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IDENTIFYING AN m-ARY PARTITION IDENTITY THROUGH AN m-ARY TREE

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Abstract
The Calkin-Wilf tree is well-known as one way to enumerate the rationals, but also may be used to count hyperbinary partitions of an integer, $h_2(n)$. We present an m-ary tree which is a generalization of the Calkin-Wilf tree and show how it may be used to count the hyper m-ary partitions of an integer, $h_m(n)$. We then use properties of the m-ary tree to prove an identity relating values of h_2 to values of h_m, showing that one sequence is a subsequence of the other. Finally, we give a bijection between the partitions to reprove our identity.

1. Introduction

Calkin and Wilf [3] defined the Calkin-Wilf tree to be a binary tree in which each vertex is labeled by a rational number as follows. The root of the tree is labeled by $\frac{1}{1}$ and each vertex $\frac{a}{b}$ has two children. The left child of vertex $\frac{a}{b}$ is $\frac{a}{a+b}$ while the right child is $\frac{a+b}{b}$. The first four levels of this tree are shown in Figure 1.

The Calkin-Wilf sequence of fractions is defined from this tree by reading consecutive levels of the tree from left to right. The sequence starts with

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{1}{4}, \frac{4}{3}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}, \ldots.$$

Calkin and Wilf [3] showed that this sequence of fractions satisfies several nice properties, culminating in the interesting result that every rational number appears exactly once on the tree, thereby giving an enumeration of the positive rational numbers. They also showed that the n^{th} rational number of this sequence is given by $\frac{h_2(n)}{h_2(n+1)}$ for $n \geq 0$, where $h_2(n)$ is the number of ways to write n in hyperbinary,

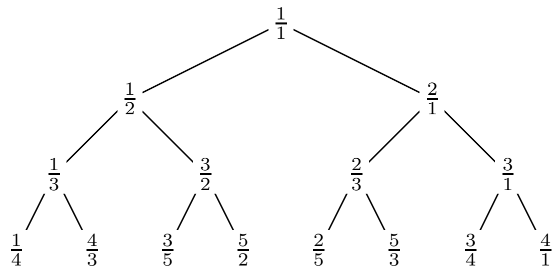

Figure 1: First four levels of Calkin-Wilf tree

that is, the number of ways to write n as the sum of powers of 2, where each power of 2 occurs no more than twice.

Since their article, several authors have further explored the Calkin-Wilf tree and the Calkin-Wilf sequence of fractions. Connections to the Stern-Brocot tree and sequence have been studied [1] as well as other interesting properties of the Calkin-Wilf tree not considered in Calkin and Wilf's original paper. Several generalizations of the tree have been given as well, including q- and (p, q)-versions [2, 5]. In addition, an m-ary version of the Calkin-Wilf tree whose vertices are labeled by rational functions can be found in [6].

In this article, we wish to bring the discussion back to the connection between the Calkin-Wilf tree and the hyperbinary partition function by extending the connection to hyper m-ary partitions. Courtright and Sellers [4] defined $h_m(n)$ to be the number of partitions of n into powers of m, where each power of m occurs no more than m times. Calling these hyper m-ary partitions, they give the following recursive formulas for $h_m(n)$.

$$h_m(mn) = h_m(n) + h_m(n-1) \tag{1}$$

$$h_m(mn+r) = h_m(n), \quad 1 \leq r \leq m-1, \tag{2}$$

with initial condition $h_m(0) = 1$. The hyperbinary sequence discussed in [3] is the hyper 2-ary partition sequence. In their paper, Courtright and Sellers prove some arithmetic properties of the hyperbinary and hyper m-ary partition functions.

In this article we give a generalization of the Calkin-Wilf tree and show that it is related to the hyper m-ary partition sequence defined by Courtwright and Sellers. In exploring this relationship, we will be able to show some interesting results regarding m-ary partitions.

2. Generalizing the Calkin-Wilf Tree

Since Calkin and Wilf first used the Calkin-Wilf tree to "recount" the rationals, several authors have introduced q-versions and (p,q)-versions of the tree in which the vertices are labeled by rational functions. In the next definition, we return to using rational numbers as labels and give an extension to a 3-ary tree.

Definition 1. The *Calkin-Wilf* 3*-ary tree* is a ternary tree in which each vertex is labeled by a rational number. The root of the tree is labeled by $\frac{1}{1}$ and each vertex $\frac{a}{b}$ has 3 children. The first child of $\frac{a}{b}$ is labeled by $\frac{a}{a}$, the middle child is labeled by $\frac{a}{a+b}$ and the last child is labeled by $\frac{a+b}{b}$.

The first four levels of the Calkin-Wilf 3-ary tree are shown in Figure 2.

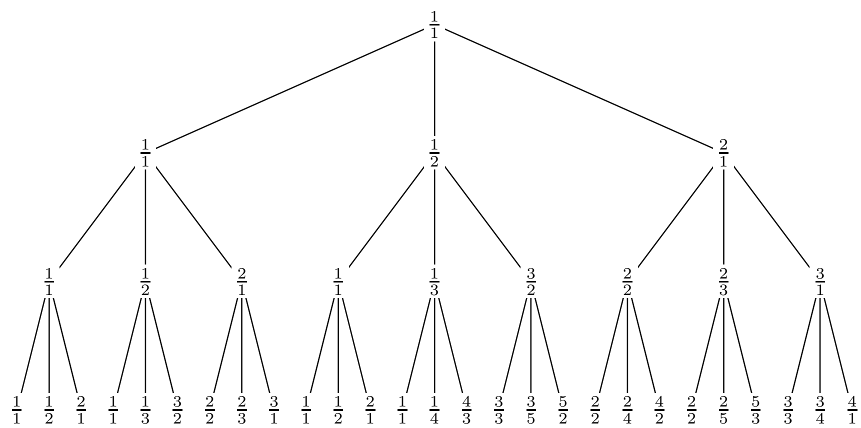

Figure 2: First four levels of the Calkin-Wilf 3-ary tree

Reading the fractions of the tree from left to right on each successive level starting with the top, we have a sequence of fractions that begins as follows:

$$\frac{1}{1},\frac{1}{1},\frac{1}{2},\frac{2}{1},\frac{1}{1},\frac{1}{2},\frac{2}{1},\frac{1}{1},\frac{1}{3},\frac{3}{2},\frac{2}{2},\frac{2}{3},\frac{3}{1},\frac{1}{1},\frac{1}{2},\frac{2}{1},\frac{1}{1},\frac{1}{3},\frac{3}{2},\frac{2}{2},\frac{2}{3},\frac{3}{1},\frac{1}{1},\frac{1}{2},\frac{2}{1},\frac{1}{1},\frac{1}{4},\frac{4}{3},\dots$$

The n^{th} rational in this sequence is also the n^{th} rational in the tree for $n \geq 1$.

We make some observations about the Calkin-Wilf 3-ary tree.

1. By comparing Figure 1 to Figure 2, one can observe that the fractions of the Calkin-Wilf tree also appear within the Calkin-Wilf 3-ary tree. The definition of each tree quickly confirms this observation and further shows that the fractions of the Calkin-Wilf tree appear as second and third children in the 3-ary tree. We will also observe that although the fractions of the Calkin-Wilf tree appear within the second and third children of the 3-ary tree, not

all second and third children of the 3-ary tree correspond to fractions in the Calkin-Wilf tree.

2. Each rational number will occur infinitely many times within the 3-ary tree. Consider the fraction $1/1$ at the root of the tree with children $1/1$ (appearing now for the second time within the tree), $1/2$, and $2/1$. The children of the second appearance of $1/1$ will also be $1/1$, $1/2$, and $2/1$. Not only are the children of the first appearance of $1/1$ also the children of the second appearance of $1/1$, but all the descendants will be the same. In fact, each appearance of $1/1$ will have the same descendants of the first appearance of $1/1$. Since $1/1$ is always the left-most child of $1/1$ by definition, this will happen infinitely many times. Furthermore, since all rational numbers occur exactly once in reduced form in the Calkin-Wilf tree, this implies that each rational number in reduced form will occur infinitely many times in the Calkin-Wilf 3-ary tree.

3. There are infinitely many nonreduced fractions in the 3-ary tree as well. By the definition of the Calkin-Wilf 3-ary tree, we see that the leftmost child of each fraction is not in reduced form except when it is $1/1$. In addition, no children of the fraction a/a will be in reduced form as the numerator and denominator of the children will have a common factor of a.

4. The denominator of one fraction in the sequence is also the numerator of the next fraction in the sequence. This is clear within the children of a particular fraction by the definition of the tree. If the fraction is the rightmost child of another fraction, then we can show that its denominator is the same as the numerator of the next fraction using induction. Assuming this holds for all levels up through the k^{th} level, consider a fraction on the $(k+1)^{st}$ level that is the rightmost child of a fraction on the previous level. Since it is the rightmost child, it has the same denominator of its parent. By our induction hypothesis, the denominator of the parent is the same as the numerator of the next fraction on that level, which is also the same as the numerator of that fraction's leftmost child. Since this fraction follows the first fraction in the sequence, we see that the denominator of the first fraction is the same as the numerator of the next fraction. Finally, the denominator of each fraction along the right edge of the tree is 1. Since the numerators of all fractions along the left edge of the tree are also 1, we see that the denominator of each fraction on the right edge is the same as the numerator of the fraction at the beginning of the next level.

This last observation implies that the n^{th} fraction in the sequence above for $n \geq 1$ is given by $f(n-1)/f(n)$ for some function f. Since the three children of $f(n-1)/f(n)$ are $f(3n-2)/f(3n-1)$, $f(3n-1)/f(3n)$, and $f(3n)/f(3n+1)$, we find

$$\begin{aligned} f(3n-2) &= f(n-1), \\ f(3n-1) &= f(n-1), \\ f(3n) &= f(n-1)+f(n), \end{aligned}$$

for all $n \geq 1$ with $f(0) = 1$. Calkin and Wilf [3] showed that the numerators and denominators of the fractions in the Calkin-Wilf tree correspond to the hyperbinary partition sequence. Similarly, we find that the numerators and denominators of the 3-ary tree correspond to the hyper 3-ary partition sequence, $h_3(n)$ as shown in the proof of Theorem 1 below.

In the following proof, observe that multiplying a number by 3 corresponds to shifting the digits of its ternary expansion to the left one place and adding an additional 0 as the last digit. Conversely, when a number is divisible by three, dividing that number by three has the effect of removing the final digit of 0, thereby shifting the remaining digits to the right.

Theorem 1. *The hyper* 3*-ary sequence* $h_3(n)$ *is the concatenation of the numerators of successive levels of the tree, that is,*

$$f(n) = h_3(n)$$

for all $n \geq 0$*, where* $\frac{f(n-1)}{f(n)}$ *is the* n^{th} *fraction in the tree for* $n \geq 1$*.*

Proof. We will show this by induction on n. Since $f(0) = h_3(0) = 1$, the theorem is true for $n = 0$. Now assume this is true for all integers less than or equal to $3n$, where $n \geq 0$.

Consider $h_3(3n+1)$. Since $3n+1$ is congruent to 1 modulo 3, we know that any hyper 3-ary expansion of $3n+1$ must contain a term with the value 1. Subtracting the one from each of the expansions gives a different hyper 3-ary expansion of $3n$. Dividing the expansion by 3, we obtain a unique hyper 3-ary expansion of n. Conversely, if we multiply a hyper 3-ary expansion of n by 3 and add 1, we will get a unique expansion of $3n+1$. Thus $h_3(3n+1) = h_3(n)$. Then since $h_3(n) = f(n) = f(3n+1)$, we have $h_3(3n+1) = f(3n+1)$.

Now consider $h_3(3n+2)$. Observe that since $3n+2$ is congruent to 2 modulo 3, any hyper 3-ary expansion of $3n+2$ must contain two 1's. Subtracting 2 from an expansion of $3n+2$ and dividing by 3 gives a unique hyper 3-ary expansion of n. Reversing the process, if we multiply a hyper 3-ary expansion of n by 3 and add 2, we obtain an expansion of $3n+2$. Thus $h_3(3n+2) = h_3(n)$ and so $h_3(3n+2) = f(3n+2)$.

Finally, consider $h_3(3n+3)$. Since a hyper 3-ary expansion of $3n+3$ can have either no 1's or three 1's, we must consider two cases. If an expansion of $3n+3$ has three 1's, we subtract 3 to get an expansion of $3n$ that has no 1's, then divide by 3 to get a unique hyper 3-ary partition of n. Reversing this, we can take a hyper 3-ary expansion of n, multiply by 3 and add three 1's to get an expansion of $3n+3$. Thus the number of hyper 3-ary expansions of $3n+3$ that have three 1's is $h_3(n)$. If a hyper 3-ary expansion of $3n+3$ has no 1's, then divide by 3 to obtain an expansion of $n+1$. Conversely, multiplying a hyper 3-ary partition of $n+1$ gives an expansion of $3n+3$. Therefore the number of hyper 3-ary expansions of $3n+3$ that have no 1's is $h_3(n+1)$. Combining the number of hyper 3-ary expansions of $3n+3$ with no 1's with the number of expansions with three 1's, we find that $h_3(3n+3) = h_3(n) + h_3(n+1)$, thus $h_3(3n+3) = f(3n+3)$.

Since $f(n)$ and $h_3(n)$ have the same initial values and the same recurrence formulas, we find that $f(n) = h_3(n)$ for all $n \geq 0$. □

This theorem along with the observation that the Calkin-Wilf tree is a subtree of the 3-ary tree implies that the hyperbinary partition sequence is a subsequence of the hyper 3-ary partition sequence. This statement will be quantified more precisely in Section 4.

3. The Hyper m-ary Partitions Grow on the m-ary Tree

The definitions and ideas of the last section extend nicely to an m-ary tree. We give the natural generalization here.

Definition 2. The *Calkin-Wilf m-ary tree* for $m \geq 3$ is an m-ary tree in which each vertex is labeled by a rational number. The root of the tree is labeled by $\frac{1}{1}$ and each vertex $\frac{a}{b}$ has m children. The first $m-2$ children of $\frac{a}{b}$ are labeled by $\frac{a}{a}$, the $m-1$ child is labeled by $\frac{a}{a+b}$ and the last child is given by $\frac{a+b}{b}$.

For each m-ary tree, we can make a sequence of fractions as we did for the 3-ary tree. Reading the fractions left to right on successive levels of the tree, we will create a sequence of fractions that begins with $\frac{1}{1}$. In a similar fashion we can show that the denominator of each fraction is the same as the numerator of the next consecutive fraction in the list. Thus the n^{th} fraction in the list, for $n \geq 1$, is given by $f(n-1)/f(n)$ for some function f. In the m-ary tree, the m children of the fraction $f(n-1)/f(n)$ are given by $f(mn-m+i)/f(mn-m+i+1)$ for $i = 1, \ldots, m$ and for $n \geq 1$. This gives the following recurrence for $f(n)$, $n \geq 1$.

$$\begin{aligned} f(mn) &= f(n) + f(n-1) \\ f(mn-r) &= f(n-1), \quad 1 \leq r \leq m-1 \end{aligned}$$

with $f(0) = 1$. With a little manipulation, this appears to be the same as the recurrence formula Courtwright and Sellers gave for the hyper m-ary partition function $h_m(n)$ given in equations (1) and (2). In fact, we can show this relationship is true.

Theorem 2. *The hyper m-ary sequence $h_m(n)$ is the concatenation of the numerators of successive levels of the tree, that is,*

$$f(n) = h_m(n)$$

for all $n \geq 0$, where $\frac{f(n-1)}{f(n)}$ is the n^{th} fraction in the m-ary Calkin-Wilf tree for $n \geq 1$.

The proof of this theorem follows the proof of Theorem 1, thus showing the fact that the numerators of the tree correspond to the number of hyper m-ary representations of a number n.

4. An Embedded Calkin-Wilf Tree

In the prior section, we observed that the m-ary tree contains repeated rationals and non-reduced rationals, but that all reduced rationals do appear at least once in the m-ary tree. We are especially interested in the *first* appearance of such fractions in the m-ary sequence of rationals. We observe the following:

1. Begin with the $\frac{1}{1}$ fraction at the root. Its first $m-2$ children are all the $\frac{1}{1}$ fraction repeated, but its $(m-1)^{st}$ child and m^{th} child are $1/2$ and $2/1$, respectively. These are reduced rationals appearing for the first time in the tree and clearly part of the embedded Calkin-Wilf tree. Any other appearance of $\frac{1}{1}$ in the m-ary tree will have the same children repeated each time.

2. Consider the first appearance of an arbitrary reduced rational (other than $\frac{1}{1}$) in the m-ary tree. The $(m-1)^{st}$ and m^{th} children of this rational will also be reduced (by applying known properties of the Calkin-Wilf tree) and will be the first appearance of these reduced rationals (otherwise, the parent is not a first appearance).

3. Let $\frac{a}{b}$ be a reduced rational in the tree which is *not* a first appearance. Each of its m children will be repeats of prior entries in the tree since they will appear as children at the first appearance of $\frac{a}{b}$.

4. Consider a non-reduced rational in the m-ary tree. Clearly, the algorithm for finding each of the m children indicates that none of the children will be reduced.

These four observations allow us to conclude that we have the first appearance of a reduced rational in the m-ary tree if and only if its $(m-1)^{st}$ and m^{th} children are the first appearance of a reduced rational. We summarize this as follows:

Lemma 1. *The Calkin-Wilf 2-ary tree is embedded in the m-ary tree beginning from the root fraction $\frac{1}{1}$. This embedded tree is exactly the subtree of the first appearance of each reduced rational in the m-ary tree. Further, this subtree is made up exactly of rationals which are a $(m-1)^{st}$ or m^{th} child and whose ancestors (other than the root) are all $(m-1)^{st}$ or m^{th} children.*

Next, recall that we may view the rational labels of the m-ary tree as a sequence of rationals where the n^{th} rational in this sequence is the n^{th} rational in the tree, $n \geq 1$. From this, we see that a rational at position number a in the m-ary tree will have m children and these children will be at the following positions:

$$ma - (m-2),\ ma - (m-3),\ \ldots,\ ma - 1,\ ma,\ \text{and}\ ma + 1\ .$$

Consider the m-ary representation (i.e. the base m representation) of these position numbers. This will depend on the value of a, but we may deduce the last digit (the m^0 digit) by viewing the positions modulo m. We observe that the last m-ary digit of every 1^{st} child position must be 2. Similarly, the last m-ary digit of all 2nd children positions is 3, etc.

Following the reasoning above, we conclude that the last m-ary digit of the position numbers of all $(m-1)^{st}$ children and m^{th} children will be 0 and 1, respectively. But we can say more about these two types of children. Write the position number a of a rational in the tree as $a = (t_k t_{k-1} \ldots t_2 t_1 t_0)_m$, then observe (as we did in Section 2) that multiplying by m corresponds to shifting the digits of the m-ary representation one place to the left. Thus, the $(m-1)^{st}$ child of the fraction in position a has position number $ma = (t_k t_{k-1} \ldots t_2 t_1 t_0 0)_m$ and the m^{th} child of the fraction in position a has position number $ma + 1 = (t_k t_{k-1} \ldots t_2 t_1 t_0 1)_m$.

These facts prove the following lemma.

Lemma 2. *Each rational in the m-ary tree has m children. The first $m-2$ of these children always have at least one digit which is neither 0 nor 1 in the m-ary expansion of their position number in the tree. The last two children will contain only the digits 0 or 1 in the m-ary expansion of their position number exactly when the position number of the parent rational has the same property.*

This conclusion extends to an entire subtree of the m-ary tree.

Lemma 3. *Consider the subtree of the m-ary tree made up entirely of rationals which are a $(m-1)^{st}$ child or m^{th} child and whose ancestors (other than the root) are all $(m-1)^{st}$ or m^{th} children. The m-ary representation of the position number of each rational in this subtree can be written using only the digits 0 and 1. The position number of every rational which is not in this subtree will have at least one digit in its m-ary representation which is neither 0 nor 1.*

Proof. The root of the m-ary tree is in position $1 = 1_m$. The root's last two children are in positions $m = 10_m$ and $m + 1 = 11_m$ and all of its prior children have a digit other than 0 or 1. Proceeding to the next level, according to Lemma 2, the last two children of positions m and $m + 1$ will have position numbers with the desired 0-1-digit property. Further, Lemma 2 ensures the property holds throughout the subtree.

Now, suppose there is a rational in the m-ary tree which is not in the subtree described above, but contains only 0 and 1 as digits in the m-ary expansion of its position number. Then, according to Lemma 2, the parent of this rational must have the same property for its position number and must be an $(m-1)^{st}$ or m^{th} child. Applying the Lemma again, we see that the parent's parent must still have the property and must also be an $(m-1)^{st}$ or m^{th} child. We may continue to trace the ancestry back to the root and these characteristics will still hold, meaning the original node chosen must, in fact, live in the desired subtree. □

5. Hyper m-ary Partition Identities

We now apply our observations about the m-ary tree to our facts about partitions from Section 3 to obtain the following theorems.

Theorem 3. *The hyperbinary partition sequence $h_2(n)$ is a subsequence of the hyper m-ary partition sequence $h_m(n)$.*

Proof. Recall that the concatenation of numerators of the sequence of rationals from the m-ary tree is $h_m(n)$. By Lemma 1, the Calkin-Wilf tree is a subtree and thus its sequence of numerators is a subsequence of $h_m(n)$. This is precisely $h_2(n)$, as shown in [3]. □

In fact, we can make the results of Theorem 3 more precise by stating the following identity:

Theorem 4. *Suppose l has binary expansion $l = a_0 2^0 + a_1 2^1 + \cdots + a_r 2^r$, where a_i is either 0 or 1 for all i. Set*

$$k = a_0 m^0 + a_1 m^1 + \cdots + a_r m^r.$$

Then

$$h_m(k) = h_2(l).$$

Proof. To begin, recall that we are numbering the root as position 1 in the tree for the Calkin-Wilf tree as we do for the m-ary tree. Now, we notice a few known features of the Calkin-Wilf tree: in row r, there are 2^{r-1} rationals; the binary expansions of the locations of these rationals in the Calkin-Wilf tree are exactly the 2^{r-1} binary numbers of length r; the first rational on row r is $\frac{1}{r}$, and its position number is $(100\dots0)_2$, with $r-1$ 0's (length r); and, the final rational on row r is $\frac{r}{1}$ with position number $(11\dots1)_2$ with r 1's.

Now, we combine the results of Lemma 1 and Lemma 3 to conclude that on any given row r of the m-ary tree, there are 2^{r-1} first appearances of reduced rationals. These are exactly the 2^{r-1} rationals from row r of the embedded Calkin-Wilf tree and the m-ary representations of these locations in the m-ary tree contain only the digits 0 and 1 (and are the only positions on that row using only 0 and 1 digits).

Because the subtree described in Lemma 1 must be the same subtree as the one described in Lemma 3, we must conclude that the m-ary position numbers in the m-ary tree "match" the binary position numbers in the Calkin-Wilf tree. For example, observe that a vertex having rational label $\frac{1}{r}$ belongs to row r in position $m^{r-1} = (100\dots0)_m$ with $r-1$ 0's, as desired. Similarly, the final entry of row r in the m-ary tree (with rational label $\frac{r}{1}$) is in position number $(11\dots1)_m$ with r 1's. The remaining positions must match as well.

Finally, we apply facts from [3] and from Theorem 2 to map the location of rationals in the respective trees to the corresponding values of h_2 and h_m. The result follows. $\square$

6. A Bijection Between Hyperbinary and Hyper m-ary Partitions

We have seen that the hyperbinary partition sequence is a subsequence of the hyperternary partition sequence and have also shown that the hyperbinary partition sequence is a subsequence of the hyper m-ary partition sequence. In fact, Theorem 4 gives the exact relationship between the sequences. As a consequence, we know that the number of hyperbinary partitions of one integer is exactly the same as the number of hyper m-ary partitions of a corresponding integer. In this section, we give a bijection between these "matching" partitions and thus reprove our prior results.

As in Section 5, let l be an integer expressed in base 2 as $l = (a_r a_{r-1} \dots a_1 a_0)_2$ and let k be an integer expressed in base m as $k = (a_r a_{r-1} \dots a_1 a_0)_m$ where $a_i \in \{0, 1\}$ for all i and must be the same in both expansions. Let B be the set of all hyperbinary partitions of l and let C_m be the set of all hyper m-ary partitions of k. For convenience, we will write hyper m-ary partitions in terms of their coefficients.

For example, $c = c_r m^r + c_{r-1} m^{r-1} + \ldots + c_0 m^0$ is written as $c = c_r c_{r-1} \ldots c_2 c_1 c_0$.

Theorem 5. *Define $g : C_3 \to B$ by setting the image of the hyperternary partition $c = c_r c_{r-1} \ldots c_2 c_1 c_0$ to be the hyperbinary partition $b = b_r b_{r-1} \ldots b_2 b_1 b_0$ according to the following rules: if $c_i = 0$, then $b_i = 0$; if $c_i = 1$, then $b_i = 1$; if $c_i = 2$, then $b_i = 1$; if $c_i = 3$, then $b_i = 2$. Then, g is a bijection.*

Proof. It is clear from the definition that g is a function. So, we first show that g is one-to-one. Suppose $x = x_r x_{r-1} \ldots x_2 x_1 x_0$ and $y = y_r y_{r-1} \ldots y_2 y_1 y_0$ are two hyperternary partitions of k such that $x \neq y$. Then there must be at least one digit, say the j^{th} digit that doesn't match, i.e., $x_j \neq y_j$. Suppose first that x_j and y_j are any two distinct numbers from the set $\{0, 1, 2, 3\}$ except for the pair $\{1, 2\}$. Then the j^{th} digit of $g(x)$ will be different than the j^{th} digit of $g(y)$. Thus $g(x) \neq g(y)$. Now suppose without loss of generality that $x_j = 1$ and $y_j = 2$. If all other digits in x and y are the same, then these two expansions can't represent the same number. So there must be at least one more digit that doesn't match. If this pair of digits is anything other than 1 and 2, we find $g(x) \neq g(y)$ as above. If the pair is a 1 and 2, then we follow the same reasoning as before and find the expansions can't represent the same number, so there must be another pair of digits that are not the same. Continuing this process, we will either find a pair of digits other than 1 and 2 that are unequal or find that all digits that don't match are a 1,2 pair. In the former case, we see that $g(x) \neq g(y)$ as argued above. In the latter case, we find that x and y cannot be partitions of the same number, giving a contradiction. Thus $g(x) \neq g(y)$ and g is one-to-one.

To show that g is onto, let y be a hyperbinary partition in B. Define $x \in C_3$ in the following way. If $y_i = 0$, then set $x_i = 0$. If $y_i = 2$, then set $x_i = 3$. We consider the situation when y_i is 1 in cases. If there is a string of 1's of length q at the end of the hyperbinary expansion y, then we let x contain a string of 1's of length q at the end. If there is a string of 1's of length q followed by a 0 in the expansion, then let the q corresponding digits in x be 1's. Finally, if there is a string of 1's of length q followed by a 2, set the q corresponding digits in x to be 2. Then y is the image of x under g. Thus g is onto. □

Before we consider the m-ary case, we need a quick lemma about the potential structures of the hyper m-ary partitions under consideration here.

Lemma 4. *If the base m representation of an integer n contains only the digits 0 or 1, then there are no hyper m-ary partitions of n which use any of the coefficients $2, 3, \ldots, m-2$.*

Proof. We show the contrapositive. Assume there is a hyper m-ary partition of n which contains at least one of $2, 3, \ldots, m-2$ as a coefficient and call this d. If none of the coefficients are m, then the partition is actually the base m representation of n and the desired result follows. So, assume there is at least one m coefficient in the partition.

Consider the rightmost m in the expansion; call this the i^{th} digit. So, $m \cdot m^i$ is part of the partition of n. If the $(i-1)^{st}$ digit is 0, then $m \cdot m^i$ may be replaced by $(m-1)m^i + m \cdot m^{i-1}$ to get another distinct hyper m-ary partition. But this does not get us closer to the base m representation of n because we still have an m coefficient.

Now, if the $(i+1)^{st}$ digit is not m, we may replace $m \cdot m^i$ by $1 \cdot m^{i+1}$ to get another distinct partition. If the d coefficient is to the right of digit i or to the left of digit $i+1$, then it will remain there in this new partition. Otherwise, the $(i+1)^{st}$ digit is now $d+1$ with $3 \leq d+1 \leq m-1$ (and thus not 0, 1, or m).

Finally, we consider the case when the $(i+1)^{st}$ digit is m. In this case, the expansion will contain a block of w m's, $w \geq 2$, where the rightmost m is in the i^{th} position and the leftmost m is in the $(i+w-1)^{st}$ position. Then we may replace

$$z \cdot m^{i+w} + m \cdot m^{i+w-1} + m \cdot m^{i+w-2} + \ldots + m \cdot m^i$$

where $0 \leq z < m$ by

$$(z+1) \cdot m^{i+w} + 1 \cdot m^{i+w-1} + 1 \cdot m^{i+w-2} + \ldots + 1 \cdot m^{i+1} + 0 \cdot m^i.$$

If $z \neq d$, then d is still a coefficient in the partition, whereas if $z = d$, then the coefficient of m^{i+w} is now $d+1$ with $3 \leq d+1 \leq m-1$ (and thus not 0, 1, or m).

In any of the above cases, we iterate by considering the rightmost m coefficient in the resulting partition of n and continue until the partition in question is actually the base m representation of n. This process will always leave at least one digit which is not 0 or 1, so the desired result follows. □

Now, using the sets B and C_m defined above, we can state the next theorem.

Theorem 6. *Define $g : C_m \to B$ by setting the image of the hyper m-ary partition $c = c_r c_{r-1} \ldots c_2 c_1 c_0$ to be the hyperbinary partition $b = b_r b_{r-1} \ldots b_2 b_1 b_0$ according to the following rules: if $c_i = 0$, then $b_i = 0$; if $c_i = 1$, then $b_i = 1$; if $c_i = m-1$, then $b_i = 1$; if $c_i = m$, then $b_i = 2$. Then, g is a bijection.*

The proof of this theorem is similar to that of Theorem 5 above. Notice that Lemma 4 ensures that all partitions in C_m are handled by the g described here.

The bijection described in this section relates to the results in the previous section via the following corollary.

Corollary 1. *Let l be an integer expressed in base 2 as $l = (a_r a_{r-1} \ldots a_1 a_0)_2$ and let k be an integer expressed in base m as $k = (a_r a_{r-1} \ldots a_1 a_0)_m$ where each a_i is the same in both and must be either 0 or 1. Then $h_2(l) = h_m(k)$.*

Proof. Let B be the set of all hyperbinary partitions of l and let C_m be the set of all hyper m-ary partitions of k. Because $g : C_m \to B$ is a bijection between finite sets, we know the sets must have the same cardinality. Thus, $h_2(l) = h_m(k)$. □

This verifies the results of Theorem 3 and Theorem 4.

7. Conclusion

The functions $h_2(n)$ and $h_m(n)$ count different partitions, so it is initially surprising that they are equal for many corresponding values. Yet we have seen that the hyperbinary partition sequence is in fact a subsequence of the hyper m-ary partition sequence. We first saw this relationship through the m-ary tree, but the bijection in the prior section gives more insight into how the structure of the different partitions (of different integers) are related in a visible way.

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