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Sampling and Interpolation on Some Nilpotent Lie Groups

Vignon Oussa
Bridgewater State University, vignon.oussa@bridgew.edu

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Research Article

Vignon Oussa

Sampling and interpolation on some nilpotent Lie groups

Abstract: Let N be a non-commutative, simply connected, connected, two-step nilpotent Lie group with Lie algebra $\mathfrak{n}$ such that $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{z}$, $[\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{z}$, the algebras $\mathfrak{a}, \mathfrak{b}, \mathfrak{z}$ are abelian, $\mathfrak{a} = \mathbb{R}\text{-span}\{X_1, X_2, \ldots, X_d\}$, and $\mathfrak{b} = \mathbb{R}\text{-span}\{Y_1, Y_2, \ldots, Y_d\}$. Also, we assume that $\det[[X_i, Y_j]]_{1 \le i, j \le d}$ is a non-vanishing homogeneous polynomial in the unknowns $Z_1, \ldots, Z_{n-2d}$ where $\{Z_1, \ldots, Z_{n-2d}\}$ is a basis for the center of the Lie algebra. Using well-known facts from time-frequency analysis, we provide some precise sufficient conditions for the existence of sampling spaces with the interpolation property, with respect to some discrete subset of N. The result obtained in this work can be seen as a direct application of time-frequency analysis to the theory of nilpotent Lie groups. Several explicit examples are computed. This work is a generalization of recent results obtained for the Heisenberg group by Currey and Mayeli in [3].

Keywords: Sampling, interpolation, nilpotent Lie groups, representations

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Vignon Oussa: Department of Mathematics & Computer Science, Bridgewater State University, Bridgewater, MA 02325, USA, e-mail: vignon.oussa@bridgew.edu

1 Introduction

Let N be a locally compact group, and let Γ be a discrete subset of N. Let $\mathbf{H}$ be a left-invariant closed subspace of $L^2(N)$ consisting of continuous functions. We call $\mathbf{H}$ a *sampling space* ([4, Section 2.6]) with respect to Γ (or Γ-sampling space) if the following properties hold.

(1) The restriction mapping $R_\Gamma : \mathbf{H} \to l^2(\Gamma)$, $R_\Gamma f = (f(\gamma))_{\gamma \in \Gamma}$ is an isometry.

(2) There exists a vector $S \in \mathbf{H}$ such that for any vector $f \in \mathbf{H}$, we have the following expansion:

$$f(x) = \sum_{\gamma \in \Gamma} f(\gamma) S(\gamma^{-1} x)$$

with convergence in the norm of $\mathbf{H}$.

The vector S is called a *sinc-type* function, and if R_Γ is surjective, we say that the sampling space $\mathbf{H}$ has the *interpolation property*.

The simplest example of a sampling space with interpolation property over a nilpotent Lie group is provided by the well-known Whittaker, Shannon, Kotel'nikov Theorem (see [4, Example 2.52]) which we recall here. Let $C(\mathbb{R})$ be the vector space of complex-valued continuous functions on the real line, and let

$$\mathbf{H} = \{f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) : \text{supp } \widehat{f} \subseteq [-0.5, 0.5]\},$$

where $f \mapsto \widehat{f}$ is the Fourier transform of f and is defined as $\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} \, dx$ whenever $f \in L^1(\mathbb{R})$. Then $\mathbf{H}$ is a sampling space which has the interpolation property with associated sinc-type function

$$S(x) = \begin{cases} \frac{\sin \pi x}{\pi x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

To the best of our knowledge, the first example of a sampling space with interpolation property on a non-commutative nilpotent Lie group, using the Plancherel transform was defined over the three-dimensional Heisenberg Lie group. This example is due to a remarkable result of Currey and Mayeli [3]. The specific

definition of bandlimited spaces by the Plancherel transform used in [3], was taken from [4, Chapter 6], where a very precise characterization of sampling spaces over the Heisenberg group was provided. Moreover, sampling spaces using a similar definition of bandlimitation were studied in [6] and [7] for a class of nilpotent Lie groups which contains the Heisenberg Lie groups. This class of groups was first introduced by the author in [6]. However, nothing was said about the interpolation property of the sampling spaces described in [6]. In fact, the question of existence of sampling spaces with interpolation property on some non-commutative nilpotent Lie groups is a challenging problem which is the central focus of this paper.

Let N be a simply connected, connected, two-step nilpotent Lie group with Lie algebra $\mathfrak{n}$ of dimension n satisfying the following conditions.

Condition 1.1. Assume that $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{z}$, where $[\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{z}\mathfrak{a}$, $\mathfrak{b}, \mathfrak{z}$ are abelian algebras such that, for $d \geq 1$, $n > 2d$,

$$\mathfrak{a} = \mathbb{R}\text{-span}\{X_1, X_2, \ldots, X_d\},$$
$$\mathfrak{b} = \mathbb{R}\text{-span}\{Y_1, Y_2, \ldots, Y_d\},$$
$$\mathfrak{z} = \mathbb{R}\text{-span}\{Z_1, Z_2, \ldots, Z_{n-2d}\},$$

and

$$\det \begin{bmatrix} [X_1, Y_1] & [X_1, Y_2] & \cdots & [X_1, Y_d] \\ [X_2, Y_1] & [X_2, Y_2] & \cdots & [X_2, Y_d] \\ \vdots & \vdots & \cdots & \vdots \\ [X_d, Y_1] & [X_d, Y_2] & \cdots & [X_d, Y_d] \end{bmatrix} \tag{1.1}$$

is a non-vanishing homogeneous polynomial in the unknowns $Z_1, \ldots, Z_{n-2d}$.

We remark that the entries of the matrix $[[X_i, Y_j]]_{1 \leq i, j \leq d}$ are linear combinations of a basis of the commutator ideal of $[\mathfrak{n}, \mathfrak{n}]$ which can be taken to be a subset of $\{Z_1, Z_2, \ldots, Z_{n-2d}\}$. The object described in (1.1) is then obtained by formally computing the determinant in the unknowns $Z_1, \ldots, Z_{n-2d}$. Also, given a Lie algebra $\mathfrak{n}$ which satisfies all assumptions in Condition 1.1, it is worth mentioning that since we require $n - 2d$ to be positive, we have $\dim \mathfrak{z} = n - 2d \geq 1$ and $\mathfrak{n}$ must necessarily be non-abelian.

One very appealing fact about these groups is the following. The infinite-dimensional irreducible representations of any group satisfying the conditions given above are related to the well-known Schrödinger representations [6, 7]. Thus, the advantage of working with this class of groups is that we are able to exploit well-known theorems from time-frequency analysis.

Let N be a nilpotent Lie group satisfying Condition 1.1. We deal with the existence of left-invariant subspaces of $L^2(N)$ which are sampling spaces which have the interpolation property. More precisely, we investigate conditions under which sampling provides an orthonormal basis which is generated by shifting a single function. The work presented here provides a natural generalization of recent results obtained for the Heisenberg group in [3]. We offer precise and explicit sufficient conditions for sampling spaces, which also have the interpolation property with respect to some discrete set $\Gamma \subset N$.

We organize this paper as follows. The second section deals with some preliminary results which can be found in [2, 6, 7]. In the third section, we introduce a natural notion of bandlimitation for the class of groups considered, and we state the main results (Theorem 3.2 and Theorem 3.3) of the paper. In the fourth section, we prove results related to sampling and frames for the class of groups considered here. The results obtained in the fourth section are crucial for the proofs of Theorem 3.2 and Theorem 3.3 which are provided in the last section. Finally, explicit examples are computed.

2 Preliminaries

Let us start by setting up some notation. In this paper, all representations are strongly continuous and unitary, unless we state otherwise. All sets are measurable, and given two equivalent unitary representations τ and π, we write $\tau \cong \pi$. We also use the same notation for isomorphic Hilbert spaces. The characteristic function of

a set $\mathbf{E}$ is written as $\chi_{\mathbf{E}}$, and the cardinal number of a set $\mathbf{I}$ is denoted by $\mathrm{card}(\mathbf{I})$. Further, V^* stands for the dual vector space of a vector space of V. Let v be a vector in $\mathbb{R}^n$. Then v^{tr} stands for the transpose of the vector v. The Fourier transform of a suitable function f defined over a commutative domain is written as $\hat{f}$, and the conjugate of a complex number z is denoted $\bar{z}$. The general linear group of $\mathbb{R}^n$ is denoted $\mathrm{GL}_n(\mathbb{R})$. Let v, w be two vectors in some Hilbert space. We write $v \perp w$ to denote that the vectors are orthogonal to each other with respect to the inner product which the given Hilbert space is endowed with.

Now, we will provide a short introduction to the theory of direct integrals which is also nicely exposed in [4, Section 3.3]. Let $\{H_\alpha\}_{\alpha \in A}$ be a family of separable Hilbert spaces indexed by a set A. Let μ be a measure defined in A. We define the direct integral of this family of Hilbert spaces with respect to μ as the space which consists of functions f defined on the parameter space A such that $f(\alpha)$ is an element of H_α for each $\alpha \in A$, and

$$\int_A \|f(\alpha)\|^2_{H_\alpha} \, d\mu(\alpha) < \infty$$

with some additional measurability conditions which we will clarify. A family of separable Hilbert spaces $\{H_\alpha\}_{\alpha \in A}$ indexed by a Borel set A is called a field of Hilbert spaces over A. Next, a map

$$f : A \to \bigcup_{\alpha \in A} H_\alpha \quad \text{such that } f(\alpha) \in H_\alpha$$

is called a vector field on A. A measurable field of Hilbert spaces over the indexing set A is a field of Hilbert spaces $\{H_\alpha\}_{\alpha \in A}$ together with a countable set $\{e_j\}_j$ of vector fields such that
(i) the functions $\alpha \mapsto \langle e_j(\alpha), e_k(\alpha) \rangle_{H_\alpha}$ are measurable for all j, k,
(ii) the linear span of $\{e_k(\alpha)\}_k$ is dense in H_α for each $\alpha \in A$.

The direct integral of the spaces H_α with respect to the measure μ is denoted by

$$\int_A^{\oplus} H_\alpha \, d\mu(\alpha)$$

and is the space of measurable vector fields f on A such that

$$\int_A \|f(\alpha)\|^2_{H_\alpha} \, d\mu(\alpha) < \infty.$$

The inner product for this Hilbert space is naturally obtained as follows. For $f, g \in \int_A^{\oplus} H_\alpha \, d\mu(\alpha)$,

$$\langle f, g \rangle = \int_A \langle f(\alpha), g(\alpha) \rangle_{H_\alpha} \, d\mu(\alpha).$$

This theory of direct integrals will play an important role in the definition of bandlimited spaces in our work.

Let N be a non-commutative connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ over the reals with some additional assumptions described in Condition 1.1.

Notice that if $\mathfrak{n}$ is the three-dimensional Heisenberg Lie algebra which is spanned by vectors X, Y, Z such that $[X, Y] = Z$, then we may define

$$\mathfrak{a} = \mathbb{R}X, \quad \mathfrak{b} = \mathbb{R}Y \quad \text{and} \quad \mathfrak{z} = \mathbb{R}Z.$$

Although this is trivial, we make the following observation: $\det[X, Y] = Z$ is a non-vanishing homogeneous polynomial in the variable Z. Therefore, the class of groups satisfying the conditions described above contains groups which can be seen as some generalization of the Heisenberg Lie groups. Let

$$\mathfrak{B} = \{T_1, T_2, \ldots, T_n\}$$

be a basis for the Lie algebra $\mathfrak{n}$. We say that $\mathfrak{B}$ is a *strong Malcev basis* (see [2, p. 10]) if and only if for each $1 \le j \le n$ the real span of $\{T_1, T_2, \ldots, T_j\}$ is an ideal of $\mathfrak{n}$. For the class of groups considered in this work, in order to obtain a strong Malcev basis, it suffices to define $Z_k = T_k$ if $1 \le k \le n - 2d$. Next, if $n - 2d + 1 \le k \le n - d$, then $k = n - 2d + j$ for some $j \in \{1, 2, \ldots, d\}$ and $T_k = Y_j$. Finally, if $n - d + 1 \le k \le n$, then $k = n - d + j$ for $j \in \{1, 2, \ldots, d\}$ and in this case $T_k = X_j$. Fixing such a strong Malcev basis of the Lie algebra $\mathfrak{n}$, a typical

element of the Lie group N is written as follows:

$$\exp\left(\sum_{k=1}^{n-2d} z_k Z_k\right) \exp\left(\sum_{k=1}^{d} y_k Y_k\right) \exp\left(\sum_{k=1}^{d} x_k X_k\right).$$

The subgroup

$$\exp\left(\sum_{k=1}^{n-2d} \mathbb{R} Z_k\right)$$

is the center of the Lie group N and the subgroup

$$\exp\left(\sum_{k=1}^{n-2d} \mathbb{R} Z_k\right) \exp\left(\sum_{k=1}^{d} \mathbb{R} Y_k\right)$$

is a maximal normal abelian subgroup of N. Moreover, N is a step-two nilpotent Lie group since the commutator ideal $[\mathfrak{n}, \mathfrak{n}]$ is central. Let us now collect some additional basic facts about groups satisfying Condition 1.1.

Proposition 2.1. *Let N be a nilpotent Lie group satisfying the conditions given above. There is a finite dimensional faithful representation of N in $\mathrm{GL}(n + 1, \mathbb{R})$ for $n \geq 3$.*

Proof. Clearly if $n < 3$, then $\mathfrak{n}$ must be abelian. Thus, we must assume that $n \geq 3$. First, let $\mathfrak{n}_1 = \mathfrak{a} \oplus \mathfrak{b} \oplus (\mathfrak{z} \ominus [\mathfrak{n}, \mathfrak{n}])$ and $\mathfrak{n}_2 = [\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z}$ such that $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$. Let α be a positive real number. Next, we define an element A_α in the outer derivation of $\mathfrak{n}$ acting by a diagonalizable action such that $[A_\alpha, U] = \ln(\alpha)U$ for all $U \in \mathfrak{n}_1$ and $[A_\alpha, Z] = 2\ln(\alpha)Z$ for all $Z \in \mathfrak{n}_2$. Using the Jacobi identity, it is fairly easy to see that indeed A_α defines a derivation. Next, we consider the linear adjoint representation of $\mathfrak{g} = \mathfrak{n} \oplus \mathbb{R} A_\alpha$, $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and we define $G = \exp(\mathrm{ad}(\mathfrak{g}))$ which is a subgroup of $\mathrm{GL}(\mathfrak{g})$. Fixing a strong Malcev basis for the Lie algebra $\mathfrak{n}$, the adjoint representation of G acting on the vector space $\mathfrak{g}$ is a faithful representation. Thus, $G = \exp(\mathrm{ad}(\mathfrak{g}))$ is a Lie subgroup of $\mathrm{GL}(\mathfrak{g}) \cong \mathrm{GL}(n + 1, \mathbb{R})$. Since N is isomorphic to $\exp(\mathrm{ad}(\mathfrak{n} \oplus \{0\}))$, it follows that $\exp(\mathrm{ad}(\mathfrak{n} \oplus \{0\}))$ is an isomorphic copy of the Lie group N inside $\mathrm{GL}(n + 1, \mathbb{R})$. $\qquad\square$

Next, in order to make this paper self-contained, we will revisit the Plancherel theory for the class of groups considered in this paper. We start by fixing a strong Malcev basis for the Lie algebra of $\mathfrak{n}$. The exponential function takes the Lebesgue measure on $\mathfrak{n}$ to a left Haar measure on N (see [2, Theorem 1.2.10]). Since N is a nilpotent Lie group, according to the orbit method (see [2]) all irreducible representations of N are parametrized by the coadjoint orbit of N in $\mathfrak{n}^*$, and it is possible to construct a smooth cross-section Σ in a Zariski open subset Ω of $\mathfrak{n}^*$ which is dense and N-invariant such that Σ meets every coadjoint orbit in Ω at exactly one point. Let $\mathcal{P}$ be the Plancherel transform on $L^2(N)$ and let $\mathcal{F}$ be the Fourier transform defined on $L^2(N) \cap L^1(N)$ by

$$\mathcal{F}(f)(\lambda) = \int_N f(n)\pi_\lambda(n)\, dn,$$

where $\{\pi_\lambda : \lambda \in \Sigma\}$ parametrizes up to a null set the unitary dual of N. In fact, the set Σ can be chosen such that for each $\lambda \in \Sigma$, the corresponding irreducible representation π_λ is realized as acting in the Hilbert space $L^2(\mathbb{R}^d)$ where d is half of the dimension of the coadjoint orbit of λ. Next, it is well known that

$$\mathcal{P} : L^2(N) \to \int_\Sigma^\oplus L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)\, d\mu(\lambda)$$

such that the Plancherel transform is the extension of the Fourier transform to $L^2(N)$ inducing the equality

$$\|f\|_{L^2(N)}^2 = \int_\Sigma \|\mathcal{P}(f)(\lambda)\|_{\mathcal{HS}}^2\, d\mu(\lambda).$$

We recall that $\|\cdot\|_{\mathcal{HS}}$ denotes the Hilbert–Schmidt norm on $L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$ and that the Hilbert space tensor product $L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$ is defined as the space of bounded linear operators $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ such that

$$\|T\|_{\mathcal{HS}} = \sum_{k \in I} \|Te_k\|_{L^2(\mathbb{R}^d)}^2$$

where $(e_k)_{k \in I}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$. Given arbitrary $S, T \in L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$, the inner product of the operators S and T is

$$\langle S, T \rangle_{\mathcal{HS}} = \sum_{k \in I} \langle Se_k, Te_k \rangle_{L^2(\mathbb{R}^d)}.$$

Also, it is useful to observe that the inner product of arbitrary rank-one operators in $L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R}^d)$ is given by

$$\langle u \otimes v, w \otimes y \rangle_{\mathcal{HS}} = \langle u, w \rangle_{L^2(\mathbb{R}^d)} \langle v, y \rangle_{L^2(\mathbb{R}^d)}.$$

Let λ be a linear functional in $\mathfrak{n}^*$. Put

$$\lambda_k = \lambda(Z_k).$$

Treating the λ_k as unknowns, we define

$$B(\lambda) = \begin{bmatrix} \lambda[X_1, Y_1] & \cdots & \lambda[X_1, Y_d] \\ \vdots & \ddots & \vdots \\ \lambda[X_d, Y_1] & \cdots & \lambda[X_d, Y_d] \end{bmatrix} \tag{2.1}$$

which is a square matrix of order d. The entries in $B(\lambda)$ are linear combinations of the unknowns $\lambda_1, \ldots, \lambda_{n-2d}$. Computing the determinant of the matrix $B(\lambda)$, we obtain a polynomial in the unknowns $\lambda_1, \ldots, \lambda_{n-2d}$. Thus, $\det[[X_i, Y_j]]_{1 \leq i,j \leq d}$ is a non-vanishing homogeneous polynomial in the unknowns $Z_1, \ldots, Z_{n-2d}$ if and only if $\det B(\lambda)$ is non-vanishing homogeneous polynomial in the unknowns $\lambda_1, \ldots, \lambda_{n-2d}$. Therefore, if the assumptions of Conditions 1.1 are met, then for $\lambda \in \mathfrak{n}^*$, $\det(B(\lambda))$ is a non-vanishing polynomials in the unknowns $\lambda_1, \ldots, \lambda_{n-2d}$.

Proposition 2.2. *Let $\mathfrak{n}$ be a Lie algebra over $\mathbb{R}$ satisfying Condition 1.1 and let L be the left regular representation of the group N.*

- *The unitary dual of N is parametrized by the smooth manifold*

$$\Sigma = \{\lambda \in \mathfrak{n}^* : \det(B(\lambda)) \neq 0, \lambda(X_1) = \cdots = \lambda(X_d) = \lambda(Y_1) = \cdots = \lambda(Y_d) = 0\}$$

  which is naturally identified with a Zariski open subset of $\mathfrak{z}^*$.

- *Let $d\lambda$ be the Lebesgue measure on Σ. The Plancherel measure for the group N is supported on Σ and is equal to*

$$d\mu(\lambda) = |\det(B(\lambda))| d\lambda. \tag{2.2}$$

- *The unitary dual of N which we denote by $\widehat{N}$ is up to a null set equal to $\{\pi_\lambda : \lambda \in \Sigma\}$ where each representation π_λ is realized as acting in $L^2(\mathbb{R}^d)$ such that*

$$\pi_\lambda \left(\exp \left(\sum_{i=1}^{n-2d} z_i Z_i \right) \right) f(t) = e^{2\pi i \lambda(\sum_{i=1}^{n-2d} z_i Z_i)} f(t),$$

$$\pi_\lambda \left(\exp \left(\sum_{i=1}^{d} y_i Y_i \right) \right) f(t) = e^{-2\pi i \langle B(\lambda) y, t \rangle} f(t),$$

$$\pi_\lambda \left(\exp \left(\sum_{i=1}^{d} x_i X_i \right) \right) f(t) = f(t - x),$$

  where $y = (y_1, \ldots, y_d)^{\mathrm{tr}}$, and $x = (x_1, \ldots, x_d)$.
- *We have*

$$L \cong \mathcal{P} \circ L \circ \mathcal{P}^{-1} = \int_\Sigma^\oplus \pi_\lambda \otimes \mathbf{1}_{L^2(\mathbb{R}^d)} d\mu(\lambda)$$

  and $\mathbf{1}_{L^2(\mathbb{R}^d)}$ is the identity operator on $L^2(\mathbb{R}^d)$. Moreover for $\lambda \in \Sigma$, we have

$$\mathcal{P}(L(x)\phi)(\lambda) = \pi_\lambda(x) \circ (\mathcal{P}\phi)(\lambda).$$

The results in the proposition above are some facts, which are well known in the theory of harmonic analysis of nilpotent Lie groups. See [6], where we specialized to the class of groups considered here. For general nilpotent Lie groups, we refer the interested reader to [2, Section 4.3] which contains a complete presentation of the Plancherel theory of nilpotent Lie groups.

We will now provide a few examples of Lie groups satisfying Condition 1.1.

Example 2.3. Let N be a nilpotent Lie group with Lie algebra $\mathfrak{n}$ spanned by the strong Malcev basis Z_1, Z_2, Y_1, Y_2, X_1, X_2 with non-trivial Lie brackets

$$[X_1, Y_1] = Z_1, \quad [X_2, Y_1] = -Z_2,$$
$$[X_1, Y_2] = Z_2, \quad [X_2, Y_2] = Z_1.$$

Clearly, N satisfies all properties described in Condition 1.1 and

$$\det([[X_i, Y_j]]_{1 \leq i, j \leq 2}) = \det \begin{bmatrix} Z_1 & Z_2 \\ -Z_2 & Z_1 \end{bmatrix} = Z_1^2 + Z_2^2.$$

Applying Proposition 2.1, we define the monomorphism $\pi : N \to \mathrm{GL}_7(\mathbb{R})$ such that for

$$p = \exp(z_1 Z_1) \exp(z_2 Z_2) \exp(y_1 Y_1) \exp(y_2 Y_2) \exp(x_1 X_1) \exp(x_2 X_2),$$

the image of p under the representation π is the following matrix:

$$\begin{bmatrix}
1 & 0 & x_1 & x_2 & -y_1 & -y_2 & 2z_1 \\
0 & 1 & -x_2 & x_1 & -y_2 & y_1 & 2z_2 \\
0 & 0 & 1 & 0 & 0 & 0 & y_1 \\
0 & 0 & 0 & 1 & 0 & 0 & y_2 \\
0 & 0 & 0 & 0 & 1 & 0 & x_1 \\
0 & 0 & 0 & 0 & 0 & 1 & x_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$

Next, referring to Proposition 2.2, the Plancherel measure is supported on the manifold

$$\Sigma = \{\lambda \in \mathfrak{n}^* : \lambda(Z_1)^2 + \lambda(Z_2)^2 \neq 0, \, \lambda(Y_j) = 0, \, \lambda(X_j) = 0 \text{ for } 1 \leq j \leq 3\}$$

and the Plancherel measure is $|\lambda_1^2 + \lambda_2^2| \, d\lambda_1 d\lambda_2$ where $\lambda_k = \lambda(Z_k)$.

The following example exhausts all elements in the class of groups considered in this paper.

Example 2.4. Fix two natural numbers n and d such that $n - 2d > 0$. Let M be a matrix of order d with entries in $\mathbb{R}Z_1 \oplus \cdots \oplus \mathbb{R}Z_{n-2d}$ such that $\det(M)$ is a non-vanishing homogeneous polynomial in the unknowns $Z_1, Z_2, \ldots, Z_{n-2d}$. Now let $\mathfrak{a} = \mathbb{R}\text{-span}\{X_1, \ldots, X_d\}$ and $\mathfrak{b} = \mathbb{R}\text{-span}\{Y_1, \ldots, Y_d\}$ such that $[X_i, Y_j] = M_{i,j}$ and $M_{i,j}$ is the entry of M located at the intersection of the i-th row and j-th column. The Lie algebra

$$\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b} \oplus (\mathbb{R}Z_1 \oplus \cdots \oplus \mathbb{R}Z_{n-2d})$$

satisfies all properties given in Condition 1.1.

Now, we define

$$\Gamma_\mathfrak{b} = \exp(\mathbb{Z}Y_1 + \cdots + \mathbb{Z}Y_d), \quad \Gamma_\mathfrak{a} = \exp(\mathbb{Z}X_1 + \cdots + \mathbb{Z}X_d), \quad \Gamma_\mathfrak{z} = \exp(\mathbb{Z}Z_1 + \cdots + \mathbb{Z}Z_{n-2d})$$

and

$$\Gamma = \Gamma_\mathfrak{z} \Gamma_\mathfrak{b} \Gamma_\mathfrak{a} \subset N. \tag{2.3}$$

Then Γ is a discrete subset of N which is not generally a subgroup of N.

3 Overview of main results

In this section, we will present an overview of the main results. In order to do so, we will need a few important definitions.

Definition 3.1. We say a function $f \in L^2(N)$ is bandlimited if its Plancherel transform is supported on a bounded measurable subset of Σ. Fix a measurable field of unit vectors $\mathbf{e} = \{\mathbf{e}_\lambda\}_{\lambda \in \Sigma}$ where $\mathbf{e}_\lambda \in L^2(\mathbb{R}^d)$. We say a Hilbert space is a multiplicity-free left-invariant subspace of $L^2(N)$ if

$$\mathbf{H}(\mathbf{e}) = \mathcal{P}^{-1}\left(\int_\Sigma^\oplus L^2(\mathbb{R}^d) \otimes \mathbf{e}_\lambda \, d\mu(\lambda) \right).$$

We observe here that the Hilbert space $\mathcal{P}(\mathbf{H}(\mathbf{e}))$ is naturally identified with $L^2(\Sigma \times \mathbb{R}^d)$. Next, we define

$$\mathbf{E} = \{\lambda \in \mathfrak{z}^* : |\det B(\lambda)| \neq 0 \text{ and } |\det B(\lambda)| \leq 1\}. \tag{3.1}$$

It is easy to see that $\mathbf{E}$ is the intersection of a Zariski open subset of $\mathfrak{z}^*$ and a closed subset of $\mathfrak{z}^*$. Also, $\mathbf{E}$ is not bounded in general and $\mathbf{E}$ is necessarily a set of positive Lebesgue measure on $\mathfrak{z}^*$. In order to develop a theory of bandlimitation, we will need to consider some bounded subset of $\mathbf{E}$.

For any given bounded set $\mathbf{A} \subset \Sigma$, we define the corresponding multiplicity-free, bandlimited, left-invariant Hilbert subspace $\mathbf{H}(\mathbf{e}, \mathbf{A})$ as follows:

$$\mathbf{H}(\mathbf{e}, \mathbf{A}) = \mathcal{P}^{-1}\left(\int_{\mathbf{A}}^\oplus L^2(\mathbb{R}^d) \otimes \mathbf{e}_\lambda |\det B(\lambda)| d\lambda \right). \tag{3.2}$$

To be more precise, for any $\phi \in \mathbf{H}(\mathbf{e}, \mathbf{A})$, there exists a measurable field of vectors $\{\mathbf{w}_\lambda^\phi\}_{\lambda \in \mathbf{A}}$, $\mathbf{w}_\lambda^\phi \in L^2(\mathbb{R}^d)$, such that

$$\mathcal{P}\phi(\lambda) = \begin{cases} \mathbf{w}_\lambda^\phi \otimes \mathbf{e}_\lambda & \text{if } \lambda \in \mathbf{A}, \\ 0 & \text{if } \lambda \notin \mathbf{A}. \end{cases}$$

Let $\phi \in \mathbf{H}(\mathbf{e}, \mathbf{A})$ and define the linear map $W_\phi : \mathbf{H}(\mathbf{e}, \mathbf{A}) \to L^2(N)$ such that $W_\phi \psi(x) = \langle \psi, L(x)\phi \rangle$. It is easy to see that the space $W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{A}))$ is a subspace of $L^2(N)$ which consists of continuous functions.

Let $\iota : \mathbb{R}^{n-2d} \to \mathfrak{z}^*$ be a map defined by

$$\iota(\lambda_1, \ldots, \lambda_{n-2d}) = \sum_{k=1}^{n-2d} \lambda_k Z_k^*$$

where $\{Z_k^* : 1 \leq k \leq n - 2d\}$ is the dual basis of $\mathfrak{z}^*$ which is associated to $\{Z_k : 1 \leq k \leq n - 2d\}$ which is a fixed basis for the central ideal $\mathfrak{z}$. Clearly, ι is a measurable bijection. Identifying $\mathbb{R}^{n-2d}$ with $\mathfrak{z}^*$ via the map ι, we slightly abuse notation when we say $\mathfrak{z}^* = \mathbb{R}^{n-2d}$. In order to make a simpler presentation, we will adopt this abuse of notation for the remainder of the paper. Now, let $\mathbf{C} \subset \mathfrak{z}^* = \mathbb{R}^{n-2d}$ be a bounded set such that

$$\{e^{2\pi i \langle k, \lambda \rangle} \chi_{\mathbf{C}}(\lambda) : k \in \mathbb{Z}^{n-2d}\}$$

is a Parseval frame for $L^2(\mathbf{C}, d\lambda)$. For example, it suffices to pick $\mathbf{C} \subseteq \mathbf{I}$ such that the collection $\{\mathbf{I} + k : k \in \mathbb{Z}^{n-2d}\}$ forms a measurable partition of $\mathbb{R}^{n-2d} = \mathfrak{z}^*$. Our main results are summarized as follows.

Theorem 3.2. *Let N be a connected, simply connected nilpotent Lie group satisfying Condition 1.1. Then there exists some $\phi \in \mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$ such that $W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}))$ is a Γ-sampling subspace of $L^2(N)$ with sinc-type function $W_\phi(\phi)$. Moreover, $W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}))$ does not generally have the interpolation property with respect to Γ.*

Now, let N be a nilpotent Lie group which satisfies all properties described in Condition 1.1 such that additionally, there exist a strong Malcev basis for the Lie algebra $\mathfrak{n}$, and a compact subset $\mathbf{R}$ of $\mathfrak{z}^*$ such that for $\mathbf{E}^\circ = \mathbf{E} \cap \mathbf{R}$,

$$\int_{\mathbf{E}^\circ} |\det B(\lambda)| \, d\lambda = 1.$$

Since $\mathbf{E}^\circ$ is a subset of $\mathbf{E}$, for each $\lambda \in \mathbf{E}^\circ$ there exists (see Remark 4.5) a corresponding set $E(\lambda)$ which tiles $\mathbb{R}^d$ by $\mathbb{Z}^d$ and packs $\mathbb{R}^d$ by $B(\lambda)^{-\mathrm{tr}}\mathbb{Z}^d$. Fix a fundamental domain Λ for the lattice $\mathbb{Z}^{n-2d}$ such that

$$\mathbf{E}^\circ = \bigcup_{k_j \in S} ((\Lambda - \kappa_j) \cap \mathbf{E}^\circ)$$

where S is a finite subset of $\mathbb{Z}^{n-2d}$ and each $(\Lambda - \kappa_j) \cap \mathbf{E}^\circ$ is a set of positive Lebesgue measure on $\mathbb{R}^{n-2d}$. For $\lambda \in \Sigma$, we define the map $\lambda \mapsto \mathbf{u}_\lambda$ on Σ such that

$$\mathbf{u}_\lambda = \begin{cases} |\det B(\lambda)|^{1/2} \chi_{E(\lambda)} & \text{if } \lambda \in \mathbf{E}^\circ, \\ 0 & \text{if } \lambda \notin \mathbf{E}^\circ, \end{cases} \tag{3.3}$$

and $\phi \in L^2(N)$ such that

$$\mathcal{P}\phi(\lambda) = \frac{\mathbf{u}_\lambda \otimes \mathbf{e}_\lambda}{\sqrt{|\det B(\lambda)|}}. \tag{3.4}$$

Theorem 3.3. *If for each $1 \le j, j' \le \mathrm{card}(S)$, $j \neq j'$, for $\lambda \in \Lambda$, for arbitrary functions $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d)$, and for distinct $\kappa_j, \kappa_{j'} \in S$ one has*

$$(\langle \mathbf{f}, \pi_{\lambda - \kappa_j}(\gamma_1) \mathbf{u}_{\lambda - \kappa_j} \rangle)_{\gamma_1 \in \Gamma_b \Gamma_a} \perp (\langle \mathbf{g}, \pi_{\lambda - \kappa_{j'}}(\gamma_1) \mathbf{u}_{\lambda - \kappa_{j'}} \rangle)_{\gamma_1 \in \Gamma_b \Gamma_a},$$

then $W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ))$ is a Γ-sampling space with sinc-type function $W_\phi(\phi)$. Moreover, $W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ))$ has the interpolation property.

The proofs of Theorem 3.2 and Theorem 3.3 will be given in the last section of this paper.

4 Results on frames and orthonormal bases

We will need to be familiar with the theory of frames (see [1, 5, 8]). Given a countable sequence $\{f_i\}_{i \in I}$ of vectors in a Hilbert space $\mathbf{H}$, we say $\{f_i\}_{i \in I}$ forms a *frame* if and only if there exist strictly positive real numbers A, B such that for any vector $f \in \mathbf{H}$,

$$A\|f\|^2 \le \sum_{i \in I} |\langle f, f_i \rangle|^2 \le B\|f\|^2.$$

In the case where $A = B$, the sequence of vectors $\{f_i\}_{i \in I}$ forms what is called a *tight frame*, and if $A = B = 1$, $\{f_i\}_{i \in I}$ is called a *Parseval frame* because it satisfies the Parseval equality

$$\sum_{i \in I} |\langle f, f_i \rangle|^2 = \|f\|^2 \quad \text{for all } f \in \mathbf{H}.$$

Also, if $\{f_i\}_{i \in I}$ is a Parseval frame such that $\|f_i\| = 1$ for all $i \in I$, then $\{f_i\}_{i \in I}$ is an orthonormal basis for $\mathbf{H}$.

Let $\Xi = A\mathbb{Z}^{2d}$ for some matrix A. We say Ξ is a *full-rank lattice* if A is non-singular. We say a lattice is *separable* if $\Xi = A\mathbb{Z}^d \times B\mathbb{Z}^d$. A *fundamental domain* D for a lattice in $\mathbb{R}^d$ is a measurable set which satisfies the following: $(D + l) \cap (D + l') \neq \emptyset$ for distinct l, l' in Ξ, and $\mathbb{R}^d = \bigcup_{l \in \Xi}(D + l)$. We say D is a *packing set* for Ξ if $(D + l) \cap (D + l')$ has Lebesgue measure zero for any $l \neq l'$. Let $\Xi = A\mathbb{Z}^d \times B\mathbb{Z}^d$ be a full-rank lattice in $\mathbb{R}^{2d}$ and $f \in L^2(\mathbb{R}^d)$. The family of functions in $L^2(\mathbb{R}^d)$

$$\mathcal{G}(f, A\mathbb{Z}^d \times B\mathbb{Z}^d) = \{e^{2\pi i \langle k, x \rangle} f(x - n) : k \in B\mathbb{Z}^d, n \in A\mathbb{Z}^d\} \tag{4.1}$$

is called a *Gabor system*. Gabor frames are a particular type of frame whose elements are generated by time-frequency shifts of a single vector. A Gabor system which is a Parseval frame is called a *Gabor Parseval frame*. Let r be a natural number. Let $\Xi = A\mathbb{Z}^r$ be a full-rank lattice in $\mathbb{R}^r$. The *volume* of Ξ is defined as

$$\mathrm{vol}(\Xi) = |\det A|,$$

and the *density* of the lattice Ξ is defined as $d(\Xi) = |\det A|^{-1}$.

Lemma 4.1 (Density condition). *Let $\Xi = A\mathbb{Z}^d \times B\mathbb{Z}^d$ be a full-rank lattice in $\mathbb{R}^{2d}$. There exists $f \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(f, A\mathbb{Z}^d \times B\mathbb{Z}^d)$ is a Parseval frame in $L^2(\mathbb{R}^d)$ if and only if $\mathrm{vol}(\Xi) = |\det A \det B| \le 1$.*

A proof of Lemma 4.1 is given in [5, Theorem 3.3].

Lemma 4.2. *Let Ξ be a full-rank lattice in $\mathbb{R}^{2d}$. There exists $f \in L^2(\mathbb{R}^d)$ such that $\mathcal{G}(f, \Xi)$ is an orthonormal basis if and only if $\mathrm{vol}(\Xi) = 1$. Also, if $\mathcal{G}(f, \Xi)$ is a Parseval frame for $L^2(\mathbb{R}^d)$, then $\|f\|^2 = \mathrm{vol}(\Xi)$.*

Lemma 4.2 is due to Theorem 1.3 and the proof of Lemma 3.2 is given in [5]. Next, from the definition of the irreducible representations of N, provided in Proposition 2.2, it is easy to see that for $f \in L^2(\mathbb{R}^d)$, $\pi_\lambda(\Gamma_b \Gamma_a) f$ is a Gabor system for each fixed $\lambda \in \Sigma$. Moreover, following the notation given in (4.1), we write

$$\pi_\lambda(\Gamma_b \Gamma_a) f = \mathcal{G}(f, \mathbb{Z}^d \times B(\lambda) \mathbb{Z}^d).$$

We recall that the set $\mathbf{C}$ satisfies the following conditions: $\mathbf{C} \subset \mathfrak{z}^* = \mathbb{R}^{n-2d}$ is a bounded set such that the system

$$\{e^{2\pi i \langle k, \lambda \rangle} \chi_{\mathbf{C}}(\lambda) : k \in \mathbb{Z}^{n-2d}\}$$

is a Parseval frame for $L^2(\mathbf{C}, d\lambda)$. Also, we recall that

$$\mathbf{E} = \{\lambda \in \mathfrak{z}^* : |\det B(\lambda)| \neq 0 \text{ and } |\det B(\lambda)| \leq 1\}.$$

Therefore, $\{e^{2\pi i \langle k, \lambda \rangle} \chi_{\mathbf{E} \cap \mathbf{C}}(\lambda) : k \in \mathbb{Z}^{n-2d}\}$ is a Parseval frame for the Hilbert space $L^2(\mathbf{E} \cap \mathbf{C}, d\lambda)$.

Lemma 4.3. *There exists a function $\phi \in \mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$ such that $L(\Gamma)\phi$ is a Parseval frame in $\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$.*

Proof. We know that by the Density Condition (see Lemma 4.1), for $\lambda \in \mathbf{E} \cap \mathbf{C}$, there exists a rank-one operator $T_\lambda = \mathbf{u}_\lambda \otimes \mathbf{e}_\lambda$ such that

$$\mathcal{P}\phi(\lambda) = \frac{T_\lambda}{\sqrt{|\det B(\lambda)|}}, \tag{4.2}$$

and the system $\mathcal{G}(\mathbf{u}_\lambda, \mathbb{Z}^d \times B(\lambda)\mathbb{Z}^d)$ is a Gabor Parseval frame in $L^2(\mathbb{R}^d)$. Next, given any vector $\psi \in \mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$, we obtain that

$$\sum_{\gamma \in \Gamma} |\langle \psi, L(\gamma)\phi \rangle_{\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})}|^2 = \sum_{\gamma \in \Gamma} \left| \int_{\mathbf{E} \cap \mathbf{C}} \langle \mathcal{P}\psi(\lambda), \pi_\lambda(\gamma) \circ \mathcal{P}\phi(\lambda) \rangle_{\mathcal{HS}} \, d\mu(\lambda) \right|^2$$

$$= \sum_{\gamma \in \Gamma} \left| \int_{\mathbf{E} \cap \mathbf{C}} \langle \mathcal{P}\psi(\lambda), \pi_\lambda(\gamma)(|\det B(\lambda)|^{-1/2} \mathbf{u}_\lambda) \otimes \mathbf{e}_\lambda \rangle_{\mathcal{HS}} |\det B(\lambda)| \, d\lambda \right|^2$$

$$= \sum_{\gamma \in \Gamma} \left| \int_{\mathbf{E} \cap \mathbf{C}} \langle \mathcal{P}\psi(\lambda), \pi_\lambda(\gamma) \mathbf{u}_\lambda \otimes \mathbf{e}_\lambda \rangle_{\mathcal{HS}} |\det B(\lambda)|^{1/2} \, d\lambda \right|^2. \tag{4.3}$$

Using the fact that $\{e^{2\pi i \langle k, \lambda \rangle} \chi_{\mathbf{E} \cap \mathbf{C}}(\lambda) : k \in \mathbb{Z}^{n-2d}\}$ is a Parseval frame for $L^2(\mathbf{E} \cap \mathbf{C}, d\lambda)$, and letting

$$\mathbf{f}(\lambda) = \langle \mathcal{P}\psi(\lambda), \pi_\lambda(\gamma_1) \mathbf{u}_\lambda \otimes \mathbf{e}_\lambda \rangle_{\mathcal{HS}} |\det B(\lambda)|^{1/2}, \tag{4.4}$$

we obtain

$$\sum_{\gamma \in \Gamma} |\langle \psi, L(\gamma)\phi \rangle_{\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})}|^2 = \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \sum_{m \in \mathbb{Z}^{n-2d}} \left| \int_{\mathbf{E} \cap \mathbf{C}} e^{-2\pi i \langle \lambda, m \rangle} \mathbf{f}(\lambda) \right|^2 d\lambda$$

$$= \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \sum_{m \in \mathbb{Z}^{n-2d}} |\hat{\mathbf{f}}(m)|^2 \, d\lambda$$

$$= \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \|\hat{\mathbf{f}}\|^2_{l^2(\mathbb{Z}^{n-2d})}$$

$$= \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \|\mathbf{f}\|^2_{L^2(\mathbf{E} \cap \mathbf{C}, d\lambda)}.$$

The last equality above is due to the Plancherel Theorem on $l^2(\mathbb{Z}^{n-2d})$. Using (4.4), letting $\mathcal{P}\psi(\lambda) = \mathbf{w}_\lambda^\psi \otimes \mathbf{e}_\lambda$, where $\mathbf{w}_\lambda^\psi = \mathbf{w}_\lambda \in L^2(\mathbb{R}^d)$, and coming back to (4.3), it follows that

$$\sum_{\gamma \in \Gamma} |\langle \psi, L(\gamma)\phi \rangle_{\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})}|^2 = \sum_{\gamma \in \Gamma} \left| \int_{\mathbf{E} \cap \mathbf{C}} \langle \mathcal{P}\psi(\lambda), \pi_\lambda(\gamma) \mathbf{u}_\lambda \otimes \mathbf{e}_\lambda \rangle_{\mathcal{HS}} |\det B(\lambda)|^{1/2} \, d\lambda \right|^2$$

$$= \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \int_{\mathbf{E} \cap \mathbf{C}} |\langle \mathbf{w}_\lambda \otimes \mathbf{e}_\lambda, \pi_\lambda(\gamma_1) \mathbf{u}_\lambda \otimes \mathbf{e}_\lambda \rangle_{\mathcal{HS}}|^2 |\det B(\lambda)| \, d\lambda$$

$$= \int_{\mathbf{E} \cap \mathbf{C}} \sum_{\gamma_1 \in \Gamma_b \Gamma_a} |\langle \mathbf{w}_\lambda \otimes \mathbf{e}_\lambda, \pi_\lambda(\gamma_1) \mathbf{u}_\lambda \otimes \mathbf{e}_\lambda \rangle_{\mathcal{HS}}|^2 |\det B(\lambda)| \, d\lambda$$

$$= \int_{\mathbf{E} \cap \mathbf{C}} \sum_{\gamma_1 \in \Gamma_b \Gamma_a} |\langle \mathbf{w}_\lambda, \pi_\lambda(\gamma_1) \mathbf{u}_\lambda \rangle_{L^2(\mathbb{R}^d)}|^2 |\det B(\lambda)| \, d\lambda.$$

Since $\pi_\lambda(\Gamma_\mathfrak{b}\Gamma_\mathfrak{a})\mathbf{u}_\lambda$ is a Parseval Gabor frame for each fixed $\lambda \in \mathbf{E} \cap \mathbf{C}$,

$$\sum_{\gamma_1 \in \Gamma_\mathfrak{b}\Gamma_\mathfrak{a}} |\langle \mathbf{w}_\lambda, \pi_\lambda(\gamma_1)\mathbf{u}_\lambda\rangle_{L^2(\mathbb{R}^d)}|^2 = \|\mathbf{w}_\lambda\|^2_{L^2(\mathbb{R}^d)},$$

we have

$$\sum_{\gamma \in \Gamma} |\langle \psi, L(\gamma)\phi\rangle_{\mathbf{H}(\mathbf{e},\mathbf{E}\cap\mathbf{C})}|^2 = \int\limits_{\mathbf{E}\cap\mathbf{C}} \|\mathbf{w}_\lambda\|^2_{L^2(\mathbb{R}^d)} |\det B(\lambda)| \, d\lambda = \int\limits_{\mathbf{E}\cap\mathbf{C}} \|\mathcal{P}\psi(\lambda)\|^2_{\mathcal{HS}} |\det B(\lambda)| \, d\lambda = \|\psi\|^2_{\mathbf{H}(\mathbf{e},\mathbf{E}\cap\mathbf{C})}.$$

Finally, we obtain that $L(\Gamma)\phi$ is a Parseval frame in $\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$. □

Lemma 4.4. *If $L(\Gamma)\phi$ is a Parseval frame in $\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$ as described in Lemma 4.3 and if*

$$\int\limits_{\mathbf{E}\cap\mathbf{C}} |\det B(\lambda)| \, d\lambda = 1,$$

then $L(\Gamma)\phi$ is an orthonormal basis.

Proof. Recall from Lemma 4.3 that for $\lambda \in \mathbf{E} \cap \mathbf{C}$,

$$\mathcal{P}\phi(\lambda) = |\det B(\lambda)|^{-1/2}\mathbf{u}_\lambda \otimes \mathbf{e}_\lambda$$

such that $\mathcal{G}(\mathbf{u}_\lambda, \mathbb{Z}^d \times B(\lambda)\mathbb{Z}^d)$ is a Gabor Parseval frame in $L^2(\mathbb{R}^d)$. Referring to [5, proof of Theorem 1.3], we have $\|\mathbf{u}_\lambda\|^2_{L^2(\mathbb{R}^d)} = |\det B(\lambda)|$ for $\lambda \in \mathbf{E} \cap \mathbf{C}$. Now,

$$\begin{aligned}
\|\phi\|^2_{\mathbf{H}(\mathbf{e},\mathbf{E}\cap\mathbf{C})} &= \int\limits_{\mathbf{E}\cap\mathbf{C}} \|\mathcal{P}\phi(\lambda)\|^2_{\mathcal{HS}} |\det B(\lambda)| \, d\lambda \\
&= \int\limits_{\mathbf{E}\cap\mathbf{C}} \||\det B(\lambda)|^{-1/2}\mathbf{u}_\lambda \otimes \mathbf{e}_\lambda\|^2_{\mathcal{HS}} |\det B(\lambda)| \, d\lambda \\
&= \int\limits_{\mathbf{E}\cap\mathbf{C}} \|\mathbf{u}_\lambda \otimes \mathbf{e}_\lambda\|^2_{\mathcal{HS}} \, d\lambda \\
&= \int\limits_{\mathbf{E}\cap\mathbf{C}} \|\mathbf{u}_\lambda\|^2_{L^2(\mathbb{R}^d)} \, d\lambda \\
&= \int\limits_{\mathbf{E}\cap\mathbf{C}} |\det B(\lambda)| \, d\lambda \\
&= \mu(\mathbf{E} \cap \mathbf{C}) = 1.
\end{aligned}$$

Since any unit-norm Parseval frame is an orthonormal basis, the proof is completed. □

Remark 4.5. Note that [8, Theorem 3.3] guarantees that for each $\lambda \in \mathbf{E} \cap \mathbf{C}$, it is possible to pick

$$\mathbf{u}_\lambda = |\det B(\lambda)|^{1/2}\chi_{E(\lambda)}$$

such that $E(\lambda)$ tiles $\mathbb{R}^d$ by $\mathbb{Z}^d$ and packs $\mathbb{R}^d$ by $B(\lambda)^{-\mathrm{tr}}\mathbb{Z}^d$ and $\mathcal{G}(\mathbf{u}_\lambda, \mathbb{Z}^d \times B(\lambda)\mathbb{Z}^d)$ is a Gabor Parseval frame in $L^2(\mathbb{R}^d)$.

5 Proof of results and examples

Recall that $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{z}$, $[\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{z}$, $\mathfrak{a}, \mathfrak{b}, \mathfrak{z}$ are abelian algebras, $\dim_{\mathbb{R}}(\mathfrak{a}) = \dim_{\mathbb{R}}(\mathfrak{b}) = d$, and $\det([[X_i, Y_j]]_{1\le i,j\le d})$ is a non-vanishing polynomial in the unknowns $Z_1, \ldots, Z_{n-2d}$. Also, we recall that

$$B(\lambda) = \begin{bmatrix} \lambda[X_1, Y_1] & \cdots & \lambda[X_1, Y_d] \\ \vdots & \ddots & \vdots \\ \lambda[X_d, Y_1] & \cdots & \lambda[X_d, Y_d] \end{bmatrix}$$

and

$$d\mu(\lambda) = |\det B(\lambda)| \, d\lambda.$$

Moreover, the unitary dual of N is parametrized by the smooth manifold

$$\Sigma = \{\lambda \in \mathfrak{n}^* : \det(B(\lambda)) \neq 0, \lambda(X_1) = \cdots = \lambda(X_d) = \lambda(Y_1) = \cdots = \lambda(Y_d) = 0\}$$

which is naturally identified with a Zariski open subset of $\mathfrak{z}^*$.

Definition 5.1. Let $(\pi, \mathbf{H}_\pi)$ denote a strongly continuous unitary representation of a locally compact group G. We say that the representation $(\pi, \mathbf{H}_\pi)$ is admissible if and only if the map $W_\phi : \mathbf{H} \to L^2(G)$, $W_\phi \psi(x) = \langle \psi, \pi(x)\phi \rangle$ defines an isometry of $\mathbf{H}$ into $L^2(G)$, and we say that ϕ is an admissible vector or a continuous wavelet.

Proposition 5.2. *Let Γ be a discrete subset of G. Let ϕ be an admissible vector for $(\pi, \mathbf{H}_\pi)$ such that $\pi(\Gamma)\phi$ is a Parseval frame for $\mathbf{H}_\pi$. Then $\mathbf{K} = W_\phi(\mathbf{H}_\pi)$ is a Γ-sampling space, and $W_\phi(\phi)$ is the associated sinc-type function for $\mathbf{K}$.*

See [4, Proposition 2.54].

5.1 Proof of Theorem 3.2

Since $\mathbf{C} \subset \mathfrak{z}^* = \mathbb{R}^{n-2d}$ is a bounded set such that the system

$$\{e^{2\pi i\langle k, \lambda\rangle} \chi_{\mathbf{C}}(\lambda) : k \in \mathbb{Z}^{n-2d}\}$$

is a Parseval frame for $L^2(\mathbf{C}, d\lambda)$ and

$$\{e^{2\pi i\langle k, \lambda\rangle} \chi_{\mathbf{E} \cap \mathbf{C}}(\lambda) : k \in \mathbb{Z}^{n-2d}\}$$

is a Parseval frame for the Hilbert space $L^2(\mathbf{E} \cap \mathbf{C}, d\lambda)$, it follows that, according to Lemma 4.3, there exists a function $\phi \in \mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$ such that $L(\Gamma)\phi$ is a Parseval frame in $\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C})$. In fact, for $\lambda \in \mathbf{E} \cap \mathbf{C}$, $T_\lambda = \mathbf{u}_\lambda \otimes \mathbf{e}_\lambda$, we define ϕ such that

$$\mathcal{P}\phi(\lambda) = \frac{T_\lambda}{\sqrt{|\det B(\lambda)|}}, \tag{5.1}$$

and the system $\mathcal{G}(\mathbf{u}_\lambda, \mathbb{Z}^d \times B(\lambda)\mathbb{Z}^d)$ is a Gabor Parseval frame in $L^2(\mathbb{R}^d)$. Next,

$$\|\mathcal{P}(\phi)(\lambda)\|^2_{\mathcal{HS}} = \left\|\frac{\mathbf{u}_\lambda \otimes \mathbf{e}_\lambda}{\sqrt{|\det B(\lambda)|}}\right\|^2_{\mathcal{HS}} = |\det B(\lambda)|^{-1}\|\mathbf{u}_\lambda\|^2_{L^2(\mathbb{R}^d)} = 1.$$

Since N is unimodular, and since $\mu(\mathbf{E} \cap \mathbf{C}) < \infty$, it follows from [4, p. 127] that $(L, \mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}))$ is an admissible representation of N. Moreover, ϕ is an admissible vector for the representation $(L, \mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}))$. Now, appealing to Proposition 5.2, then $\mathbf{K} = W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}))$ is a sampling space, and $W_\phi(\phi)$ is the associated sinc-type function for $\mathbf{K}$. To see that in general $W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{E} \cap \mathbf{C}))$ does not have the interpolation property, it suffices to observe that the condition

$$\|\phi\|^2_{\mathbf{H}(\mathbf{e},\mathbf{E}\cap\mathbf{C})} = \mu(\mathbf{E} \cap \mathbf{C}) = 1$$

does not always hold. This completes the proof of Theorem 3.2.

5.2 Proof of Theorem 3.3

In order to prove Theorem 3.3, we will need a series of lemmas first. Let us assume throughout this subsection that there exist a basis for the Lie algebra $\mathfrak{n}$ and a compact subset $\mathbf{R}$ of $\mathfrak{z}^*$ such that $\mu(\mathbf{E} \cap \mathbf{R}) = 1$. Put $\mathbf{E}^\circ = \mathbf{E} \cap \mathbf{R}$ and

$$\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ) = \mathcal{P}^{-1}\left(\int_{\mathbf{E}^\circ}^{\oplus} L^2(\mathbb{R}^d) \otimes \mathbf{e}_\lambda |\det B(\lambda)|\, d\lambda\right).$$

Lemma 5.3. *Let $\mathbf{R}$ be given such that*

$$\int_{\mathbf{E}^\circ} |\det B(\lambda)|\, d\lambda = 1.$$

Then the set $\mathbf{E}^\circ$ cannot be contained in a fundamental domain of the lattice $\mathbb{Z}^{n-2d}$.

Proof. Assume that

$$\mu(\mathbf{E}^\circ) = \int_{\mathbf{E}^\circ} |\det B(\lambda)| \, d\lambda = 1.$$

We observe that the function $\lambda \mapsto |\det B(\lambda)|$ is a non-constant continuous function which is bounded above by 1 on $\mathbf{E}^\circ$. Therefore,

$$1 = \int_{\mathbf{E}^\circ} |\det B(\lambda)| \, d\lambda < \int_{\mathbf{E}^\circ} d\lambda = m(\mathbf{E}^\circ)$$

where m is the Lebesgue measure on Σ. By contradiction, let us assume that $\mathbf{E}^\circ$ is contained in a fundamental domain of a lattice $\mathbb{Z}^{n-2d}$. Then $1 < m(\mathbf{E}^\circ) \le 1$ and we reach a contradiction. $\qquad\square$

Lemma 5.4. *There exists a finite partition of $\mathbf{E}^\circ$,*

$$\mathbf{P} = \{A_1, A_2, \dots, A_{\mathrm{card}(\mathbf{P})}\},$$

such that:

(1) *We have*

$$\mathbf{E}^\circ = \bigcup_{A_j \in \mathbf{P}} A_j \quad and \quad \mathbf{H}(\mathbf{e}, \mathbf{E}^\circ) = \bigoplus_{j=1}^{\mathrm{card}(\mathbf{P})} \mathbf{H}(\mathbf{e}, A_j).$$

(2) *For each j where $1 \le j \le \mathrm{card}(\mathbf{P})$, A_j is contained in a fundamental domain for $\mathbb{Z}^{n-2d}$.*

(3) *For each j where $1 \le j \le \mathrm{card}(\mathbf{P})$, there exists a Parseval frame of the type $L(\Gamma)\phi_j$ for the Hilbert space*

$$\mathbf{H}(\mathbf{e}, A_j) = \mathcal{P}^{-1}\left(\int_{A_j}^{\oplus} L^2(\mathbb{R}^d) \otimes \mathbf{e}_\lambda |\det B(\lambda)| \, d\lambda \right).$$

Proof. Parts (1)–(2) are obviously true.

For the proof for part (3), we observe that if A_j is contained in a fundamental domain of $\mathbb{Z}^{n-2d}$, then

$$\{e^{2\pi i \langle k, \lambda \rangle} \chi_{A_j}(\lambda) : k \in \mathbb{Z}^{n-2d}\}$$

is a Parseval frame for the Hilbert space $L^2(A_j, d\lambda)$. Thus Lemma 4.3 gives us part 3. $\qquad\square$

Lemma 5.5. *For each $1 \le j \le \mathrm{card}(\mathbf{P})$, we can construct a Parseval frame of the type $L(\Gamma)\phi_j$ such that*

$$\left\| \sum_{j=1}^{\mathrm{card}(\mathbf{P})} \phi_j \right\|_{\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)}^2 = 1.$$

Proof. The construction of a Parseval frame for each $\mathbf{H}(\mathbf{e}, A_j)$, $1 \le j \le \mathrm{card}(\mathbf{P})$, of the type $L(\Gamma)\phi_j$ is given in Lemma 4.3, and

$$\left\| \sum_{j=1}^{\mathrm{card}(\mathbf{P})} \phi_j \right\|_{\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)}^2 = \sum_{j=1}^{\mathrm{card}(\mathbf{P})} \|\phi_j\|_{\mathbf{H}(\mathbf{e}, A_j)}^2 = \int_{\bigcup_{j=1}^{\mathrm{card}(\mathbf{P})} A_j} |\det B(\lambda)| \, d\lambda = \int_{\mathbf{E}^\circ} |\det B(\lambda)| \, d\lambda = 1. \qquad\square$$

Lemma 5.6. *Let $\phi = \sum_{j=1}^{\mathrm{card}(\mathbf{P})} \phi_j$ such that for each $1 \le j \le \mathrm{card}(\mathbf{P})$, $L(\Gamma)\phi_j$ is a Parseval frame for $\mathbf{H}(\mathbf{e}, A_j)$, and $\|\phi\|_{\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)}^2 = 1$. If $L(\Gamma)(\phi)$ is a Parseval frame, then $L(\Gamma)\phi$ is an orthonormal basis for $\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)$.*

Proof. If $L(\Gamma)\phi$ is a Parseval frame for $\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)$, then $L(\Gamma)\phi$ is an orthonormal basis since $\|\phi\|_{\mathbf{H}(\mathbf{e}, \mathbf{E} \cap C)}^2 = 1$. $\qquad\square$

We would like to remark that in general the direct sum of Parseval frames is not a Parseval frame.

Next, let us fix a fundamental domain Λ of $\mathbb{Z}^{n-2d}$ such that

$$\mathbf{E}^\circ = \bigcup_{\kappa_j \in S} ((\Lambda - \kappa_j) \cap \mathbf{E}^\circ),$$

each $A_j = \mathbf{E}^\circ \cap (\Lambda - \kappa_j)$ is a set of positive Lebesgue measure for all $\kappa_j \in S$ and S is a finite subset of $\mathbb{Z}^{n-2d}$. Clearly, the collection of sets

$$\mathbf{P} = \{A_j : 1 \le j \le \mathrm{card}(S)\}$$

provides us with a partition of $\mathbf{E}^\circ$ as described in Lemma 5.4.

Lemma 5.7. *For each $1 \le j \le \text{card}(\mathbf{P})$, there exists some $\phi_j \in \mathbf{H}(\mathbf{e}, A_j)$ such that the following holds:*
(1) *$L(\Gamma)\phi_j$ is a Parseval frame for $\mathbf{H}(\mathbf{e}, A_j)$,*
(2) *$\mathcal{P}\phi_j(\lambda) = (\mathbf{u}_\lambda \otimes \mathbf{e}_\lambda)|\det B(\lambda)|^{-1/2}$ where*

$$
\mathbf{u}_\lambda = \begin{cases} |\det B(\lambda)|^{1/2}\chi_{E(\lambda)} & \text{if } \lambda \in A_j, \\ 0 & \text{if } \lambda \notin \Sigma - A_j, \end{cases}
$$

such that $E(\lambda)$ tiles $\mathbb{R}^d$ by $\mathbb{Z}^d$ and packs $\mathbb{R}^d$ by $B(\lambda)^{-\text{tr}}\mathbb{Z}^d$.

Proof. See Lemma 4.3 and Remark 4.5. □

Let us now define

$$
\phi = \phi_1 + \cdots + \phi_{\text{card}(\mathbf{P})}
$$

such that each ϕ_j is as described in Lemma 5.7. Then clearly,

$$
\mathcal{P}\phi(\lambda) = \begin{cases} \chi_{E(\lambda)} \otimes \mathbf{e}_\lambda & \text{if } \lambda \in \mathbf{E}^\circ, \\ 0 & \text{if } \lambda \notin \Sigma - \mathbf{E}^\circ. \end{cases}
$$

Lemma 5.8. *If for each $1 \le j, j' \le \text{card}(\mathbf{P})$, $j \ne j'$, and for arbitrary functions $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d)$, $\kappa_j, \kappa_{j'} \in S$ one has*

$$
(\langle \mathbf{f}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j} \rangle)_{\gamma_1 \in \Gamma_\mathfrak{b}\Gamma_\mathfrak{a}} \perp (\langle \mathbf{g}, \pi_{\lambda-\kappa_{j'}}(\gamma_1)\mathbf{u}_{\lambda-\kappa_{j'}} \rangle)_{\gamma_1 \in \Gamma_\mathfrak{b}\Gamma_\mathfrak{a}}
$$

for $\lambda \in \Lambda$, then $L(\Gamma)(\phi)$ is an orthonormal basis for the Hilbert space $\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)$.

Proof. Let ψ be any arbitrary element in

$$
\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ) = \bigoplus_{j=1}^{\text{card}(\mathbf{P})} \mathbf{H}(\mathbf{e}, A_j)
$$

such that $\psi = \sum_{j=1}^{\text{card}(\mathbf{P})} \psi_j$ for $\psi_j \in \mathbf{H}(\mathbf{e}, A_j)$. Let $r(\lambda) = |\det B(\lambda)|$. Then

$$
\|\psi_j\|_{\mathbf{H}(\mathbf{e}, A_j)}^2 = \int_{A_j} \|\mathcal{P}\psi_j(\sigma)\|_{\mathcal{HS}}^2 r(\sigma)\, d\sigma.
$$

Next, it is easy to see that

$$
\|\psi_j\|_{\mathbf{H}(\mathbf{e}, A_j)}^2 = \int_\Lambda \|\mathcal{P}\psi_j(\lambda - \kappa_j)\|_{\mathcal{HS}}^2 r(\lambda - \kappa_j)\, d\lambda.
$$

Let $\mathcal{P}\psi_j(\lambda - \kappa_j) = (\mathbf{w}_{\lambda-\kappa_j} \otimes \mathbf{e}_{\lambda-\kappa_j}) \in L^2(\mathbb{R}^d) \otimes \mathbf{e}_{\lambda-\kappa_j}$ for $\lambda \in \Lambda$. Then

$$
\sum_{j=1}^{\text{card}(\mathbf{P})} \|\psi_j\|_{\mathbf{H}(\mathbf{e}, A_j)}^2 = \sum_{j=1}^{\text{card}(\mathbf{P})} \int_\Lambda \sum_{\gamma_1 \in \Gamma_\mathfrak{b}\Gamma_\mathfrak{a}} |\langle \mathbf{w}_{\lambda-\kappa_j}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j} \rangle|^2 r(\lambda - \kappa_j)\, d\lambda
$$

$$
= \int_\Lambda \sum_{\gamma_1 \in \Gamma_\mathfrak{b}\Gamma_\mathfrak{a}} \sum_{j=1}^{\text{card}(\mathbf{P})} |\langle \mathbf{w}_{\lambda-\kappa_j}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j} \rangle|^2 r(\lambda - \kappa_j)\, d\lambda. \tag{5.2}
$$

We would like to be able to state that for $\lambda \in \mathbf{E}^\circ$,

$$
\sum_{\gamma_1 \in \Gamma_\mathfrak{b}\Gamma_\mathfrak{a}} \sum_{j=1}^{\text{card}(\mathbf{P})} |\langle \mathbf{w}_{\lambda-\kappa_j}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j} \rangle|^2 = \sum_{\gamma_1 \in \Gamma_\mathfrak{b}\Gamma_\mathfrak{a}} \left| \sum_{j=1}^{\text{card}(\mathbf{P})} \langle \mathbf{w}_{\lambda-\kappa_j}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j} \rangle \right|^2. \tag{5.3}
$$

Indeed, letting $(b_{\gamma_1}(\lambda))_{\gamma_1 \in \Gamma_\mathfrak{b}\Gamma_\mathfrak{a}} \in l^2(\Gamma_\mathfrak{b}\Gamma_\mathfrak{a})$ such that $(b_{\gamma_1}(\lambda))_{\gamma_1 \in \Gamma_\mathfrak{b}\Gamma_\mathfrak{a}}$ is a sum of $\text{card}(\mathbf{P})$-many sequences of the type $(b_{\gamma_1}^j(\lambda))_{\gamma_1 \in \Gamma_\mathfrak{b}\Gamma_\mathfrak{a}}$ such that

$$
(b_{\gamma_1}(\lambda))_{\gamma_1 \in \Gamma_\mathfrak{b}\Gamma_\mathfrak{a}} = \sum_{j=1}^{\text{card}(\mathbf{P})} (\langle \mathbf{w}_{\lambda-\kappa_j}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j} \rangle)_{\gamma_1 \in \Gamma_\mathfrak{b}\Gamma_\mathfrak{a}} = \sum_{j=1}^{\text{card}(\mathbf{P})} (b_{\gamma_1}^j(\lambda))_{\gamma_1 \in \Gamma_\mathfrak{b}\Gamma_\mathfrak{a}},
$$

we compute the norm of the sequence $(b_{\gamma_1}(\lambda))_{\gamma_1 \in \Gamma_\mathfrak{b}\Gamma_\mathfrak{a}}$ in two different ways. First,

$$
\|(b_{\gamma_1}(\lambda))_{\gamma_1 \in \Gamma_\mathfrak{b}\Gamma_\mathfrak{a}}\|^2 = \sum_{\gamma_1 \in \Gamma_\mathfrak{b}\Gamma_\mathfrak{a}} |b_{\gamma_1}(\lambda)|^2 = \sum_{\gamma_1 \in \Gamma_\mathfrak{b}\Gamma_\mathfrak{a}} \left| \sum_{j=1}^{\text{card}(\mathbf{P})} \langle \mathbf{w}_{\lambda-\kappa_j}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j} \rangle \right|^2.
$$

Second

$$\|(b_{\gamma_1}(\lambda))_{\gamma_1 \in \Gamma_b \Gamma_a}\|^2 = \left\|\sum_{j=1}^{\mathrm{card}(\mathbf{P})} (b_{\gamma_1}^j(\lambda))_{\gamma_1 \in \Gamma_b \Gamma_a}\right\|^2$$

$$= \sum_{j=1}^{\mathrm{card}(\mathbf{P})} \|(b_{\gamma_1}^j(\lambda))_{\gamma_1 \in \Gamma_b \Gamma_a}\|^2$$

$$= \sum_{j=1}^{\mathrm{card}(\mathbf{P})} \|(\langle \mathbf{w}_{\lambda-\kappa_j}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j}\rangle)_{\gamma_1 \in \Gamma_b \Gamma_a}\|^2$$

$$= \sum_{j=1}^{\mathrm{card}(\mathbf{P})} \sum_{\gamma_1 \in \Gamma_b \Gamma_a} |\langle \mathbf{w}_{\lambda-\kappa_j}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j}\rangle|^2.$$

The second equality above is due to the fact that we assume that for $j \neq j'$,

$$(\langle \mathbf{w}_{\lambda-\kappa_j}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j}\rangle)_{\gamma_1 \in \Gamma_b \Gamma_a} \perp (\langle \mathbf{w}_{\lambda-\kappa_{j'}}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_{j'}}\rangle)_{\gamma_1 \in \Gamma_b \Gamma_a} \quad \text{for } \lambda \in \Lambda.$$

Thus, the equality given in (5.3) holds.

Next, coming back to (5.2), we obtain

$$\sum_{j=1}^{\mathrm{card}(\mathbf{P})} \|\psi_j\|^2_{\mathbf{H}(\mathbf{e},A_j)} = \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \int_\Lambda |a_{\gamma_1}(\lambda)|^2 \, d\lambda \quad \text{where} \quad a_{\gamma_1}(\lambda) = \sum_{j=1}^{\mathrm{card}(\mathbf{P})} \langle \mathbf{w}_{\lambda-\kappa_j}, \overbrace{\pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j}r(\lambda-\kappa_j)^{1/2}}^{\mathbf{q}_{j,\gamma_1}(\lambda)}\rangle.$$

Writing $\lambda(m) = e^{2\pi i \langle \lambda, m \rangle}$, it follows that

$$\sum_{j=1}^{\mathrm{card}(\mathbf{P})} \|\psi_j\|^2_{\mathbf{H}(\mathbf{e},A_j)} = \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \|a_{\gamma_1}\|^2_{L^2(\Lambda)}$$

$$= \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \|\widehat{a_{\gamma_1}}\|^2_{l^2(\mathbb{Z}^{n-2d})}$$

$$= \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \sum_{m \in \mathbb{Z}^{n-2d}} |\widehat{a_{\gamma_1}}(m)|^2$$

$$= \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \sum_{m \in \mathbb{Z}^{n-2d}} \left|\int_\Lambda a_{\gamma_1}(\lambda)\lambda(m) \, d\lambda\right|^2$$

$$= \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \sum_{m \in \mathbb{Z}^{n-2d}} \left|\sum_{j=1}^{\mathrm{card}(\mathbf{P})} \int_\Lambda \langle \mathbf{w}_{\lambda-\kappa_j}, \mathbf{q}_{j,\gamma_1}(\lambda)\rangle\lambda(m) \, d\lambda\right|^2.$$

Next, letting

$$\overline{\lambda(m)} = \overline{e^{2\pi i \langle \lambda, m \rangle}} = e^{-2\pi i \langle \lambda, m \rangle},$$

we obtain that

$$\sum_{j=1}^{\mathrm{card}(\mathbf{P})} \|\psi_j\|^2_{\mathbf{H}(\mathbf{e},A_j)} = \sum_{\gamma_1 \in \Gamma_b \Gamma_a} \sum_{m \in \mathbb{Z}^{n-2d}} \left|\int_\Lambda \sum_{j=1}^{\mathrm{card}(\mathbf{P})} \langle \mathbf{w}_{\lambda-\kappa_j}, \overline{\lambda(m)}\mathbf{q}_{j,\gamma_1}(\lambda)\rangle \, d\lambda\right|^2$$

$$= \sum_{\gamma \in \Gamma} \left|\int_{\mathbf{E}^\circ} \langle \mathcal{P}\psi(\sigma), \pi_\sigma(\gamma)\overbrace{|\sigma|^{-1/2}\mathbf{u}_\sigma \otimes \mathbf{e}_\sigma}^{\mathcal{P}\phi(\sigma)}\rangle_{\mathcal{HS}} |\sigma| \, d\sigma\right|^2$$

$$= \sum_{\gamma \in \Gamma} \left|\int_{\mathbf{E}^\circ} \langle \mathcal{P}\psi(\sigma), \pi_\sigma(\gamma)\mathcal{P}\phi(\sigma)\rangle_{\mathcal{HS}} |\sigma| \, d\sigma\right|^2$$

$$= \sum_{\gamma \in \Gamma} |\langle \psi, L(\gamma)\phi\rangle_{\mathbf{H}(\mathbf{e},\mathbf{E}^\circ)}|^2.$$

Finally, we arrive at the following fact:

$$\|\psi\|^2_{\mathbf{H}(\mathbf{e},\mathbf{E}^\circ)} = \sum_{\gamma \in \Gamma} |\langle \psi, L(\gamma)\phi\rangle_{\mathbf{H}(\mathbf{e},\mathbf{E}^\circ)}|^2.$$

Thus, $L(\Gamma)\phi$ is a Parseval frame for $\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)$.

Now, we compute the norm of the vector ϕ. Since

$$\mathcal{P}\phi(\lambda) = (\mathbf{u}_\lambda \otimes \mathbf{e}_\lambda)|\det B(\lambda)|^{-1/2} \quad \text{and} \quad \mathbf{u}_\lambda = |\det B(\lambda)|^{1/2}\chi_{E(\lambda)},$$

we have

$$\mathcal{P}\phi(\lambda) = (|\det B(\lambda)|^{1/2}\chi_{E(\lambda)} \otimes \mathbf{e}_\lambda)|\det B(\lambda)|^{-1/2} = \chi_{E(\lambda)} \otimes \mathbf{e}_\lambda.$$

Since $E(\lambda)$ is a fundamental domain for $\mathbb{Z}^d$, it follows that

$$\|\phi\|^2_{\mathbf{H}(\mathbf{e},\mathbf{E}^\circ)} = 1.$$

Finally, because L is a unitary representation, and using the fact that $L(\Gamma)\phi$ forms a unit norm Parseval frame, it follows that $L(\Gamma)\phi$ forms an orthonormal basis in $\mathbf{H}(\mathbf{e},\mathbf{E}^\circ)$. □

Lemma 5.9. *If ϕ satisfies all conditions given in Lemma 5.8, then $\|\mathcal{P}\phi(\lambda)\|_{\mathcal{HS}} = 1$ for $\lambda \in \mathbf{E}^\circ$ and ϕ is an admissible vector for the representation $(L, \mathbf{H}(\mathbf{e}, \mathbf{E}^\circ))$.*

Proof. For any given $\lambda \in \mathbf{E}^\circ$,

$$\|\mathcal{P}(\phi)(\lambda)\|^2_{\mathcal{HS}} = \|\mathbf{u}_\lambda \otimes \mathbf{e}_\lambda|\det B(\lambda)|^{-1/2}\|^2_{\mathcal{HS}} = |\det B(\lambda)|^{-1}\|\mathbf{u}_\lambda\|^2_{L^2(\mathbb{R}^d)} = 1.$$

Since N is unimodular, and since $\mu(\mathbf{E}^\circ) < \infty$, it follows from [4, p. 126] that $(L, \mathbf{H}(\mathbf{e}, \mathbf{E}^\circ))$ is an admissible representation of N and ϕ is an admissible vector for the representation $(L, \mathbf{H}(\mathbf{e}, \mathbf{E}^\circ))$. □

Remark 5.10. A proof of Theorem 3.3 is derived by applying Proposition 5.2, Lemma 5.8 and Lemma 5.9.

5.3 Additional observations

Let m be the Lebesgue measure on $\mathbb{R}^{n-2d}$. Given two measurable sets $A, B \subseteq \mathbb{R}^{n-2d}$,

$$A \,\Delta\, B = (A - B) \cup (B - A)$$

is the symmetric difference of the sets. Now, let us assume that there exists a fundamental domain Λ of $\mathbb{Z}^{n-2d}$ such that

$$m\left(\mathbf{E}^\circ \,\Delta\, \left(\bigcup_{j\in S}(\Lambda + k_j)\right)\right) = 0.$$

That is, up to a set of Lebesgue measure zero, $\mathbf{E}^\circ$ is a finite disjoint union of sets which are $\mathbb{Z}^{n-2d}$-congruent to a fundamental domain of $\mathbb{R}^{n-2d}$. We acknowledge that this is a very strong condition to impose. However, under this condition, we would like to present some simple sufficient conditions for the statement

$$(\langle \mathbf{f}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j}\rangle)_{\gamma_1\in\Gamma_b\Gamma_a} \perp (\langle \mathbf{g}, \pi_{\lambda-\kappa_{j'}}(\gamma_1)\mathbf{u}_{\lambda-\kappa_{j'}}\rangle)_{\gamma_1\in\Gamma_b\Gamma_a}$$

given in Lemma 5.8.

Lemma 5.11. *Let us assume that there exists a fundamental domain Λ of $\mathbb{Z}^{n-2d}$ such that*

$$m\left(\mathbf{E}^\circ \,\Delta\, \left(\bigcup_{j\in S}(\Lambda + k_j)\right)\right) = 0.$$

For $j \neq j'$, $\lambda \in \Lambda$ and $\mathbf{u}_{\lambda-\kappa_j}, \mathbf{u}_{\lambda-\kappa_{j'}} \in L^2(\mathbb{R}^d)$ as given in Lemma 5.8, if for any fixed $m \in \mathbb{Z}^d$,

$$\bigcup_{\kappa_s\in S} B(\lambda - \kappa_s)^{\mathrm{tr}}(E(\lambda - \kappa_s) + m) \text{ is a subset of a fundamental domain for } \mathbb{Z}^d$$

and if

$$B(\lambda - \kappa_j)^{\mathrm{tr}}(E(\lambda - \kappa_j) + m) \cap B(\lambda - \kappa_{j'})^{\mathrm{tr}}(E(\lambda - \kappa_{j'}) + m)$$

is a null set, then

$$(\langle \mathbf{f}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j}\rangle)_{\gamma_1\in\Gamma_b\Gamma_a} \perp (\langle \mathbf{g}, \pi_{\lambda-\kappa_{j'}}(\gamma_1)\mathbf{u}_{\lambda-\kappa_{j'}}\rangle)_{\gamma_1\in\Gamma_b\Gamma_a}$$

for all $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d)$.

Proof. Clearly, in order to compute the inner product of the sequences

$$(\langle \mathbf{f}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j}\rangle)_{\gamma_1 \in \Gamma_b \Gamma_a}, (\langle \mathbf{g}, \pi_{\lambda-\kappa_{j'}}(\gamma_1)\mathbf{u}_{\lambda-\kappa_{j'}}\rangle)_{\gamma_1 \in \Gamma_b \Gamma_a} \in l^2(\Gamma_b \Gamma_a),$$

we need to calculate a formula for the following sum:

$$\sum_{\gamma_1 \in \Gamma_b \Gamma_a} (\langle \mathbf{f}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j}\rangle \overline{\langle \mathbf{g}, \pi_{\lambda-\kappa_{j'}}(\gamma_1)\mathbf{u}_{\lambda-\kappa_{j'}}\rangle}).$$

First,

$$\langle \mathbf{f}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j}\rangle = \int_{\mathbb{R}} \mathbf{f}(t)\overline{\pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j}(t)}\, dt = \int_{\mathbb{R}} \mathbf{f}(t)\overline{e^{2\pi i\langle l, B(\lambda-\kappa_j)^{tr}t\rangle}\mathbf{u}_{\lambda-\kappa_j}(t-m)}\, dt.$$

Put $s = B(\lambda-\kappa_j)^{tr}t$. We recall that $\mathbf{u}_\lambda = |\det B(\lambda)|^{1/2}\chi_{E(\lambda)}$. So,

$$\langle \mathbf{f}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j}\rangle = \int_{B(\lambda-\kappa_j)^{tr}(E(\lambda-\kappa_j)+m)} \frac{\mathbf{f}(B(\lambda-\kappa_j)^{-tr}s)}{|\det B(\lambda-\kappa_j)|^{1/2}}e^{-2\pi i\langle l,s\rangle}\, ds.$$

Similarly,

$$\langle \mathbf{g}, \pi_{\lambda-\kappa_{j'}}(\gamma_1)\mathbf{u}_{\lambda-\kappa_{j'}}\rangle = \int_{B(\lambda-\kappa_{j'})^{tr}(E(\lambda-\kappa_{j'})+m)} \frac{\mathbf{g}(B(\lambda-\kappa_{j'})^{-tr}s)}{|\det B(\lambda-\kappa_{j'})|^{1/2}}e^{-2\pi i\langle l,s\rangle}\, ds.$$

If for $\lambda - \kappa_j \in A_j$,

$$\bigcup_{\kappa_s \in S} B(\lambda-\kappa_s)^{tr}(E(\lambda-\kappa_s)+m)$$

is a subset of a fundamental domain of $\mathbb{Z}^d$ for distinct $\kappa_j, \kappa_{j'} \in S$, and if

$$B(\lambda-\kappa_j)^{tr}(E(\lambda-\kappa_j)+m) \cap B(\lambda-\kappa_{j'})^{tr}(E(\lambda-\kappa_{j'})+m)$$

is a null set, then

$$(\langle \mathbf{f}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j}\rangle)_{\gamma_1 \in \Gamma_b \Gamma_a} \perp (\langle \mathbf{g}, \pi_{\lambda-\kappa_{j'}}(\gamma_1)\mathbf{u}_{\lambda-\kappa_{j'}}\rangle)_{\gamma_1 \in \Gamma_b \Gamma_a}$$

because $(\langle \mathbf{f}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j}\rangle)_{\gamma_1 \in \Gamma_b \Gamma_a}$ and $(\langle \mathbf{g}, \pi_{\lambda-\kappa_{j'}}(\gamma_1)\mathbf{u}_{\lambda-\kappa_{j'}}\rangle)_{\gamma_1 \in \Gamma_b \Gamma_a}$ are Fourier inverses of the following orthogonal functions:

$$\theta_{\mathbf{f},\lambda,j,m}(s) = \chi_{B(\lambda-\kappa_j)^{tr}(E(\lambda-\kappa_j)+m)}(s)\frac{\mathbf{f}(B(\lambda-\kappa_j)^{-tr}s)}{|\det B(\lambda-\kappa_j)|^{1/2}}, \tag{5.4}$$

respectively

$$\theta_{\mathbf{g},\lambda,j',m}(s) = \chi_{B(\lambda-\kappa_{j'})^{tr}(E(\lambda-\kappa_{j'})+m)}(s)\frac{\mathbf{g}(B(\lambda-\kappa_{j'})^{-tr}s)}{|\det B(\lambda-\kappa_{j'})|^{1/2}}. \tag{5.5}$$

In fact, we think of the functions above (5.4), (5.5) as being elements of $L^2(\mathbf{I}_{m,\lambda})$ such that $\mathbf{I}_{m,\lambda}$ is a fundamental domain for $\mathbb{Z}^d$. Combining the observations made above, we obtain that for any $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^d)$,

$$\sum_{\gamma_1 \in \Gamma_b \Gamma_a} \langle \mathbf{f}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j}\rangle \overline{\langle \mathbf{g}, \pi_{\lambda-\kappa_{j'}}(\gamma_1)\mathbf{u}_{\lambda-\kappa_{j'}}\rangle} = \sum_{m\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d}\left(\int_{\mathbf{I}_{m,\lambda}} \theta_{\mathbf{f},\lambda,j,m}(s)e^{-2\pi i\langle l,s\rangle}\, ds\right)\overline{\left(\int_{\mathbf{I}_{m,\lambda}} \theta_{\mathbf{g},\lambda,j',m}(s)e^{-2\pi i\langle l,x\rangle}\, dx\right)}$$

$$= \sum_{m\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^d} \widehat{\theta_{\mathbf{f},\lambda,j,m}}(l)\overline{\widehat{\theta_{\mathbf{g},\lambda,j',m}}(l)}$$

$$= \sum_{m\in\mathbb{Z}^d}\overline{\sum_{l\in\mathbb{Z}^d} \overset{=0}{\overbrace{\widehat{\theta_{\mathbf{f},\lambda,j,m}}(l)\overline{\widehat{\theta_{\mathbf{g},\lambda,j',m}}(l)}}}}$$

$$= 0.$$

Thus

$$(\langle \mathbf{f}, \pi_{\lambda-\kappa_j}(\gamma_1)\mathbf{u}_{\lambda-\kappa_j}\rangle)_{\gamma_1 \in \Gamma_b \Gamma_a} \perp (\langle \mathbf{g}, \pi_{\lambda-\kappa_{j'}}(\gamma_1)\mathbf{u}_{\lambda-\kappa_{j'}}\rangle)_{\gamma_1 \in \Gamma_b \Gamma_a}.$$

This concludes the proof. □

In light of Lemma 5.8 and Lemma 5.11, the following holds true.

Proposition 5.12. *Let us assume that there exists a fundamental domain Λ of $\mathbb{Z}^{n-2d}$ such that*

$$m\left(\mathbf{E}^\circ \, \Delta \left(\bigcup_{j \in S}(\Lambda + k_j)\right)\right) = 0.$$

If $\bigcup_{\kappa_s \in S} B(\lambda - \kappa_s)^{\mathrm{tr}}(E(\lambda - \kappa_s) + m)$ is a subset of a fundamental domain of $\mathbb{Z}^d$ and if

$$B(\lambda - \kappa_j)^{\mathrm{tr}}(E(\lambda - \kappa_j) + m) \cap B(\lambda - \kappa_{j'})^{\mathrm{tr}}(E(\lambda - \kappa_{j'}) + m)$$

is a null set for $\lambda \in \Lambda$ for $m \in \mathbb{Z}^d$ and for distinct $\kappa_j, \kappa_{j'} \in S$, then $L(\Gamma)(\phi)$ is an orthonormal basis for the Hilbert space $\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)$.

5.4 Examples

Example 5.13. Let N be a nilpotent Lie group with Lie algebra spanned by the vectors $Z_1, Z_2, Y_1, Y_2, X_1, X_2$ with the following non-trivial Lie brackets:

$$[X_1, Y_1] = Z_1, \quad [X_2, Y_2] = Z_2.$$

In this example, the discrete set

$$\Gamma = \exp(\mathbb{Z}Z_1 + \mathbb{Z}Z_2) \exp(\mathbb{Z}Y_1 + \mathbb{Z}Y_2) \exp(\mathbb{Z}X_1 + \mathbb{Z}X_2)$$

is actually a uniform subgroup of the Lie group N. Moreover, N is a direct product of two Heisenberg groups and satisfies all properties described in Condition 1.1. Next, we will apply Theorem 3.3 to show that there exists a left-invariant subspace of $L^2(N)$ which is a Γ-sampling space with the interpolation property. First, it is easy to check that

$$\mu(\mathbf{E}) = \mu(\{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : \lambda_1 \lambda_2 \neq 0 \text{ and } |\lambda_1 \lambda_2| \leq 1\}) = \infty.$$

Now, let $\mathbf{R} = [-1, 1]^2$. Then

$$\mu(\mathbf{E} \cap \mathbf{R}) = \int_{-1}^{1} \int_{-1}^{1} |\lambda_1 \lambda_2| \, d\lambda_1 d\lambda_2 = 1.$$

Next, we observe for each $\lambda \in \mathbf{E} \cap \mathbf{R}$, $[0, 1)^2$ tiles $\mathbb{R}^2$ by $\mathbb{Z}^2$ and packs $\mathbb{R}^2$ by

$$B(\lambda)^{-\mathrm{tr}} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^{-1} \mathbb{Z}^2.$$

Next, it is not too hard to check that

$$\mathbf{E}^\circ = \{(\lambda_1, \lambda_2) \in [-1, 1]^2 : \lambda_1 \lambda_2 \neq 0\}$$

and

$$m\left(\mathbf{E}^\circ \, \Delta \left(\bigcup_{j \in S}([0, 1)^2 + j)\right)\right) = 0$$

where

$$S = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}.$$

Moreover, for $\lambda \in [0, 1)^2$ and for $m \in \mathbb{Z}^2$, we define

$$M_{\lambda,1} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad M_{\lambda,2} = \begin{bmatrix} \lambda_1 - 1 & 0 \\ 0 & \lambda_2 - 1 \end{bmatrix}, \quad M_{\lambda,3} = \begin{bmatrix} \lambda_1 - 1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \qquad M_{\lambda,4} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 - 1 \end{bmatrix},$$

$$\mathbf{S}_{\lambda,1,j} = M_{\lambda,1}((0, 1)^2 + j), \quad \mathbf{S}_{\lambda,2,j} = M_{\lambda,2}((0, 1)^2 + j), \qquad \mathbf{S}_{\lambda,3,j} = M_{\lambda,3}((0, 1)^2 + j), \quad \mathbf{S}_{\lambda,4,m} = M_{\lambda,4}((0, 1)^2 + j).$$

We observe that for all $j \in \mathbb{Z}^2$, $\bigcup_{k=1}^4 \mathbf{S}_{\lambda,k,j}$ is a subset of a fundamental domain for $\mathbb{Z}^2$ and $\mathbf{S}_{\lambda,\ell_1,j} \cap \mathbf{S}_{\lambda,\ell_2,j}$ is a null set for distinct ℓ_1, ℓ_2 and for almost every $\lambda \in (0,1)^2$. Let $\phi \in \mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)$ such that

$$\mathcal{P}\phi(\lambda) = \chi_{[0,1)^2}(t) \otimes \chi_{[0,1)^2}(t).$$

According to Theorem 3.3 and Proposition 5.12, $L(\Gamma)\phi$ is an orthonormal basis for $\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ)$ and $W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ))$ is a Γ-sampling space with the interpolation property.

Example 5.14. Put $\alpha = (\frac{\pi}{2})^{1/4}$. Let N be a nilpotent Lie group with Lie algebra spanned by the vectors $Z_1, Z_2, Y_1, Y_2, X_1, X_2$ with the following non-trivial Lie brackets:

$$[X_1, Y_1] = \alpha Z_1, \quad [X_1, Y_2] = -\alpha Z_2, \quad [X_2, Y_1] = \alpha Z_2, \quad [X_2, Y_2] = \alpha Z_1.$$

Here

$$B(\lambda) = \begin{bmatrix} \alpha\lambda_1 & -\alpha\lambda_2 \\ \alpha\lambda_2 & \alpha\lambda_1 \end{bmatrix}$$

and

$$\Gamma = \exp(\mathbb{Z}Z_1 + \mathbb{Z}Z_2)\exp(\mathbb{Z}Y_1 + \mathbb{Z}Y_2)\exp(\mathbb{Z}X_1 + \mathbb{Z}X_2).$$

Next, the Plancherel measure is $\alpha^2(\lambda_1^2 + \lambda_2^2)\,d\lambda_1 d\lambda_2$ and it is supported on the manifold

$$\Sigma = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : (\lambda_1, \lambda_2) \neq (0,0)\}.$$

Now, we have

$$\mathbf{E} = \left\{ (\lambda_1, \lambda_2) \in \Sigma : \lambda_1^2 + \lambda_2^2 \le \frac{1}{\alpha^2} \right\}$$

and

$$\mu(\mathbf{E}) = \int_0^{2\pi} \int_0^{(\frac{\pi}{2})^{-1/4}} r^3\,dr\,d\theta = 1.$$

Thus, in this example $\mathbf{E}^\circ = \mathbf{E}$. Next, we partition the set $\mathbf{E}^\circ$ such that

$$\mathbf{E}^\circ = \bigcup_{j \in S}\left(([0,1)^2 + j) \cap \mathbf{E}^\circ\right)$$

where

$$S = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}.$$

For each $j \in S$, put $A_j = ([0,1)^2 \cap \mathbf{E}^\circ) + j$. Next, appealing to Theorem 3.2, for each $j \in S$, there exists some $\phi_j \in \mathbf{H}(\mathbf{e}, A_j)$ such that $W_{\phi_j}(\mathbf{H}(\mathbf{e}, A_j))$ is a Γ-sampling subspace of $L^2(N)$ with sinc-type function $W_{\phi_j}(\phi_j)$. Also for each $j \in S$, $W_{\phi_j}(\mathbf{H}(\mathbf{e}, A_j))$ does not have the interpolation property with respect to Γ since $\mu(A_j) < 1$ for each $j \in S$. Although

$$\left\| \sum_{j \in S} \phi_j \right\|^2 = 1,$$

we cannot say that $L(\Gamma)(\sum_{j \in S}\phi_j)$ is a Parseval frame in

$$\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ) = \bigoplus_{j \in S} \mathbf{H}(\mathbf{e}, A_j).$$

Suppose that we define the map $\lambda \mapsto \mathbf{u}_\lambda$ on Σ such that

$$\mathbf{u}_\lambda \begin{cases} |\det B(\lambda)|^{1/2}\chi_{E(\lambda)} & \text{if } \lambda \in \mathbf{E}^\circ, \\ 0 & \text{if } \lambda \notin \Sigma - \mathbf{E}^\circ, \end{cases}$$

where $E(\lambda) \subset \mathbb{R}^2$ tiles $\mathbb{R}^2$ by $\mathbb{Z}^2$ and packs $\mathbb{R}^2$ by

$$B(\lambda)^{-\mathrm{tr}} \begin{bmatrix} \frac{\lambda_1}{\alpha\lambda_1^2 + \alpha\lambda_2^2} & -\frac{\lambda_2}{\alpha\lambda_1^2 + \alpha\lambda_2^2} \\ \frac{\lambda_2}{\alpha\lambda_1^2 + \alpha\lambda_2^2} & \frac{\lambda_1}{\alpha\lambda_1^2 + \alpha\lambda_2^2} \end{bmatrix} \mathbb{Z}^2.$$

Finally, we define $\phi \in L^2(N)$ such that

$$\mathcal{P}\phi(\lambda) = \frac{\mathbf{u}_\lambda \otimes \mathbf{e}_\lambda}{\sqrt{|\det B(\lambda)|}}.$$

According to Theorem 3.3, if for each $1 \leq j, j' \leq \mathrm{card}(S)$ with $j \neq j'$, for $\lambda \in [0,1)^2$, for two arbitrary functions $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^2)$, and for distinct $j, j' \in S$ one has

$$(\langle \mathbf{f}, \pi_{\lambda - j}(\gamma_1)\mathbf{u}_{\lambda - j}\rangle)_{\gamma_1 \in \Gamma_b \Gamma_a} \perp (\langle \mathbf{g}, \pi_{\lambda - j'}(\gamma_1)\mathbf{u}_{\lambda - j'}\rangle)_{\gamma_1 \in \Gamma_b \Gamma_a},$$

then $W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ))$ is a Γ-sampling space with sinc-type function $W_\phi(\phi)$. Moreover, $W_\phi(\mathbf{H}(\mathbf{e}, \mathbf{E}^\circ))$ has the interpolation property as well.

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