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The E.P.R. Paradox

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George graduated from Bridgewater State College with majors in Physics, Mathematics, Criminal Justice, and Sociology. This piece is his Honors project for Electricity and Magnetism advised by Dr. Edward Deveney. George ruminated to help the reader formulate, and accept, why quantum mechanics, though weird, is valid.

This paper intends to discuss the E.P.R. paradox and its implications for quantum mechanics. In order to do so, this paper will discuss the features of intrinsic spin of a particle, the Stern-Gerlach experiment, the E.P.R. paradox itself and the views it portrays. In addition, we will consider where such a classical picture succeeds and, eventually, as we will see in Bell's inequality, fails in the strange world we live in – the world of quantum mechanics.

Intrinsic Spin

Intrinsic spin angular momentum is odd to describe by any normal terms. It is unlike, and often entirely unrelated to, the classical "orbital angular momentum." But luckily we can describe the intrinsic spin by its relationship to the magnetic moment of the particle being considered. The magnetic moment can be given by:

$$\mu = \frac{IA}{c} = \left(\frac{q}{T}\right)\frac{\pi r^2}{c} = \frac{qvr}{2c} = \frac{q}{2mc}L$$

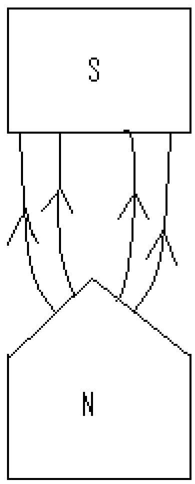
This brief derivation can be seen to apply where mass and charge coincide in space. More generally, we tend to consider:

$$\mu = \frac{gq}{2mc}S$$

where g is an experimentally determined number (depending on the particle used like g = 2.00 for an electron). This is essentially useful background for when we put our particle into a Stern-Gerlach device.

Stern-Gerlach Experiments

Stern-Gerlach devices utilize the magnetic moment of a particle by placing it in a non-uniform magnetic field as depicted here:



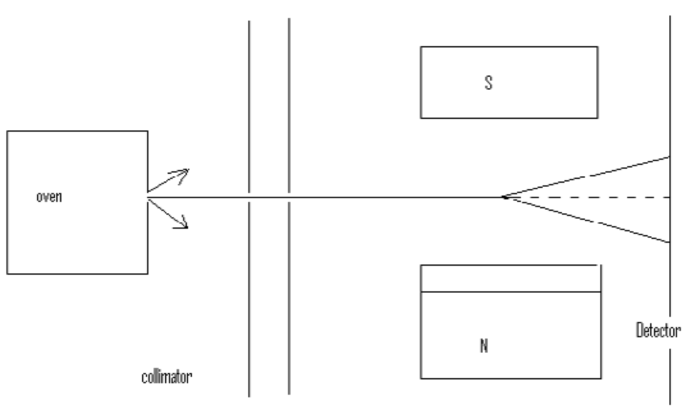
While $F = \nabla(\mu \bullet B)$ for a neutral atom entering the device, since $-(\mu \bullet B)$ is the energy of a magnetic dipole placed in a magnetic field), the magnetic field is largely in one direction (here we can call it the z-direction). So,

$$F_z = \mu \circ \frac{\partial B}{\partial z} \cong \mu_z \frac{\partial B_z}{\partial z}$$

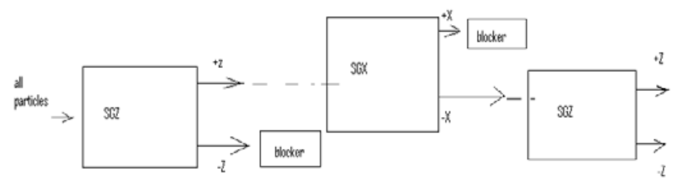
Normally, a statement like this may not raise too many eyebrows but there is quantum weirdness here as well. Classically, the magnetic moment will take on a continuum of values (from $\mu_z = |\mu| \cos\theta$ where theta is the azimuthal angle). But in our experiment it takes on the values of *only*

$$S_z = \pm \frac{\hbar}{2}$$

Our assumption may be that the particles are oriented in this way from the start but the experimenters shot the particles from an oven (as to acquire a random and expectedly continuous distribution of magnetic moments) as below:



So there is little doubt that a non-continuous distribution is unexpected. The result is something that is famously known as "space quantization" and was indeed a "big deal" at the time of its discovery. As a result, there was much focus on this experiment and its implications for science. There were adaptations of this experiment to get a deeper understanding of nature. One of these is diagramed below:



Note how this experiment reveals that we can send all particles through a Z-axis oriented device and get a 50% / 50% distribution and then remove one state completely from the system (with a blocker), and send it through an X-axis oriented SG (Stern-Gerlach) device and get 50/50 again. Most amazingly we can take half of these away and put the remaining ones through another SGZ and wind up with a 50/50 distribution all over again.

In other words, not only do we get some sort of binary nature out of what was thought to be a random orientation of particles, but making another measurement on the particles (in this case with an SGX) destroys the information that preceded it so that we can wind up with a 50/50 distribution all over again.

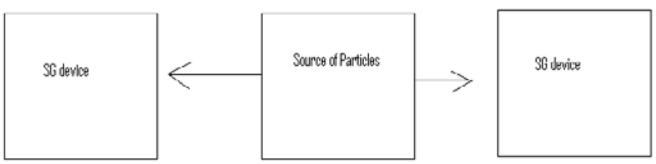
It is the result of these experiments that leads to many important quantum mechanical ideas. First off, the notion that particles do not exist in either one state or another but exist in a "superposition" of states (or "both states simultaneously"). This is evident in the notation of quantum. (For example, $|\varphi\rangle = \frac{1}{\sqrt{2}}|+z\rangle + \frac{1}{\sqrt{2}}|-z\rangle$ shows us that the orientation of a particle can be seen as some probability (when squared just like the psi – or wave – function) of both states being existent simultaneously.) Also, it is the idea of quantum mechanics that the particle exists in this superposition until a measurement is made. Without, or after, an interaction it returns to this superposition. This type of experimental result led to the questioning of and investigation about the world on the quantum mechanical level. During this time there was much questioning and discontent. It is out of this discontent for experimental results that we get the E.P.R. paradox.

The E.P.R. Paradox

In order to cast a shade of doubt on the quantum mechanical world, three scientists (A. Einstein, B. Podolsky, and N. Rosen) proposed a thought experiment that raised a reasonable doubt about the beliefs of quantum mechanics. You see, it brought great

discontent to the "realists" that were Einstein, Podolsky, and Rosen and the idea that the measurement created the particle to exist in some state was appalling. In their view, it was quite apparent that the particle existed in this state before the measurement and that the measurement was only the future observation of this state. (There is one notable quotation from Einstein to another scientist, A. Pais when they were out on a walk talking on this subject and Einstein asked whether he believed the moon was there when he wasn't looking).

So our three scientists devised an experiment that can be diagramed as below:



Here we have a set of particles, emitted two at a time, from some common origin such that their combined orbital angular momentum is zero. Here it is seen that, if the SG devices are oriented similarly, if one device measures a particle in the +z state then the other will have to note a –z state (in order to conserve angular momentum).

But Einstein, Podolsky, and Rosen propose that we assume these SG devices have different settings (or directions of orientation). This way we could consider a hidden-variable theory of quantum mechanics. More explicitly, if we were handed particles in the state |+x>, there would be half which (when measured in a SGz) would be in the |+z> state and half in the |-z> state. Further, if we were handed these |+x> particles (prior to measurement), while they would have the attribute to be either |+z> or |-z>, we would be unable to distinguish them unless we measured them.

But the biggest implication of the E.P.R. experimental design is the following "paradox:" if quantum mechanics is right (and one particles measurement would force it to be in one state and, thus, slam its pair particle to be in the other to conserve angular momentum before it is measured), then two vastly far SG devices in our experimental design would force some sort of faster-than-the-speed-of-light communication between them - an awkward and ugly conclusion that would drive some to question the beliefs of quantum mechanics.

Moving Toward Bell's Inequality

For a single SG device, the realists would say that 50% of the particles exist in one state (like {+z}) and 50% in the other (like {-z}). The followers of quantum mechanics would say:

If $\left|\varphi\right\rangle = \frac{1}{\sqrt{2}}\left|+z\right\rangle - \frac{1}{\sqrt{2}}\left|-z\right\rangle$

then

$$\left\langle +z\middle|\varphi\right\rangle = \frac{1}{\sqrt{2}}\left\langle +z\middle|+z\right\rangle - \frac{1}{\sqrt{2}}\left\langle +z\middle|-z\right\rangle = \frac{1}{\sqrt{2}}\left\langle +z\middle|+z\right\rangle - 0 = \frac{1}{\sqrt{2}}$$

So, the probability is $\left|\left\langle +z\middle|\varphi\right\rangle\right|^2 = \frac{1}{2}$ for this state.

And

$$\left\langle -z\middle|\varphi\right\rangle = \frac{1}{\sqrt{2}}\left\langle -z\middle|+z\right\rangle - \frac{1}{\sqrt{2}}\left\langle -z\middle|-z\right\rangle = \frac{1}{\sqrt{2}}\left\langle -z\middle|-z\right\rangle - 0 = \frac{1}{\sqrt{2}}$$

So, the probability is $\left|\left\langle -z\middle|\varphi\right\rangle\right|^2 = \frac{1}{2}$ for this state.

Since there is no contradiction, we continue to advance our experimental method, in search of some sort of contradiction to test. For the two particle SG device, the realist would say that half of the particles are in the {+z, -z} state and half are in the {-z,+z} state (where this notation is the states of particles 1 and 2 respectively). The followers of quantum mechanics would say:

If $\left|0,0\right\rangle = \frac{1}{\sqrt{2}}\left|+z,-z\right\rangle - \frac{1}{\sqrt{2}}\left|-z,+z\right\rangle$

(or a state of conservative angular momentum) then

$$\left\langle +z,-z\middle|0,0\right\rangle = \frac{1}{\sqrt{2}}\left\langle +z,-z\middle|+z,-z\right\rangle - \frac{1}{\sqrt{2}}\left\langle +z,-z\middle|-z,+z\right\rangle =$$

$$= \frac{1}{\sqrt{2}}\left\langle +z\middle|+z\right\rangle\left\langle -z\middle|-z\right\rangle - \frac{1}{\sqrt{2}}\left\langle +z\middle|+z\right\rangle\left\langle -z\middle|-z\right\rangle = \frac{1}{\sqrt{2}}(1) - 0 = \frac{1}{\sqrt{2}}$$

So, the probability is $\left|\left\langle +z,-z\middle|0,0\right\rangle\right|^2 = \frac{1}{2}$ for this state.

And then,

$$\left\langle -z,+z\middle|0,0\right\rangle = \frac{1}{\sqrt{2}}\left\langle -z,+z\middle|+z,-z\right\rangle - \frac{1}{\sqrt{2}}\left\langle -z,+z\middle|-z,+z\right\rangle =$$

$$= \frac{1}{\sqrt{2}}\left\langle -z\middle|+z\right\rangle\left\langle +z\middle|-z\right\rangle - \frac{1}{\sqrt{2}}\left\langle -z\middle|-z\right\rangle\left\langle +z\middle|+z\right\rangle = 0 - \frac{1}{\sqrt{2}}(1) = -\frac{1}{\sqrt{2}}$$

So, the probability is $\left|\left\langle -z,+z\middle|0,0\right\rangle\right|^2 = \frac{1}{2}$ for this state.

Now let us kick up the level of difficulty another notch. Let us say that we maintain a two particle device but now we can set our SG devices in either the z-axis (an SGz device) or the x axis (SGx device). Now since we have two states the local realist would say that every particle emitted would have a two-part instruction set (one in case it reaches an SGx and one for an SGz). The realist might say there exists

Population	Particle 1	Particle 2
1	{+x, +z}	{-x, -z}
2	{+x, -z}	{-x, +z}

3	{-x, +z}	{+x, -z}
4	{-x, -z}	{+x, +z}

Here these populations occur equally so, for populations 1 and 4 measurements in either x or z, for either device, will always yield particles oriented in opposing states. For populations 2 and 3, randomly oriented devices will yield spin down and spin up measurements only half the time. The other half 2 and 3 will yield similarly oriented particles (spin up or spin down but along different axis). So in total, if measurements are taken in different axis, there will be opposite signs 2/4*1=50% of the time.

Now the follower of quantum mechanics would say,

knowing

$$\left|0,0\right\rangle = \frac{1}{\sqrt{2}}\left|+z,-z\right\rangle - \frac{1}{\sqrt{2}}\left|-z,+z\right\rangle$$

and also that

$$\left|+n\right\rangle = \cos\frac{\theta}{2}\left|+z\right\rangle + e^{i\phi}\sin\frac{\theta}{2}\left|-z\right\rangle.$$

$$\left\langle +z,+x\|0,0\right\rangle = \frac{1}{\sqrt{2}}\left\langle +z,+x|+z,-z\right\rangle - \frac{1}{\sqrt{2}}\left\langle +z,+x|-z,+z\right\rangle =$$

$$= \frac{1}{\sqrt{2}}\left\langle +z|+z\right\rangle\left\langle +x|-z\right\rangle - \frac{1}{\sqrt{2}}\left\langle +z|+z\right\rangle\left\langle -z|-z\right\rangle = \frac{1}{\sqrt{2}}(1)\sin\frac{\theta}{2} - 0 = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{2}$$

So, the probability is $\left|\left\langle +z,+x|0,0\right\rangle\right|^2 = \frac{1}{4}$ for this state.

$$\left\langle +z,-x\|0,0\right\rangle = \frac{1}{\sqrt{2}}\left\langle +z,-x|+z,-z\right\rangle - \frac{1}{\sqrt{2}}\left\langle +z,-x|-z,+z\right\rangle =$$

$$= \frac{1}{\sqrt{2}}\left\langle +z|+z\right\rangle\left\langle -x|-z\right\rangle - \frac{1}{\sqrt{2}}\left\langle +z|+z\right\rangle\left\langle -x|-z\right\rangle = \frac{1}{\sqrt{2}}(1)\cos\frac{\theta}{2} - 0 = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{1}{2}$$

So, the probability is $\left|\left\langle +z,-x|0,0\right\rangle\right|^2 = \frac{1}{4}$ for this state.

$$\left\langle -z,+x\|0,0\right\rangle = \frac{1}{\sqrt{2}}\left\langle -z,+x|+z,-z\right\rangle - \frac{1}{\sqrt{2}}\left\langle -z,+x|-z,+z\right\rangle =$$

$$= \frac{1}{\sqrt{2}}\left\langle -z|+z\right\rangle\left\langle +x|-z\right\rangle - \frac{1}{\sqrt{2}}\left\langle -z|-z\right\rangle\left\langle +x|-z\right\rangle = 0 - \frac{1}{\sqrt{2}}(1)\sin\frac{\theta}{2} = -\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = -\frac{1}{2}$$

So, the probability is $\left|\left\langle -z,+x|0,0\right\rangle\right|^2 = \frac{1}{4}$ for this state.

$$\left\langle -z,-x\|0,0\right\rangle = \frac{1}{\sqrt{2}}\left\langle -z,-x|+z,-z\right\rangle - \frac{1}{\sqrt{2}}\left\langle -z,-x|-z,+z\right\rangle =$$

$$= \frac{1}{\sqrt{2}}\left\langle -z|+z\right\rangle\left\langle -x|-z\right\rangle - \frac{1}{\sqrt{2}}\left\langle -z|-z\right\rangle\left\langle -x|-z\right\rangle = 0 - \frac{1}{\sqrt{2}}(1)\cos\frac{\theta}{2} = \frac{-1}{\sqrt{2}}\frac{1}{\sqrt{2}} = \frac{-1}{2}$$

So, the probability is $\left|\left\langle -z,-x|0,0\right\rangle\right|^2 = \frac{1}{4}$ for this state.

Opposite signs in opposite axis still amount to ¼ + ¼ = ½ of the time.

All right, let us kick it up just one more notch of difficulty. Consider the two particle SG experiment with different settings but this time make it three separate (coplanar) orientations. (Note: none of these may be the x or z axis so we will give them general vector labels of a, b, and c.)

Well, let us see what our local realist has to say.

Population	Particle 1	Particle 2
N1	{+a, +b, +c}	{-a, -b, -c}
N2	{+a, +b, -c}	{-a, -b, +c}
N3	{+a, -b, +c}	{-a, +b, -c}
N4	{+a, -b, -c}	{-a, +b, +c}
N5	{-a, +b, +c}	{+a, -b, -c}
N6	{-a, +b, -c}	{+a, -b, +c}
N7	{-a, -b, +c}	{+a, +b, -c}
N8	{-a, -b, -c}	{+a, +b, +c}

Consider all of the possible state "bra's" of the two states measured by A and B respectively (representative of states to be projected onto another state in usual ket form):

<+a, +a| <+b, +a| <+c, +a|
<+a, -a| <+b, -a| <+c, -a|
<-a, +a| <-b, +a| <-c, +a|
<-a, -a| <-b, -a| <-c, -a|
<+a, +b| <+b, +b| <+c, +b|
<+a, -b| <+b, -b| <+c, -b|
<-a, +b| <-b, +b| <-c, +b|
<-a, -b| <-b, -b| <-c, -b|
<+a, +c| <+b, +c| <+c, +c|
<+a, -c| <+b, -c| <+c, -c|
<-a, +c| <-b, +c| <-c, +c|
<-a, -c| <-b, -c| <-c, -c|

Now consider those just in different orientations and that also have opposite signs:

X   X   X
X   <+b, -a| <+c, -a|
X   <-b, +a| <-c, +a|
X   X   X
X   X   X
<+a, -b| X   <+c, -b|
<-a, +b| X   <-c, +b|
X   X   X
X   X   X
<+a, -c| <+b, -c| X
<-a, +c| <-b, +c| X
X   X   X

Looking at the remaining states here, we could say (without much thought) that these occur a third of the time for populations N2 through N7. We also could see that if we measure opposite orientations each time that populations N1 and N8 will *always*

yield one of the kets above. So we can say that these states occur *at least a third of the time* (as illustrated by just N2 through N7) and if all of the populations occur the same amount then these states happen 1/3*(3/4) +1*(1/4) = ½ of the time.

Now let us move to the follower of quantum mechanics.

Let us use $|0,0\rangle = \frac{1}{\sqrt{2}}|+a,-a\rangle - \frac{1}{\sqrt{2}}|-a,+a\rangle$ and also that

$|+n\rangle = \cos\frac{\theta}{2}|+z\rangle + e^{i\phi}\sin\frac{\theta}{2}|-z\rangle$ applies generally.

Now let us compute:

$$\langle +a,-b|0,0\rangle = \frac{1}{\sqrt{2}}\langle +a,-b|+a,-a\rangle - \frac{1}{\sqrt{2}}\langle +a,-b|-a,+a\rangle = \frac{1}{\sqrt{2}}\langle +a|+a\rangle\langle -b|-a\rangle - \frac{1}{\sqrt{2}}\langle +a|-a\rangle\langle -b|+a\rangle$$

$$= \frac{1}{\sqrt{2}}\langle -b|-a\rangle$$

where $\langle -b|-a\rangle = \cos\frac{\theta_{ab}}{2}$

So, the probability $\left|\langle +a,-b|0,0\rangle\right|^2 = \frac{1}{2}\cos^2\frac{\theta_{ab}}{2}$. Now

$$\langle -a,+b|0,0\rangle = \frac{1}{\sqrt{2}}\langle -a,+b|+a,-a\rangle - \frac{1}{\sqrt{2}}\langle -a,+b|-a,+a\rangle = \frac{1}{\sqrt{2}}\langle -a|+a\rangle\langle +b|-a\rangle - \frac{1}{\sqrt{2}}\langle -a|-a\rangle\langle +b|+a\rangle$$

$$= -\frac{1}{\sqrt{2}}\langle +b|+a\rangle$$

where $\langle +b|+a\rangle = \cos\frac{\theta_{ab}}{2}$.

So, the probability is $\left|\langle -a,+b|0,0\rangle\right|^2 = \frac{1}{2}\cos^2\frac{\theta_{ab}}{2}$.

Now,

$$\langle +\mathrm{a},-\mathrm{c}|0,0\rangle = \frac{1}{\sqrt{2}}\langle +\mathrm{a},-\mathrm{c}|+\mathrm{a},-\mathrm{a}\rangle - \frac{1}{\sqrt{2}}\langle +\mathrm{a},-\mathrm{c}|-\mathrm{a},+\mathrm{a}\rangle = \frac{1}{\sqrt{2}}\langle +\mathrm{a}|+\mathrm{a}\rangle\langle -\mathrm{c}|-\mathrm{a}\rangle - \frac{1}{\sqrt{2}}\langle +\mathrm{a}|-\mathrm{a}\rangle\langle -\mathrm{c}|+\mathrm{a}\rangle$$

$$= \frac{1}{\sqrt{2}}\langle -\mathrm{c}|-\mathrm{a}\rangle$$

where $\langle -c|-a\rangle = \cos\frac{\theta_{ac}}{2}$. So, the probability is $\left|\langle +a,-c|0,0\rangle\right|^2 = \frac{1}{2}\cos^2\frac{\theta_{ac}}{2}$. Now,

$$\langle -a,+c|0,0\rangle = \frac{1}{\sqrt{2}}\langle -a,+c|+a,-a\rangle - \frac{1}{\sqrt{2}}\langle -a,+c|-a,+a\rangle = \frac{1}{\sqrt{2}}\langle -a|+a\rangle\langle +c|-a\rangle - \frac{1}{\sqrt{2}}\langle -a|-a\rangle\langle +c|+a\rangle$$

$$= -\frac{1}{\sqrt{2}}\langle +c|+a\rangle$$

where $\langle +c|+a\rangle = \cos\frac{\theta_{ac}}{2}$. So, the probability is $\left|\langle -a,+c|0,0\rangle\right|^2 = \frac{1}{2}\cos^2\frac{\theta_{ab}}{2}$.

Now for something slightly different. Here we will consider a different basis vector of the form

$|0,0\rangle = \frac{1}{\sqrt{2}}|+c,-c\rangle - \frac{1}{\sqrt{2}}|-c,+c\rangle$ for simplicity.

Now let us calculate,

$$\langle -b,+c|0,0\rangle = \frac{1}{\sqrt{2}}\langle -b,+c|+c,-c\rangle - \frac{1}{\sqrt{2}}\langle -b,+c|-c,+c\rangle = \frac{1}{\sqrt{2}}\langle -b|+c\rangle\langle +c|-c\rangle - \frac{1}{\sqrt{2}}\langle -b|-c\rangle\langle +c|+c\rangle$$

$$= -\frac{1}{\sqrt{2}}\langle -b|-c\rangle$$

$$= -\frac{1}{\sqrt{2}}\langle -b|-c\rangle$$

where $\langle -b|-c\rangle = \cos\frac{\theta_{bc}}{2}$. So, the probability is $\left|\langle -b,+c|0,0\rangle\right|^2 = \frac{1}{2}\cos^2\frac{\theta_{bc}}{2}$. Similarly,

$$\langle +b,-c|0,0\rangle = \frac{1}{\sqrt{2}}\langle +b,-c|+c,-c\rangle - \frac{1}{\sqrt{2}}\langle +b,-c|-c,+c\rangle = \frac{1}{\sqrt{2}}\langle +b|+c\rangle\langle -c|-c\rangle - \frac{1}{\sqrt{2}}\langle +b|-c\rangle\langle -c|+c\rangle$$

$$= \frac{1}{\sqrt{2}}\langle +b|+c\rangle$$

where $\langle +b|+c\rangle = \cos\frac{\theta_{bc}}{2}$. So, the probability is $\left|\langle +b,-c|0,0\rangle\right|^2 = \frac{1}{2}\cos^2\frac{\theta_{bc}}{2}$.

While we calculated all the probabilities,

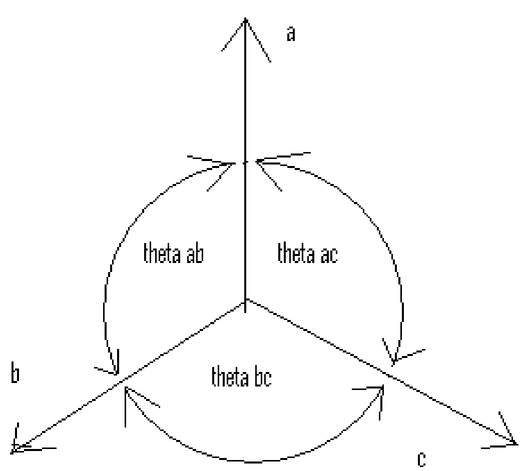
$$\left|\langle -b,+c|0,0\rangle\right|^2, \left|\langle +b,-c|0,0\rangle\right|^2, \left|\langle -a,+c|0,0\rangle\right|^2, \left|\langle +a,-c|0,0\rangle\right|^2, \left|\langle +a,-b|0,0\rangle\right|^2, and \left|\langle -a,+b|0,0\rangle\right|^2$$

we did not compute the probabilities for their reverse states

$$\left|\langle +c,-b|0,0\rangle\right|^2, \left|\langle -c,+b|0,0\rangle\right|^2, \left|\langle +c,-a|0,0\rangle\right|^2, \left|\langle -c,+a|0,0\rangle\right|^2, \left|\langle -b,+a|0,0\rangle\right|^2, and \left|\langle +b,-a|0,0\rangle\right|^2,$$

respectively. It should be noted that the calculations are nearly exactly the same but with a slightly different order. A physicist should note that calculating the probability of particle 1 to be in the |+a> state and particle 2 to be in the |-b> state is nearly identical to calculating the probability of having particle 1 in the |-b> state and particle 2 in the |+a> state. (Or, more simply, deciding which particle is named "particle 1" or "2" is arbitrary.)

Now, with all of these probabilities in the form of some function of theta we must choose some orientation of the vectors a, b, and c. Let us choose the one below:



Here all of the angles depicted are 120 degrees. This makes some exemplary calculations simple as all of the probabilities, which are of the form $\frac{1}{2}\cos^2\frac{\theta}{2}$, yield ¼. As a result, the occurrence of any different setting SG's yielding opposing values is ¼.

But notice . . . this probability is distinctly different from the one from the realist perspective (of "at least one third"). This grounds of difference becomes the playing grounds for experimentation. In the end the real world results will determine which theory is correct. But first let us generalize our quantum mechanical calculations into an inequality that tests infinitely many orientations of a, b, and c. We call this Bell's inequality.

Bell's Inequality

Recalling our prior realist's populations N1 through N8, we can create many inequalities. A prime example may be

$$N3 + N4 \le (N2 + N4) + (N3 + N7)$$

(as two additional populations will certainly yield amounts greater than the previous unless the occurrence of these additional populations is nonexistent – then there is no effect).

We can also look at which populations will create particle one and two ; like

$$\frac{N3 + N4}{\sum N_i} = P(+a;+b)$$

$$\frac{N2 + N4}{\sum N_i} = P(+a;+c)$$

$$\frac{N3 + N7}{\sum N_i} = P(+c;+b)$$

(or in other words the probability of particle 1 and 2 to end up in specific states is equivalent to the sum of the populations that they occur in divided by the total number of populations).

Simply, this leads us down the garden path to the following substitution,

$$P(+a;+b) \le P(+a;+c) + P(+c;+b)$$

These probabilities invite us to take advantage of our knowledge of quantum mechanics for another substitution. So, once again we say let us use

$$|0,0\rangle = \frac{1}{\sqrt{2}}|+a,-a\rangle - \frac{1}{\sqrt{2}}|-a,+a\rangle$$ and also that

$$|+n\rangle = \cos\frac{\theta}{2}|+z\rangle + e^{i\phi}\sin\frac{\theta}{2}|-z\rangle$$ applies generally.

Now let us compute:

$$\langle +a,+b|0,0\rangle = \frac{1}{\sqrt{2}}\langle +a,+b|+a,-a\rangle - \frac{1}{\sqrt{2}}\langle +a,+b|-a,+a\rangle = \frac{1}{\sqrt{2}}\langle +a|+a\rangle\langle +b|-a\rangle - \frac{1}{\sqrt{2}}\langle +a|-a\rangle\langle +b|+a\rangle$$

$$= \frac{1}{\sqrt{2}}\langle +b|-a\rangle$$

where $\langle +b|-a\rangle = \sin\frac{\theta_{ab}}{2}$. So, the probability P(+a;+b) is $\left|\langle +a,-b|0,0\rangle\right|^2 = \frac{1}{2}\sin^2\frac{\theta_{ab}}{2}$.

Also, let's compute P(+a;+c):

$$\langle +\mathrm{a},+\mathrm{c}|0,0\rangle = \frac{1}{\sqrt{2}}\langle +\mathrm{a},+\mathrm{c}|+\mathrm{a},-\mathrm{a}\rangle - \frac{1}{\sqrt{2}}\langle +\mathrm{a},+\mathrm{c}|-\mathrm{a},+\mathrm{a}\rangle = \frac{1}{\sqrt{2}}\langle +\mathrm{a}|+\mathrm{a}\rangle\langle +\mathrm{c}|-\mathrm{a}\rangle - \frac{1}{\sqrt{2}}\langle +\mathrm{a}|-\mathrm{a}\rangle\langle +\mathrm{c}|+\mathrm{a}\rangle$$

$$= \frac{1}{\sqrt{2}}\langle +\mathrm{c}|-\mathrm{a}\rangle$$

where $\langle +c|-a\rangle = \sin\frac{\theta_{ac}}{2}$. So, the probability P(+a;+c) is $\left|\langle +a,+c|0,0\rangle\right|^2 = \frac{1}{2}\sin^2\frac{\theta_{ac}}{2}$.

Finally we will compute P(+c;+b). Once again, we will consider a different basis vector of the form

$$\left|0,0\right\rangle = \frac{1}{\sqrt{2}}\left|+c,-c\right\rangle - \frac{1}{\sqrt{2}}\left|-c,+c\right\rangle,$$ for simplicity.

$$\left\langle +c,+b \middle| 0,0\right\rangle = \frac{1}{\sqrt{2}}\left\langle +c,+b \middle| +c,-c\right\rangle - \frac{1}{\sqrt{2}}\left\langle +c,+b \middle| -c,+c\right\rangle = \frac{1}{\sqrt{2}}\left\langle +c \middle| +c\right\rangle\left\langle +b \middle| -c\right\rangle - \frac{1}{\sqrt{2}}\left\langle +c \middle| -c\right\rangle\left\langle +b \middle| +c\right\rangle$$
$$= \frac{1}{\sqrt{2}}\left\langle +b \middle| -c\right\rangle$$

Where $\left\langle +b \middle| -c\right\rangle = \sin\frac{\theta_{bc}}{2}$.

So, the probability P(+c;+b) is $\left|\left\langle +c,+b \middle| 0,0\right\rangle\right|^2 = \frac{1}{2}\sin^2\frac{\theta_{bc}}{2}$.

Now with these functions of probability we can substitute into $P(+a;+b) \leq P(+a;+c) + P(+c;+b)$ to get

$$\sin^2\frac{\theta_{ab}}{2} \leq \sin^2\frac{\theta_{ac}}{2} + \sin^2\frac{\theta_{cb}}{2}$$

It is this inequality that is recognized as Bell's inequality as it is accredited to John S. Bell in 1964. Notice what this derivation implies. Since we started with an initial assumption (made by the realist and how his/her proposed populations with defined attributes should be related to one another), then any violations of this inequality could be attributed to our initial assumption – that of the realists "hidden-variable theory" of quantum mechanics that created the populations. (Please note that the inequality is now dependent on the angles that are between our three arbitrary vectors. This is valuable since it is testable.)

If we were to wonder at which angles Bell's inequality is violated, we could easily quench our prying curiosity with a simple computer program to step through all possible angles that any three vectors can take with respect to one another. (Please see the attached computer program for an example). The results might be more continuous than one would think (once again see attached).

Experimental Results

But so far as this discussion goes, no proof of whether the realist or the avid determinist is right has been given. Indeed, we did go through some effort just to come up with a disagreement between the two theories. In order to see who is right and who is wrong we turn to experiment. So here we merely do the experiment as we have already outlined (two Stern-Gerlach devices with random orientations and particles from some common origin to conserve momentum).

Enter Aspect et. al. who conclude that there are certainly correlations that violate Bell's inequality from two standard deviations (99% confidence level) and even all the way up to nine standard deviations (nearly 100% confidence level). In the end it is quantum mechanics that comes out on top and regarded as correct with empirical support.

Implications

So, what does it all mean? Well, quite plainly the realists (even with the notable Einstein himself) were wrong. But more even more disgustingly awkward it what this implies. This means that the particle before it is measured really does exist in some sort of superposition of states and afterward chooses a state that we can predict (somehow) using accurate probabilistic methods. (In a different light, this means that particles do not carry around some sort of instruction set or obey the proposed hidden-variable theory of quantum mechanics).

Also, this implies that the particles really do have some way to communicate to each other in order to preserve the conservative laws of momentum for us (the observers). These particles can "communicate" with one another at rates that are faster than the speed of light. So in the end, the E.P.R. paradox is no longer seen as a questionable doubt but an actual fact – both a quality and quandary of our quantum mechanical world.

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