} - \xi - 2\bar{\xi} + 2\bar{\xi}^2\xi \right\rangle,$

$= \frac{\dot{a}}{2a^2(1+\xi\bar{\xi})^2} \langle \bar{\xi} + \bar{\xi} - 2\bar{\xi} + \xi - \xi - \bar{\xi}^3 + \bar{\xi}^3 - \xi\bar{\xi}^2 - \xi\bar{\xi}^2 + 2\xi\bar{\xi}^2 \rangle ,$

$$\therefore \Gamma^0_{ii} l^i m^i = \Gamma^0_{11} l^1 m^1 + \Gamma^0_{22} l^2 m^2 + \Gamma^0_{33} l^3 m^3 = 0,$$

which means that for the case where $b = 0$ equation (6.6) becomes:

$$l^a \nabla_a m^0 = l^a \frac{\partial}{\partial x^a} m^0 + \Gamma^0_{ac} l^a m^c = l^0 \frac{\partial}{\partial x^0} m^0 + \Gamma^0_{11} l^1 m^1 + \Gamma^0_{22} l^2 m^2 + \Gamma^0_{33} l^3 m^3 = 0. \tag{6.7}$$

For the case where $\underline{b = 1}$ equation (6.6) becomes:

$$l^a \nabla_a m^1 = l^a \tfrac{\partial}{\partial x^a} m^1 + \Gamma^1_{ac} l^a m^c = l^0 \tfrac{\partial}{\partial t} m^1 + \Gamma^1_{01} l^0 m^1 + \Gamma^1_{10} l^1 m^0$$

and since $m^0 = 0$, when $b = 1$ we can drop the last term and write:

$$l^a \nabla_a m^1 = l^a \tfrac{\partial}{\partial x^a} m^1 + \Gamma^1_{ac} l^a m^c = l^0 \tfrac{\partial}{\partial t} m^1 + \Gamma^1_{01} l^0 m^1 = l^0 (\tfrac{\partial}{\partial t} m^1 + \Gamma^1_{01} m^1)$$

Where after substitutions we have:
$\frac{\partial}{\partial t} m^1 = \frac{\partial}{\partial t} \frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})}(1 - \bar{\xi}^2) = \frac{-\dot{a}(1-\bar{\xi}^2)}{a^2\sqrt{2}(1+\xi\bar{\xi})}$, and $\Gamma^1_{01} m^1 = \frac{\dot{a}}{a} \frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})}(1 - \bar{\xi}^2) = \frac{\dot{a}(1-\bar{\xi}^2)}{a^2\sqrt{2}(1+\xi\bar{\xi})}$,
which means that for the case where $b = 1$ equation (6.6) becomes:

$$l^a \nabla_a m^1 = l^0 (\frac{\partial}{\partial t} m^1 + \Gamma^1_{01} m^1) = l^0 (\frac{-\dot{a}(1 - \bar{\xi}^2)}{a^2\sqrt{2}(1+\xi\bar{\xi})} + \frac{\dot{a}(1 - \bar{\xi}^2)}{a^2\sqrt{2}(1+\xi\bar{\xi})}) = l^0(0) = 0. \tag{6.8}$$

For the case where $\underline{b = 2}$ equation (6.6) becomes:

$$l^a \nabla_a m^2 = l^a \tfrac{\partial}{\partial x^a} m^2 + \Gamma^2_{ac} l^a m^c = l^0 \tfrac{\partial}{\partial t} m^2 + \Gamma^2_{02} l^0 m^2 + \Gamma^2_{20} l^2 m^0$$

and since $m^0 = 0$, when $b = 2$ we can drop the last term and write:

$$l^a \nabla_a m^2 = l^a \tfrac{\partial}{\partial x^a} m^2 + \Gamma^2_{ac} l^a m^c = l^0 \tfrac{\partial}{\partial t} m^2 + \Gamma^2_{02} l^0 m^2 = l^0 (\tfrac{\partial}{\partial t} m^2 + \Gamma^2_{02} m^2)$$

Where after substitutions we have:
$\frac{\partial}{\partial t} m^2 = \frac{\partial}{\partial t} \frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})}(-i(\bar{\xi}^2 + 1)) = \frac{\dot{a}i(\bar{\xi}^2+1)}{a^2\sqrt{2}(1+\xi\bar{\xi})}$, and $\Gamma^1_{01} m^1 = \frac{\dot{a}}{a} \frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})}(-i(\bar{\xi}^2 + 1)) = \frac{-\dot{a}i(\bar{\xi}^2+1)}{a^2\sqrt{2}(1+\xi\bar{\xi})}$,
which means that for the case where $b = 2$ equation (6.6) becomes:

$$l^a \nabla_a m^2 = l^0 (\frac{\partial}{\partial t} m^2 + \Gamma^2_{02} m^2) = l^0 (\frac{\dot{a}i(\bar{\xi}^2 + 1)}{a^2\sqrt{2}(1+\xi\bar{\xi})} + \frac{-\dot{a}i(\bar{\xi}^2 + 1)}{a^2\sqrt{2}(1+\xi\bar{\xi})}) = l^0(0) = 0. \tag{6.9}$$

For the case where $\underline{b=3}$ equation (6.6) becomes:

$$l^a\nabla_a m^3 = l^a \tfrac{\partial}{\partial x^a} m^3 + \Gamma^3_{ac} l^a m^c = l^0 \tfrac{\partial}{\partial t} m^3 + \Gamma^3_{03} l^0 m^3 + \Gamma^3_{30} l^3 m^0$$

and since $m^0 = 0$, when $b = 3$ we can drop the last term and write:

$$l^a\nabla_a m^3 = l^a \tfrac{\partial}{\partial x^a} m^3 + \Gamma^3_{ac} l^a m^c = l^0 \tfrac{\partial}{\partial t} m^3 + \Gamma^3_{03} l^0 m^3 = l^0 (\tfrac{\partial}{\partial t} m^3 + \Gamma^3_{03} m^3)$$

Where after substitutions we have:

$\frac{\partial}{\partial t} m^3 = \frac{\partial}{\partial t} \frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})}(2\bar{\xi}) = \frac{-\dot{a}2\bar{\xi}}{a^2\sqrt{2}(1+\xi\bar{\xi})}$, and $\Gamma^3_{03} m^3 = \frac{\dot{a}}{a} \frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})}(2\bar{\xi}) = \frac{\dot{a}2\bar{\xi}}{a^2\sqrt{2}(1+\xi\bar{\xi})}$,
which means that for the case where $b = 3$ equation (6.6) becomes:

$$l^a\nabla_a m^3 = l^0 (\frac{\partial}{\partial t} m^2 + \Gamma^2_{02} m^2) = l^0 (\frac{-\dot{a}2\bar{\xi}}{a^2\sqrt{2}(1+\xi\bar{\xi})} + \frac{\dot{a}2\bar{\xi}}{a^2\sqrt{2}(1+\xi\bar{\xi})}) = l^0(0) = 0. \tag{6.10}$$

6.1.4 3^{rd} Condition

The third normalization condition that our complex null tetrad needs to meet can be expressed as:

$$g_{ab} l^a l^b = 0, \tag{6.11}$$

where:

$$g_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ 0 & 0 & 0 & -a^2(t) \end{bmatrix},$$

for a flat unperturbed FLRW metric. We now write out all the terms in equation (6.11). They are:
$g_{ab} l^a l^b = g_{00} l^0 l^0 + g_{11} l^1 l^1 + g_{22} l^2 l^2 + g_{33} l^3 l^3 = g_{00}(l^0)^2 + g_{11}(l^1)^2 + g_{22}(l^2)^2 + g_{33}(l^3)^2$,

then by substituting in our null tetrad and the the relevant metric terms we get:

$g_{ab}l^a l^b = (1)(\frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})}(-1-\xi\bar{\xi}))^2 + (-a^2)(\frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})}\frac{1}{a(t)}(\bar{\xi}+\xi))^2 + (-a^2)(\frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})}\frac{i}{a(t)}(\bar{\xi}-\xi))^2 + (-a^2)(\frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})}\frac{1}{a(t)}(-1+\xi\bar{\xi}))^2,$

$= (\frac{(-1-\xi\bar{\xi})}{a(t)\sqrt{2}(1+\xi\bar{\xi})})^2 + (-a^2)(\frac{(\bar{\xi}+\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})})^2 + (-a^2)(\frac{i(\bar{\xi}-\xi)}{a^2\sqrt{2}(1+\xi\bar{\xi})})^2 + (-a^2)(\frac{(-1+\xi\bar{\xi})}{a^2\sqrt{2}(1+\xi\bar{\xi})})^2,$

$= (\frac{-(1+\xi\bar{\xi})}{a(t)\sqrt{2}(1+\xi\bar{\xi})})^2 + (-a^2\frac{(\bar{\xi}+\xi)^2}{2a^4(1+\xi\bar{\xi})^2}) + (-a^2\frac{i^2(\bar{\xi}-\xi)^2}{2a^4(1+\xi\bar{\xi})^2}) + (-a^2\frac{(-1+\xi\bar{\xi})^2}{2a^4(1+\xi\bar{\xi})^2}),$

$= (\frac{(1+\xi\bar{\xi})^2}{2a^2(1+\xi\bar{\xi})^2}) - (\frac{(\bar{\xi}+\xi)^2}{2a^2(1+\xi\bar{\xi})^2}) - (\frac{-(\bar{\xi}-\xi)^2}{2a^2(1+\xi\bar{\xi})^2}) - (\frac{(-1+\xi\bar{\xi})^2}{2a^2(1+\xi\bar{\xi})^2}),$

$= \frac{1}{2a^2(1+\xi\bar{\xi})^2}((1+\xi\bar{\xi})^2 - (\bar{\xi}+\xi)^2 + (\bar{\xi}-\xi)^2 - (-1+\xi\bar{\xi})^2),$

$= \frac{1}{2a^2(1+\xi\bar{\xi})^2}((1+2\xi\bar{\xi}+(\xi\bar{\xi})^2) - (\bar{\xi}^2+2\xi\bar{\xi}+\xi^2) + (\bar{\xi}^2-2\xi\bar{\xi}+\xi^2) - (1-2\xi\bar{\xi}+(\xi\bar{\xi})^2)),$

$= \frac{1}{2a^2(1+\xi\bar{\xi})^2}(1+2\xi\bar{\xi}+(\xi\bar{\xi})^2 - \bar{\xi}^2 - 2\xi\bar{\xi} - \xi^2 + \bar{\xi}^2 - 2\xi\bar{\xi} + \xi^2 - 1 + 2\xi\bar{\xi} - (\xi\bar{\xi})^2),$

$= \frac{1}{2a^2(1+\xi\bar{\xi})^2}(1-1+2\xi\bar{\xi}+2\xi\bar{\xi}-2\xi\bar{\xi}-2\xi\bar{\xi}+(\xi\bar{\xi})^2-(\xi\bar{\xi})^2-\bar{\xi}^2+\bar{\xi}^2-\xi^2+\xi^2),$

which means that equation (6.11) becomes:

$$g_{ab}l^a l^b = \frac{1}{2a^2(1+\xi\bar{\xi})^2}(0) = 0, \tag{6.12}$$

and our tetrad meets the third normalization condition for a flat unperturbed FLRW metric.

6.1.5 4^{th} Condition

The fourth normalization condition that our complex null tetrad needs to meet can be expressed as:

$$g_{ab}n^a n^b = 0, \tag{6.13}$$

where using similar substitutions as those used in the third condition, we can write equation (6.13) as:

$g_{ab}n^a n^b = g_{00}n^0 n^0 + g_{11}n^1 n^1 + g_{22}n^2 n^2 + g_{33}n^3 n^3 = g_{00}(n^0)^2 + g_{11}(n^1)^2 + g_{22}(n^2)^2 + g_{33}(n^3)^2,$

and:

$g_{ab}n^a n^b = (1)(\frac{a(t)}{\sqrt{2}(1+\xi\bar{\xi})}(-1-\xi\bar{\xi}))^2+(-a^2)(\frac{a(t)}{\sqrt{2}(1+\xi\bar{\xi})}\frac{-1}{a(t)}(\bar{\xi}+\xi))^2+(-a^2)(\frac{a(t)}{\sqrt{2}(1+\xi\bar{\xi})}\frac{i}{a(t)}(\xi-\bar{\xi}))^2+(-a^2)(\frac{a(t)}{\sqrt{2}(1+\xi\bar{\xi})}\frac{-1}{a(t)}(-1+\xi\bar{\xi}))^2$,

$= (\frac{a^2(-1-\xi\bar{\xi})^2}{2(1+\xi\bar{\xi})^2}) - (\frac{(a^2)(\bar{\xi}+\xi)^2}{2(1+\xi\bar{\xi})^2}) - (\frac{(a^2)i^2(\xi-\bar{\xi})^2}{2(1+\xi\bar{\xi})^2}) - (\frac{(a^2)(-1+\xi\bar{\xi})^2}{2(1+\xi\bar{\xi})^2})$,

$= (\frac{a^2(-1-\xi\bar{\xi})^2}{2(1+\xi\bar{\xi})^2}) - (\frac{(a^2)(\bar{\xi}+\xi)^2}{2(1+\xi\bar{\xi})^2}) - (\frac{(a^2)i^2(\xi-\bar{\xi})^2}{2(1+\xi\bar{\xi})^2}) - (\frac{(a^2)(-1+\xi\bar{\xi})^2}{2(1+\xi\bar{\xi})^2})$,

$= (\frac{a^2}{2(1+\xi\bar{\xi})^2})((-1-\xi\bar{\xi})^2 - (\bar{\xi}+\xi)^2 - ((-1)(\xi-\bar{\xi})^2) - (-1+\xi\bar{\xi})^2)$,

$= (\frac{a^2}{2(1+\xi\bar{\xi})^2})((1+2\xi\bar{\xi}+(\xi\bar{\xi})^2) - (\bar{\xi}^2+2\xi\bar{\xi}+\xi^2) + (\xi^2-2\xi\bar{\xi}+\bar{\xi}^2) - (1-2\xi\bar{\xi}+(\xi\bar{\xi})^2)$,

$= (\frac{a^2}{2(1+\xi\bar{\xi})^2})(1+2\xi\bar{\xi}+(\xi\bar{\xi})^2 - \bar{\xi}^2 - 2\xi\bar{\xi} - \xi^2 + \xi^2 - 2\xi\bar{\xi} + \bar{\xi}^2 - 1 + 2\xi\bar{\xi} - (\xi\bar{\xi})^2)$,

$= (\frac{a^2}{2(1+\xi\bar{\xi})^2})(1-1+2\xi\bar{\xi}-2\xi\bar{\xi}+2\xi\bar{\xi}-2\xi\bar{\xi}+2\xi\bar{\xi}+(\xi\bar{\xi})^2-(\xi\bar{\xi})^2-\bar{\xi}^2+\bar{\xi}^2-\xi^2+\xi^2)$,

which means that equation (6.13) becomes:

$$g_{ab}n^a n^b = (\frac{a^2}{2(1+\xi\bar{\xi})^2})(0) = 0, \tag{6.14}$$

and our tetrad meets the fourth normalization condition for a flat unperturbed FLRW metric.

6.1.6 5^{th} Condition

The fifth normalization condition that our complex null tetrad needs to meet can be expressed as:

$$g_{ab}m^a m^b = 0, \tag{6.15}$$

where using similar substitutions as those used in the third condition, we can write equation (6.15) as:

$g_{ab}m^a m^b = g_{00}m^0 m^0 + g_{11}m^1 m^1 + g_{22}m^2 m^2 + g_{33}m^3 m^3 = g_{00}(m^0)^2 + g_{11}(m^1)^2 + g_{22}(m^2)^2 + g_{33}(m^3)^2$,

and:

$g_{ab}m^a m^b = (1)(\frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})}(0)^2+(-a^2)(\frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})}(1-\bar{\xi}^2))^2+(-a^2)(\frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})}((-i)(\bar{\xi}^2+1))^2+(-a^2)(\frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})}(2\bar{\xi}))^2$,

$= \frac{-a^2}{2a^2(1+\xi\bar{\xi})^2}\left[(1-\bar{\xi}^2)^2 + ((-i)(\bar{\xi}^2+1))^2 + (2\bar{\xi})^2\right] = \frac{-1}{2(1+\xi\bar{\xi})^2}\left[(1-2\bar{\xi}^2+\bar{\xi}^4) + ((-1)(\bar{\xi}^4+2\bar{\xi}^2+1)) + 4\bar{\xi}^2\right]$,

$= \frac{-1}{2(1+\xi\overline{\xi})^2}\left[1 - 2\overline{\xi}^2 + \overline{\xi}^4 - \overline{\xi}^4 - 2\overline{\xi}^2 - 1 + 4\overline{\xi}^2\right] = \frac{-1}{2(1+\xi\overline{\xi})^2}\left[1 - 1 - 2\overline{\xi}^2 - 2\overline{\xi}^2 + 4\overline{\xi}^2 + \overline{\xi}^4 - \overline{\xi}^4\right],$

which means that equation (6.15) becomes:

$$g_{ab}m^a m^b = \frac{-1}{2(1+\xi\overline{\xi})^2}\,[0] = 0, \tag{6.16}$$

and our tetrad meets the fifth normalization condition for a flat unperturbed FLRW metric.

6.1.7 6^{th} Condition

The sixth normalization condition that our complex null tetrad needs to meet can be expressed as:

$$g_{ab}\overline{m}^a m^b = -1, \tag{6.17}$$

where using similar substitutions as those used in the third condition, we can write equation (6.17) as: $g_{ab}\overline{m}^a m^b = g_{00}\overline{m}^0 m^0 + g_{11}\overline{m}^1 m^1 + g_{22}\overline{m}^2 m^2 + g_{33}\overline{m}^3 m^3$,

and:

$g_{ab}\overline{m}^a m^b = (1)(\frac{1}{a(t)\sqrt{2}(1+\xi\overline{\xi})})^2(0)(0) + (-a^2)(\frac{1}{a(t)\sqrt{2}(1+\xi\overline{\xi})})^2(1-\xi^2)(1-\overline{\xi}^2) + (-a^2)(\frac{1}{a(t)\sqrt{2}(1+\xi\overline{\xi})})^2(i(\xi^2+1))((-i)(\overline{\xi}^2+1)) + (-a^2)(\frac{1}{a(t)\sqrt{2}(1+\xi\overline{\xi})})^2(2\xi)(2\overline{\xi}),$

$= (-a^2)(\frac{1}{a(t)\sqrt{2}(1+\xi\overline{\xi})})^2\left[(1-\xi^2)(1-\overline{\xi}^2) + (i(\xi^2+1))((-i)(\overline{\xi}^2+1)) + (2\xi)(2\overline{\xi})\right],$

$= \frac{-1}{2(1+\xi\overline{\xi})^2}\left[(1-\xi^2)(1-\overline{\xi}^2) + (i(\xi^2+1))((-i)(\overline{\xi}^2+1)) + (2\xi)(2\overline{\xi})\right],$

$= \frac{-1}{2(1+\xi\overline{\xi})^2}\left[(1-\xi^2-\overline{\xi}^2+(\xi\overline{\xi})^2) + (\xi^2+1)(\overline{\xi}^2+1) + (4\xi\overline{\xi})\right],$

$= \frac{-1}{2(1+\xi\overline{\xi})^2}\left[(1-\xi^2-\overline{\xi}^2+(\xi\overline{\xi})^2) + (1+\xi^2+\overline{\xi}^2+(\xi\overline{\xi})^2) + (4\xi\overline{\xi})\right],$

$= \frac{-1}{2(1+\xi\overline{\xi})^2}\left[1-\xi^2-\overline{\xi}^2+(\xi\overline{\xi})^2+1+\xi^2+\overline{\xi}^2+(\xi\overline{\xi})^2+4\xi\overline{\xi}\right],$

$= \frac{-1}{2(1+\xi\overline{\xi})^2}\left[-\xi^2+\xi^2-\overline{\xi}^2+\overline{\xi}^2+1+1+4\xi\overline{\xi}+(\xi\overline{\xi})^2+(\xi\overline{\xi})^2\right] = \frac{-1}{2(1+\xi\overline{\xi})^2}\left[2+4\xi\overline{\xi}+2(\xi\overline{\xi})^2\right],$

which means equation (6.17) becomes:

$$g_{ab}\overline{m}^a m^b = \frac{-1}{(1+\xi\overline{\xi})^2}\left[1 + 2\xi\overline{\xi} + (\xi\overline{\xi})^2\right] = -1 \tag{6.18}$$

,

and our tetrad meets the sixth normalization condition for our metric.

6.1.8 7^{th} Condition

The seventh normalization condition that our complex null tetrad needs to meet can be expressed as:

$$g_{ab}l^a n^b = 1, \tag{6.19}$$

where using similar substitutions as those used in the third condition, we can write equation (6.19) as: $g_{ab}l^a n^b = g_{00}l^0 n^0 + g_{11}l^1 n^1 + g_{22}l^2 n^2 + g_{33}l^3 n^3$,

and:

$g_{ab}l^a n^b = (1)(\frac{1}{a(t)\sqrt{2}(1+\xi\overline{\xi})})(\frac{a(t)}{\sqrt{2}(1+\xi\overline{\xi})})(-1-\xi\overline{\xi})(-1-\xi\overline{\xi}) + (-a^2)(\frac{1}{a(t)\sqrt{2}(1+\xi\overline{\xi})})(\frac{a(t)}{\sqrt{2}(1+\xi\overline{\xi})})(\frac{1}{a(t)}(\overline{\xi}+\xi))(\frac{-1}{a(t)}(\overline{\xi}+\xi)) + (-a^2)(\frac{1}{a(t)\sqrt{2}(1+\xi\overline{\xi})})(\frac{a(t)}{\sqrt{2}(1+\xi\overline{\xi})})(\frac{i}{a(t)}(\overline{\xi}-\xi))(\frac{i}{a(t)}(\xi-\overline{\xi})) + (-a^2)(\frac{1}{a(t)\sqrt{2}(1+\xi\overline{\xi})})(\frac{a(t)}{\sqrt{2}(1+\xi\overline{\xi})})(\frac{1}{a(t)}(-1+\xi\overline{\xi}))(\frac{-1}{a(t)}(-1+\xi\overline{\xi}))$,

$= (\frac{1}{2(1+\xi\overline{\xi})^2})[(1+\xi\overline{\xi})^2 + (-a^2)(\frac{1}{a(t)}(\overline{\xi}+\xi))(\frac{-1}{a(t)}(\overline{\xi}+\xi)) + (-a^2)(\frac{i}{a(t)}(\overline{\xi}-\xi))(\frac{i}{a(t)}(\xi-\overline{\xi})) + (-a^2)(\frac{1}{a(t)}(-1+\xi\overline{\xi}))(\frac{-1}{a(t)}(-1+\xi\overline{\xi}))]$,

$= (\frac{1}{2(1+\xi\overline{\xi})^2})\left[(1+\xi\overline{\xi})^2 + (\overline{\xi}+\xi)^2 - i^2(\overline{\xi}-\xi)(\xi-\overline{\xi}) + (-1+\xi\overline{\xi})^2)\right]$,

$= (\frac{1}{2(1+\xi\overline{\xi})^2})\left[(1+\xi\overline{\xi})^2 + (\overline{\xi}+\xi)^2 + (\overline{\xi}-\xi)(\xi-\overline{\xi}) + (-1+\xi\overline{\xi})^2)\right]$,

$= (\frac{1}{2(1+\xi\overline{\xi})^2})\left[(1 + 2\xi\overline{\xi} + \xi\overline{\xi}^2) + (\overline{\xi}^2 + 2\xi\overline{\xi} + \xi^2) + (-\overline{\xi}^2 + 2\xi\overline{\xi} - \xi^2) + (1 - 2\xi\overline{\xi} + \xi\overline{\xi}^2)\right]$,

$= (\frac{1}{2(1+\xi\overline{\xi})^2})\left[1 + 2\xi\overline{\xi} + \xi\overline{\xi}^2 + \overline{\xi}^2 + 2\xi\overline{\xi} + \xi^2 - \overline{\xi}^2 + 2\xi\overline{\xi} - \xi^2 + 1 - 2\xi\overline{\xi} + \xi\overline{\xi}^2\right]$,

$= (\frac{1}{2(1+\xi\overline{\xi})^2})\left[1 + 1 + 2\xi\overline{\xi} - 2\xi\overline{\xi} + 2\xi\overline{\xi} + 2\xi\overline{\xi} + (\xi\overline{\xi})^2 + (\xi\overline{\xi})^2 + \overline{\xi}^2 - \overline{\xi}^2 + \xi^2 - \xi^2\right]$,

$= (\frac{1}{2(1+\xi\bar{\xi})^2}) \left[2 + 4\xi\bar{\xi} + 2(\xi\bar{\xi})^2\right] = (\frac{1}{(1+\xi\bar{\xi})^2}) \left[1 + 2\xi\bar{\xi} + (\xi\bar{\xi})^2\right] = (\frac{1}{(1+\xi\bar{\xi})^2})(1 + \xi\bar{\xi})^2,$

which means equation (6.19) becomes:

$$g_{ab}l^a n^b = (\frac{1}{(1+\xi\bar{\xi})^2})(1 + \xi\bar{\xi})^2 = 1 \tag{6.20}$$

,

and our tetrad meets the seventh normalization condition for the flat FLRW metric.

6.1.9 8^{th} Condition

The eighth normalization condition can be expressed as:

$$l \cdot m = 0, \tag{6.21}$$

where the operator ($\cdot$) is the dot product, we can write equation (6.21) as:

$l \cdot m = l^0 m^0 + l^1 m^1 + l^2 m^2 + l^3 m^3,$

and by substituiting in our null tetrad we get:

$l \cdot m = (\frac{1}{a(t)\sqrt{2}(1+\xi\bar{\xi})})^2 \left[(-1 - \xi\bar{\xi})(0) + \frac{1}{a(t)}(\bar{\xi} + \xi)(1 - \bar{\xi}^2) - \frac{i}{a(t)}(\bar{\xi} - \xi)i(\bar{\xi}^2 + 1) + \frac{1}{a(t)}(-1 + \xi\bar{\xi})(2\bar{\xi})\right],$

$= (\frac{1}{2a^2(1+\xi\bar{\xi})^2}) \left[\frac{1}{a(t)}(\bar{\xi} + \xi)(1 - \bar{\xi}^2) - \frac{i}{a(t)}(\bar{\xi} - \xi)i(\bar{\xi}^2 + 1) + \frac{1}{a(t)}(-1 + \xi\bar{\xi})(2\bar{\xi})\right],$

$= (\frac{1}{2a^3(1+\xi\bar{\xi})^2}) \left[(\bar{\xi} + \xi)(1 - \bar{\xi}^2) - (-1)(\bar{\xi} - \xi)(\bar{\xi}^2 + 1) + (-1 + \xi\bar{\xi})(2\bar{\xi})\right],$

$= (\frac{1}{2a^3(1+\xi\bar{\xi})^2}) \left[(\bar{\xi} + \xi - \bar{\xi}^3 - \xi\bar{\xi}^2) + (\bar{\xi}^3 - \xi\bar{\xi}^2 + \bar{\xi} - \xi) + (-2\bar{\xi} + 2\xi\bar{\xi}^2)\right],$

$= (\frac{1}{2a^3(1+\xi\bar{\xi})^2}) \left[\bar{\xi} + \xi - \bar{\xi}^3 - \xi\bar{\xi}^2 + \bar{\xi}^3 - \xi\bar{\xi}^2 + \bar{\xi} - \xi + -2\bar{\xi} + 2\xi\bar{\xi}^2\right],$

$= (\frac{1}{2a^3(1+\xi\bar{\xi})^2}) \left[\bar{\xi} + \bar{\xi} - 2\bar{\xi} + \xi - \xi - \bar{\xi}^3 + \bar{\xi}^3 - \xi\bar{\xi}^2 - \xi\bar{\xi}^2 + 2\xi\bar{\xi}^2\right],$

or equation (6.21) becomes:

$$l \cdot m = (\frac{1}{2a^3(1+\xi\bar{\xi})^2})\,[0] = 0 \tag{6.22}$$

and our complex null tetrad meets the eighth normalization condition .

Note: this condition DOES NOT dependent on our metric.

6.1.10 9^{th} Condition

The ninth normalization condition can be expressed as:

$$l \cdot \overline{m} = 0, \tag{6.23}$$

where by computing the dot product we can write equation (6.23) as:

$l \cdot \overline{m} = l^0\overline{m}^0 + l^1\overline{m}^1 + l^2\overline{m}^2 + l^3\overline{m}^3$,

and by substituiting in our null tetrad we get:

$l \cdot \overline{m} = (\frac{1}{a(t)\sqrt{2}(1+\xi\overline{\xi})})^2 \Big[(-1-\xi\overline{\xi})(0) + \frac{1}{a(t)}(\overline{\xi}+\xi)(1-\xi^2) + \frac{i}{a(t)}(\overline{\xi}-\xi)i(\xi^2+1) + \frac{1}{a(t)}(-1+\xi\overline{\xi})(2\xi)\Big]$,

$= (\frac{1}{2a^2(1+\xi\overline{\xi})^2}) \Big[\frac{1}{a(t)}(\overline{\xi}+\xi)(1-\xi^2) + \frac{i}{a(t)}(\overline{\xi}-\xi)i(\xi^2+1) + \frac{1}{a(t)}(-1+\xi\overline{\xi})(2\xi)\Big]$,

$= (\frac{1}{2a^3(1+\xi\overline{\xi})^2}) \left[(\overline{\xi}+\xi)(1-\xi^2) + (-1)(\overline{\xi}-\xi)(\xi^2+1) + (-1+\xi\overline{\xi})(2\xi)\right]$,

$= (\frac{1}{2a^3(1+\xi\overline{\xi})^2}) \left[(\overline{\xi}+\xi-\xi^3-\overline{\xi}\xi^2) - (\overline{\xi}\xi^2-\xi^3+\overline{\xi}-\xi) + (-2\xi+2\overline{\xi}\xi^2)\right]$,

$= (\frac{1}{2a^3(1+\xi\overline{\xi})^2}) \left[(\overline{\xi}+\xi-\xi^3-\overline{\xi}\xi^2) - (\overline{\xi}\xi^2-\xi^3+\overline{\xi}-\xi) + (-2\xi+2\overline{\xi}\xi^2)\right]$,

$= (\frac{1}{2a^3(1+\xi\overline{\xi})^2}) \left[\overline{\xi}+\xi-\xi^3-\overline{\xi}\xi^2-\overline{\xi}\xi^2+\xi^3-\overline{\xi}+\xi-2\xi+2\overline{\xi}\xi^2\right]$,

$= (\frac{1}{2a^3(1+\xi\overline{\xi})^2}) \left[\overline{\xi}-\overline{\xi}+\xi+\xi-2\xi-\xi^3+\xi^3-\overline{\xi}\xi^2-\overline{\xi}\xi^2+2\overline{\xi}\xi^2\right]$,

or equation (6.23) becomes:

$$l \cdot \overline{m} = (\frac{1}{2a^3(1+\xi\overline{\xi})^2})\,[0] = 0 \tag{6.24}$$

and our complex null tetrad meets the ninth normalization condition .

6.1.11 10^{th} Condition

The tenth condition is:

$$n \cdot m = 0, \tag{6.25}$$

where by computing the dot product we can write equation (6.25) as:

$n \cdot m = n^0 m^0 + n^1 m^1 + n^2 m^2 + n^3 m^3,$

and by substituiting in our null tetrad we get:

$n \cdot m = (\frac{a(t)}{\sqrt{2}(1+\xi\overline{\xi})})(\frac{1}{a(t)\sqrt{2}(1+\xi\overline{\xi})})[(-1-\xi\overline{\xi})(0) + \frac{-1}{a(t)}(\overline{\xi}+\xi)(1-\overline{\xi}^2) + \frac{i}{a(t)}(\xi-\overline{\xi})(-i(\overline{\xi}^2+1)) + \frac{-1}{a(t)}(-1+\xi\overline{\xi})(2\overline{\xi})],$

$= (\frac{1}{2a(t)(1+\xi\overline{\xi})^2}) \left[(-1)(\overline{\xi}+\xi)(1-\overline{\xi}^2) - (i^2)(\xi-\overline{\xi})(\overline{\xi}^2+1) - (-1+\xi\overline{\xi})(2\overline{\xi})\right],$

$= (\frac{1}{2a(t)(1+\xi\overline{\xi})^2}) \left[(\overline{\xi}+\xi)(-1+\overline{\xi}^2) + (\xi-\overline{\xi})(\overline{\xi}^2+1) - (-2\overline{\xi}+2\xi\overline{\xi}^2)\right],$

$= (\frac{1}{2a(t)(1+\xi\overline{\xi})^2}) \left[(-\overline{\xi}-\xi+\overline{\xi}^3+\xi\overline{\xi}^2) + (\xi\overline{\xi}^2-\overline{\xi}^3+\xi-\overline{\xi}) + 2\overline{\xi} - 2\xi\overline{\xi}^2\right],$

$= (\frac{1}{2a(t)(1+\xi\overline{\xi})^2}) \left[-\overline{\xi}-\xi+\overline{\xi}^3+\xi\overline{\xi}^2+\xi\overline{\xi}^2-\overline{\xi}^3+\xi-\overline{\xi}+2\overline{\xi}-2\xi\overline{\xi}^2\right],$

$= (\frac{1}{2a(t)(1+\xi\overline{\xi})^2}) \left[-\overline{\xi}-\overline{\xi}+2\overline{\xi}-\xi+\xi+\overline{\xi}^3-\overline{\xi}^3+\xi\overline{\xi}^2+\xi\overline{\xi}^2-2\xi\overline{\xi}^2\right],$

or equation (6.25) becomes:

$$n \cdot m = (\frac{1}{2a(t)(1+\xi\overline{\xi})^2})\,[0] = 0 \tag{6.26}$$

and our complex null tetrad meets the tenth normalization condition .

6.1.12 11^{th} Condition

The eleventh normalization condition is:

$$n \cdot \overline{m} = 0, \tag{6.27}$$

where by computing the dot product we can write equation (6.27) as:

$n \cdot \overline{m} = n^0\overline{m}^0 + n^1\overline{m}^1 + n^2\overline{m}^2 + n^3\overline{m}^3,$

and by substituiting in our null tetrad we get:

$n{\cdot}\overline{m} = (\frac{a(t)}{\sqrt{2}(1+\xi\overline{\xi})})(\frac{1}{a(t)\sqrt{2}(1+\xi\overline{\xi})})\left[(-1-\xi\overline{\xi})(0) + \frac{-1}{a(t)}(\overline{\xi}+\xi)(1-\xi^2) + \frac{i}{a(t)}(\xi-\overline{\xi})i(\xi^2+1) + \frac{-1}{a(t)}(-1+\xi\overline{\xi})(2\xi)\right],$

$= (\frac{1}{2a(t)(1+\xi\overline{\xi})^2})\left[(-1)(\overline{\xi}+\xi)(1-\xi^2) + (i^2)(\xi-\overline{\xi})(\xi^2+1) + (-1)(-1+\xi\overline{\xi})(2\xi)\right],$

$= (\frac{1}{2a(t)(1+\xi\overline{\xi})^2})\left[(\overline{\xi}+\xi)(\xi^2-1) - (\xi-\overline{\xi})(\xi^2+1) - (-1+\xi\overline{\xi})(2\xi)\right],$

$= (\frac{1}{2a(t)(1+\xi\overline{\xi})^2})\left[(\overline{\xi}\xi^2+\xi^3-\overline{\xi}-\xi) - (\xi^3-\overline{\xi}\xi^2+\xi-\overline{\xi}) - (-2\xi+2\overline{\xi}\xi^2)\right],$

$= (\frac{1}{2a(t)(1+\xi\overline{\xi})^2})\left[\overline{\xi}\xi^2+\xi^3-\overline{\xi}-\xi-\xi^3+\overline{\xi}\xi^2-\xi+\overline{\xi}+2\xi-2\overline{\xi}\xi^2\right],$

$= (\frac{1}{2a(t)(1+\xi\overline{\xi})^2})\left[\overline{\xi}\xi^2+\overline{\xi}\xi^2-2\overline{\xi}\xi^2+\xi^3-\xi^3-\overline{\xi}+\overline{\xi}-\xi-\xi+2\xi\right],$

or equation (6.27) becomes:

$$n \cdot \overline{m} = (\frac{1}{2a(t)(1+\xi\overline{\xi})^2})\,[0] = 0 \tag{6.28}$$

and our complex null tetrad meets the tenth normalization condition .

This concludes the set of normalization and metric conditions that our complex null-tetrad needs meet in order to use the Newman-Penrose formalism in a flat FLRW cosmology with no gravitational perturbations from local clumps of matter.

Bibliography

Newman, E.T. & Penrose, R., J. Math Phys. **3**, 566-78 (1962)

Kling, T.P. & Campbell, B., Phys. Rev. D. **12**, 123-77 (2008)

Wald, R. M., *General Relativity*, (University Of Chicago Press, Chigago, IL, 1984)

Schneider, P., Ehlers, J. & Falco, E.E., *Gravitational Lenses,* (Springer-Verlag, Berlin, Heildelerg, 1992)

Stephani, H., & Stewart, J., *General relativity: an introduction to the theory of the gravitational field,* (Cambridge University Press, Cambridgeshire,Cambridge, 1982)

Ryden, B. S., *Introduction to cosmology,* (Addison-Wesley, San Francisco, 2003)

Carroll, S. M., *Spacetime and geometry: an introduction to general relativity,* (Addison-Wesley, San Francisco, 2004)

Foster, J., & Nightingale, J. D., *A short course in general relativity , 2nd ed* (Springer-Verlag, Berlin, Heildelerg, 1995)

McEvoy, J. P., *A brief history of the universe*, (Robinson, London,2010)

Whittaker, E. T., *A history of the theories of aether and electricity,* (Nelson, London, 1951,)

Xu, G., *GPS: Theory, algorithms, and applications,* (Springer, Berlin, 2003)

Brown, L. M., Pais, A., & Pippard, A. B., *Twentieth century physics,* (Institute of Physics Pub., Bristol, 1995)

Earman, J., Janssen, M., & Norton, J. D.,

Will the Universe expand forever? (n.d.), Retrieved December 10, 2014, from $http : //map.gsfc.nasa.gov/universe/uni_shape.html$