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# Making Algebra More Accessible: How Steep Can it be for Teachers?

Diana Cheng
*Towson University*

Polina Sabinin
*Bridgewater State University*, polina.sabinin@bridgew.edu

# CHAPTER 6

## MAKING ALGEBRA MORE ACCESSIBLE: HOW STEEP CAN IT BE FOR TEACHERS?

**Diana Cheng & Polina Sabinin**

*Towson University, Maryland, USA*
*Bridgewater State University, Massachusetts, USA*

**dcheng@towson.edu**
**polina.sabinin@bridgew.edu**

## Abstract

Teacher educators need to support middle grades teachers in developing mathematical knowledge for teaching algebraic concepts. In particular, teachers should become familiar with possible introductions and sequencing to the concept of slope, and common middle school students' limited conceptions about measuring the steepness of an incline. Steepness can be expressed directly in terms of an angle or indirectly as a slope. Encouraging middle school students to find a measure of steepness using a ratio may help support students' transition to multiplicative thinking. This mixed – methods study explores middle school students' responses in solving a comparison problem involving the steepness of two inclines, in order to gain insight into common student strategies. The quantitative portion of the study involved written surveys distributed to 256 Grade 7 participants in the United States. We examined the frequency and types of solutions offered by these participants. We found that 27% of the participants provided an incorrect solution which was consistent with additive reasoning. The qualitative portion of this study consisted of small group interviews of 19 Grade 7 participants, who were conflicted in the different solutions they produced from using additive reasoning and their geometric knowledge.

## Keywords

Steepness – Slope – Multiplicative reasoning

## 1. Mathematical Knowledge for Teaching Slope

The preparation of middle school students to learn algebraic concepts such as slope is of international concern. Many researchers have uncovered middle school students' lack of preparation to learn algebraic concepts such as distinguishing between word problems involving additive or multiplicative reasoning (Van Dooren, De Bock, Vleugels, & Verschaffel, 2008), determining characteristics of a valid measure of the steepness of an incline (Lobato & Thanheiser, 2002), determining what measurements are involved in the "rise over run" procedure (Lobato, 1996), and understanding what the slope of a linear function means (Yerushalmy, 1997). In the United States, mathematics educators, business specialists, and policy makers nationwide collaborated to produce the Common Core State Standards (2011), which suggests that between sixth through eighth grades, students should learn a progression of concepts leading into the learning about slopes of lines. The progression includes constructing ratios for multiplicative relationships in a variety of contexts, finding equivalent ratios, understanding constants of proportionality and their relationships to the steepnesses of graphed proportional relationships, and interpreting the slopes of these lines. We discuss middle school students' responses to a steepness task in an effort to shed light upon ways that teacher educators can help increase preservice teachers' mathematical knowledge for teaching the connections between proportional reasoning, steepness, and slope.

Much research on teacher knowledge has mapped out the kinds of subject matter knowledge teachers need in their work of teaching mathematics (Ball, Thames, & Phelps, 2008). Specifically, in the domain of subject matter knowledge, teacher's mathematical knowledge for teaching (Hill, Rowan, & Ball, 2005), researchers have identified a specialized content knowledge that only teachers will need in their tasks of teaching students. Another important sub-domain of subject matter knowledge is the Horizon Content Knowledge (HCK), which refers to knowledge that "supports a kind of awareness, sensibility, disposition that informs, orients and culturally frames instructional practice" (Ball & Bass, 2009). This kind of knowledge involves being cognizant of the large mathematical landscape in which the present experience and instruction is situated (Ball & Bass, 2009). HCK plays a crucial role in teachers' knowledge and influence their instructional practices, which in turn affect students' learning (at the moment and future possibilities) and their learning trajectories. Such knowledge necessarily influences the nature of the tasks teachers set and how they are implement them in the classroom, in particular with respect to regulating the mathematical demands involved (Charalambous, 2008).

How, then, might teachers become better prepared to teach their middle school students about slope? Simon and Blume (1994) suggest that teacher educators need to help preservice teachers become more familiar with the content, that is, to understand how to use proportional relationships to find a measure of an incline's steepness. Preservice teachers often do not understand that a mathematical measure must be reproducible, that is, the measure alone should be sufficient for producing an incline with a given steepness. They also often have difficulties distinguishing between additive relationships and multiplicative relationships. Simon and Blume's (1994) findings about preservice teachers' conceptual difficulties are similar to the middle school students' difficulties identified by researchers such as Lobato & Thanheiser (2002) and Van Dooren et. al., (2008). It is unsurprising that Hill, Rowan & Ball (2005) found that teachers' mathematical knowledge for teaching correlated their students' achievements; thus it is especially important to focus on preservice teachers' understandings of the content at hand, measuring steepness.

It is not only imperative that preservice teachers themselves understand how to measure steepness using a ratio, but also preservice teachers should have knowledge of the students and how they might respond to questions regarding the steepness of inclines. If preservice teachers acquire the skill of anticipating student responses, they will be better equipped to address their future students' difficulties (e.g., Wallace, 2007)). In an effort to reveal common student understandings for the purposes of teacher preparation, this article presents information about how seventh grade students responded to a comparison question asking which of two inclines is steeper.

## 2. Methods

The sample for the survey study consisted of 256 students in grade 7 who attended one public middle school. Teachers handed participants the instrument. All students were given unlimited time but most finished in about twenty minutes, on average. Participants did not receive incentives for participating in the study and were told that their participation would not impact their mathematics course grades. The authors had prior relationships with the school and the mathematics teachers; teachers mentioned to participants that this was part of a research study and they expected students to try their best.

In addition, group interviews were conducted during one mathematics period for a class of seventh graders in a small private school. Discussions in each of the groups were facilitated by the authors and the classroom teachers. Facilitators were provided with a list of prompts which would ensure that each student had a chance to be heard but did not guide the discussion in any particular direction.

### *2.1. Instrument*

To assess middle school students' abilities to compare the steepness of lines, the Spider Web Steepness Test was developed, and face validity on the test was confirmed by mathematics and mathematics education experts. The participants in this study could be expected to correctly answer all of the items on the test and they were introduced to nonstandard units of measurement in elementary school.

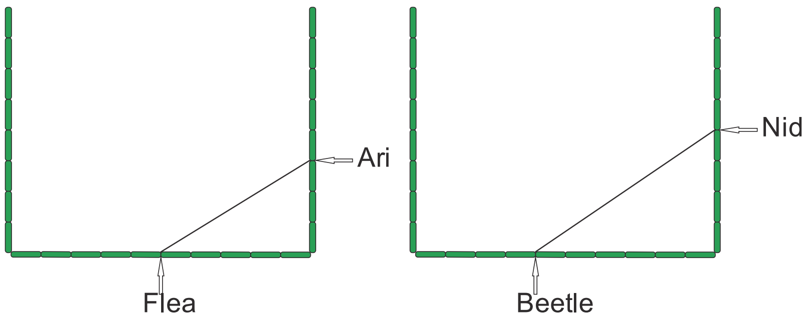
To assess middle school students' responses to steepness problems, the Spider Web Steepness Test was developed, drawing on past research and piloting by the authors (Cheng & Sabinin, 2008; Sabinin & Cheng, 2009) as well as prior research by Noelting (1980). Since Moyer, Cai and Grampp (1997) recommend that instruction on slope begins with comparison activities, 90% of the problems on the

Spider Web Steepness Test were comparison problems. The test includes 9 problems that asked participants to determine which of two drawings was steeper. Each comparison problem asked participants to compare the steepness of two inclines and had three answer choices: 1) left incline is steeper, 2) right incline is steeper, 3) the inclines have the same steepness, or 4) it was not possible to tell. Correct responses earned 1 point and incorrect responses earned 0 points. Students' correct responses indicated that they found productive ways of solving the steepness problems, although the strategies may have been only applicable to specific contexts or structural difficulties. The pairs of slopes of the lines presented in the context of webs are in a variety of difficulty levels, as found by Noelting (1980) in empirical studies. For example, easier pairings of slopes include having equal vertical dimensions in both inclines but different horizontal dimensions. A more difficult pairing of slopes would involve having relatively prime horizontal measures and relatively prime vertical measures. The tenth problem on the Spider Web Steepness Test was a missing value problem involving steepness where participants were asked to create an incline with the same steepness as a given incline.

## 3. A problem involving steepness

One possible way to lead participants to think about steepness using proportions is by providing them with tasks in which it is difficult to determine steepness solely by looking at angles. The following question was given to grade 7 participants:

Two spiders, Ari and Nid, live in rooms with tiles on the two walls and on the floor.
Ari wants to catch a flea from 3 tiles high,
while Nid wants to catch a beetle from 4 tiles high.
Whose web will be steeper?




Circle the best answer:

A. Ari's web to the flea is steeper.
B. Nid's web to the beetle is steeper.
C. Both webs are the same steepness.
D. You cannot tell which web is steeper.

Figure 1: Spider Web Steepness Survey Question 6

This question asked participants to determine which of the two spiders' webs was steeper, and there were four answer choices: Ari's, Nid's, both had the same steepness, or you cannot tell which is steeper. Visually, it was difficult to identify whether the webs had the same steepness by "eye-balling it," because the angles to be compared were close in value. Looking at the angles that the spiders' webs make with the floor, Ari's web is 59.0 degrees and Nid's web is 56.3 degrees. The slopes of the two webs are 3/5 and 4/6, which are non-integral ratios.

Using the vertical wall as the reference line, the left and right angles created are 31.0 and 33.7 degrees respectively, and the slopes are 5/3 and 6/4.

The frequencies of 256 urban and suburban public school participants' responses are recorded in Figure 3.

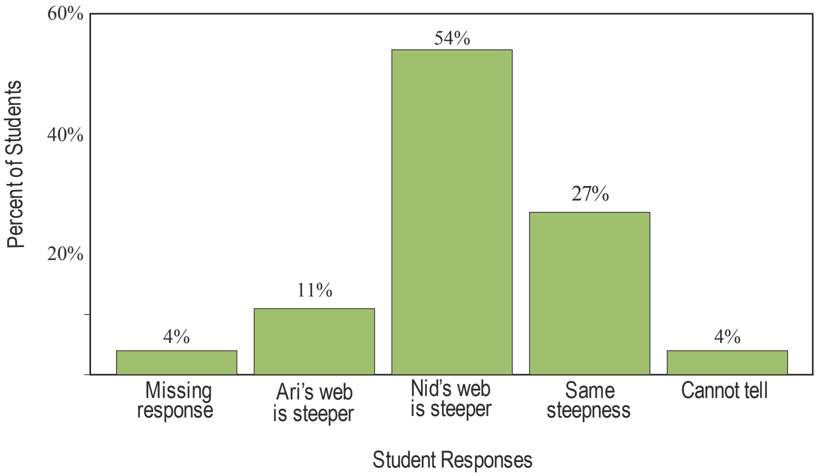



Figure 3: Seventh Grade participants' responses to Steepness Survey Question 6

There are two possible 'correct' answers depending upon the reference line used:

- Nid's web is steeper, using the bottom horizontal line or 'floor' as the reference line, which is reasonable from the flea's and beetle's perspectives.
- Ari's web is steeper, using the vertical line or the 'wall' as the reference line, which is reasonable from Ari's and Nid's perspectives.

Using the flea's and beetle's perspectives is reasonable from a traditional viewpoint of slope. Using Ari's and

Nid's perspectives is reasonable since Ari and Nid are shooting the webs.

Sixty-five percent (65%) of the surveyed participants were able to determine that the two webs do not have the same steepness, and selected either Ari's or Nid's web as steeper. Twenty-seven percent (27%) of the surveyed participants thought that the two webs were equally steep, and the remaining 8% either did not respond or answered that they couldn't tell which web was steeper.

## 4. Observations of participants' reasoning regarding steepness

To observe how participants might justify their answers to these questions, interviews of nineteen 7th graders were conducted. The participants were different students than those who had taken the survey and they had learned to compare fractions and find equivalent fractions in grades 5 and 6. In grade 7, they learned about scaling using proportional models, but had not yet been formally introduced to the idea of slope as a proportion.

During the interviews, groups of three or four participants discussed the same question. The participants used the horizontal floor as their reference line.

Upon first glance, some of the participants thought that the two lines might have the same steepness. In one group interview, one participant supported this claim because she thought that adding one tile to the horizontal and one tile to the vertical would "show more of the same angle." Then another participant in the group drew both lines on the same coordinate plane, "continued the lines" from the bottom and found that they "met." Using the reasoning that "any unparallel lines are eventually going to cross," he correctly stated that the two lines were unparallel and therefore did not have the same steepness.

In another group interview, one participant challenged this first idea that the lines had the same steepness, because he extended the two lines from the top (not on the same coordinate plane) and the lines intersected. Another participant was confused because she extended these two lines from the bottom, and the lines did not intersect but appeared to go further and further apart. After much discussion, these participants concluded that they could determine that the two lines had the same steepness if they never intersected, by extending the lines from either the top or the bottom. The participants also reasoned that the lines did not have the same steepness if they appeared to get closer or further apart.

The participants used two general approaches to solve the problem: geometric and analytic. Participants using a geometric approach compared the two lines to determine whether or not they were parallel in one of two ways: 1) by seeing if the two lines drawn on the same coordinate axes would intersect beyond the page, resulting in the conclusion that two lines do not have the same steepness, 2) by seeing if the two lines not drawn on the same coordinate axes would come closer or further apart, resulting in the conclusion that the two lines do not have the same steepness. In addition, some participants tried to determine which of the angles between their reference lines and the webs was larger, where a larger angle would indicate a steeper line. This was difficult because the angles were visually so close together. A third geometric strategy was to determine which of the triangles underneath the webs had larger area, resulting in the conclusion that the line forming a triangle with larger area is steeper. This can be a problematic way of generalizing steepness because the steepest line, a vertical line, will have zero area underneath it and two similar triangles may have different areas.

Participants using an analytic approach made several comparisons. One comparison was of the ratios 3/5 and 4/6 as the slopes of the lines, resulting in the correct line being identified as steeper. Another comparison was of the differences of the vertical and horizontal changes for each line: $5 - 3 = 2$, $6 - 4 = 2$. Participants using this strategy incorrectly concluded that the lines have the same steepness. Other participants observed that there was a constant difference of one tile between the horizontal dimensions ($3 + 1 = 4$) and the vertical dimensions ($5 + 1 = 6$), resulting in their incorrectly concluding that the two lines had the same steepness. In the coordinate plane, having equal differences between the vertical and horizontal changes for each line will only result in the correct identification of parallel lines in the case that the lines have a slope of 1. This was not the case in Question 6, so using additive reasoning results in an erroneous conclusion.

## Discussion

Comparing extended lines may guide seventh graders to the use the ratio of the vertical and horizontal changes as a measure of steepness. Geometrically, lines which never intersect are parallel and lines which intersect at one point are never parallel. Analytically, these lines can be distinguished by their slopes, and a line whose slope has a higher magnitude is steeper.

Observing participants' discussions was interesting not only because the question had two possible correct solutions, but also because these grade 7 participants' geometric intuitions were often more accurate than their analytic explanations. Using visual cues can help participants connect their geometric and analytic knowledge in situations involving proportional reasoning. Although participants should have numerous opportunities to work with both dimensions and build connections between them, the geometric dimension appears to be more intuitive for children and can begin to develop at an earlier age with the development of the concept of angle. This may be the case because children are generally more comfortable with reasoning involving solely one measurement than reasoning involving multiple measurements (Halford, 1993). In preparing middle school participants for the algebraic study of slope, it may be helpful to connect their understanding of steepness from its angular representation to its fractional representation.

The Common Core State Standards (2011) state that seventh grade students are expected to "*Explain what a point (x, y) on the graph of a proportional relationship means in terms of the situation, with special attention to the points (0, 0) and (1, r) where r is the unit rate*." There is an emphasis on the idea of proportional relationships as a major type of linear function, i.e. they are linear functions that have a positive rate of change through the origin. This knowledge is then built upon in eighth grade, where students are expected to "*Understand the connections between proportional relationships, lines, and linear equations*," particularly between constant of proportionality and slope. The CCSS seems to endorse this learning progression of starting with using graphical representations to explore the idea of proportionality in a simple linear graph prior to students learning formally the concept of slopes in a straight line. However, such connection may not be unveiled in teacher preparation programs and not explicitly be made in textbooks that are often the main source of teacher knowledge. Therefore, it may not be reasonable to expect teachers to be able to make that connection between the two concepts for their students. We contend that the teacher educators play a crucial role in fostering these connections with preservice and in-service teachers.

The results of this study have implications for the teaching of preservice teachers, the design of curriculum in the middle grades, as well as for the choice of curriculum that may help students more fully understand proportional reasoning in light of connections between geometric and analytic representations.

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