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Pythagorean Triples Challenge

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by Tom Moore

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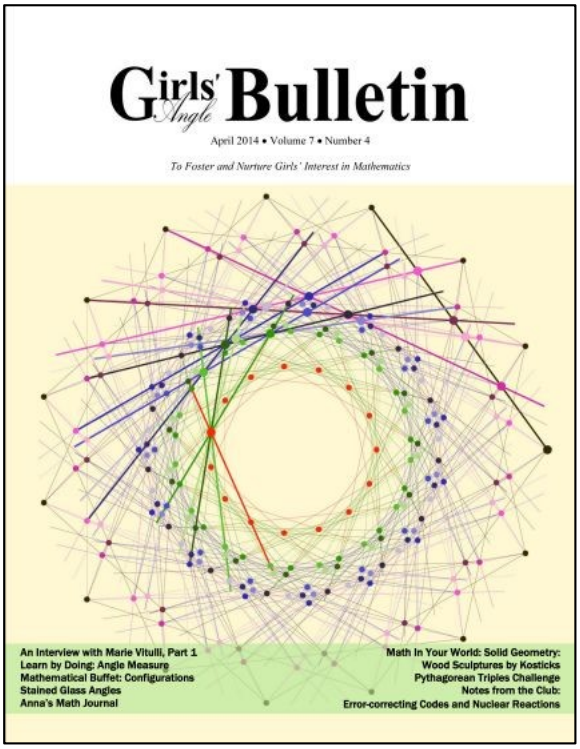

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A Math Club for Girls

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Pythagorean Triples Challenge

by Tom Moore[1] | edited by Jennifer Silva

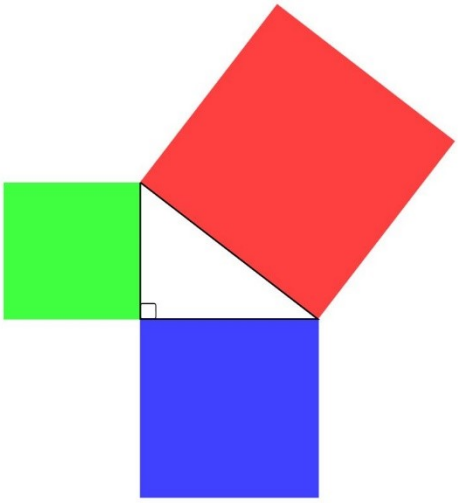
If you know a lot about Pythagorean triples, feel free to skip right to the 5 challenge problems at the end of this article. Otherwise, here is some background information.

A **Pythagorean triple** (a, b, c) is a triple of positive integers that can be used to form the sides of a right triangle with legs of lengths a and b and hypotenuse of length c. According to the Pythagorean theorem, $c^2 = a^2 + b^2$. Conversely, if a, b, and c are positive integers that satisfy the Pythagorean equation $c^2 = a^2 + b^2$, then a, b, and c can be used as the lengths of the sides of a right triangle.

A Pythagorean triple (a, b, c) is called **primitive** if a and b share no common factor other than 1. For example, (3, 4, 5), (5, 12, 13), (8, 15, 17), and (7, 24, 25) are all primitive Pythagorean triples, but (6, 8, 10) is not primitive, even though it is a Pythagorean triple.

Suppose (a, b, c) is a primitive Pythagorean triple. Show that a and b have opposite parity.

A Little History



From the ancient Greek manuscript *Elements*, which was written by Euclid over 2,000 years ago, we learn both the statement and proof of Pythagoras's theorem. In Book I of the *Elements*, we find Proposition 47: *In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle*.[2] That is, the area of the red square is equal to the combined areas of the green and blue squares in the figure at right.

In Book X, Proposition 29, Lemma 1[3], we learn how to generate all primitive Pythagorean triples. Euclid's statement is geometric, but we can put it algebraically, like this: if $m > n$ are positive integers of opposite parity and their highest common factor is 1, then $(2mn, m^2 - n^2, m^2 + n^2)$ is a primitive Pythagorean triple. Furthermore, all primitive Pythagorean triples may be obtained in this manner (switching the leg lengths if necessary so that the even one comes first).

Since every Pythagorean triple can be obtained by scaling a primitive Pythagorean triple by an integer scale factor, the problem of finding all Pythagorean triples is reduced to that of finding all primitive Pythagorean triples. So Euclid's proposition 29 solves the problem of finding Pythagorean triples.

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[1] Thomas Moore is professor emeritus at Bridgewater State University.
[2] See aleph0.clarku.edu/~djoyce/java/elements/bookI/propI47.html.
[3] See aleph0.clarku.edu/~djoyce/java/elements/bookX/propX29.html.

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The first few primitive Pythagorean triples derived using Euclid's Proposition 29 are:

m	n	$2mn$	$m^2 - n^2$	$m^2 + n^2$
2	1	4	3	5
3	2	12	5	13
4	1	8	15	17
4	3	24	7	25
5	2	20	21	29
5	4	40	9	41
6	1	12	35	37
6	5	60	11	61

During the years 800 through 200 BCE, books emerged in India that are now grouped under the name **Sulbasutras**. The following primitive Pythagorean triples appear in the Sulbasutras:

$$(3, 4, 5), \quad (5, 12, 13), \quad (8, 15, 17), \quad (7, 24, 25), \quad (12, 35, 37).$$

There are geometric constructions in the Sulbasutras that lead to algebraic formulas that can be used to produce Pythagorean triples. For example, in one construction of a square with a given area t, the algebraic identity[4]

$$t = \left(\frac{t+1}{2}\right)^2 - \left(\frac{t-1}{2}\right)^2$$

can be inferred. If we substitute $t = (2x + 1)^2$ into this identity, we get the identity

$$(2x + 1)^2 = (2x^2 + 2x + 1)^2 - (2x^2 + 2x)^2.$$

This identity yields the Pythagorean triples $(2x^2 + 2x, 2x + 1, 2x^2 + 2x + 1)$, where x is a positive integer. Similarly, substituting $t = x^2$ into the identity yields Pythagorean triples of the form $(2x, x^2 - 1, x^2 + 1)$, where x is any integer greater than 1. Here's a table of some of the Pythagorean triples produced by these formulas:

x	$(2x^2 + 2x, 2x + 1, 2x^2 + 2x + 1)$
1	(4, 3, 5)
2	(12, 5, 13)
3	(24, 7, 25)
4	(40, 9, 41)
5	(60, 11, 61)

x	$(2x, x^2 - 1, x^2 + 1)$
2	(4, 3, 5)
3	(6, 8, 10)
4	(8, 15, 17)
5	(10, 24, 26)
6	(12, 35, 37)

In ancient Babylon, archeologists unearthed thousands of clay tablets with writing on them from a system called **cuneiform**. One of these tablets, known as Plimpton 322, lists some Pythagorean triples, although it is not known whether the Babylonians interpreted these numbers as the sides of a right triangle. The cuneiform system used a base 60 number system. To learn more about this and try your hand at deciphering the contents of Plimpton 322, look up Plimpton 322 on the internet.

[4] See www.math.tifr.res.in/~dani/pyth.pdf

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Pythagorean Triples Challenges

I have gathered a few problems that I have published over the years related to this topic. You are hereby challenged to try and solve some of them! The last one is a new problem created especially for Girls' Angle.

Girls' Angle thanks Professor Moore for problem #5!

We welcome you to submit your solutions! Send them to girlsangle@gmail.com.

1. From the *Pi Mu Epsilon Journal,* 1993 (used with permission from Steve Miller):

For $a < b < c$ positive integers, if gcd(a, b) = 1 and $a^2 + b^2 = c^2$, then we say (a, b, c) is a primitive Pythagorean. If both a and c are primes, we call it a prime primitive Pythagorean triple. (i) If (a, b, c) is a prime primitive Pythagorean triple, deduce that $b = c - 1$. (ii) Find all prime primitive Pythagorean triples in which a) a and c are twin primes; b) both are Mersenne primes; c) both are Fermat primes; d) one is Mersenne and the other Fermat.

2. From *The Pentagon*, 2012 (used with permission from Pat Costello):

Prove that there are infinitely many primitive Pythagorean triples (a, b, c), such as (5, 12, 13), with hypotenuse c such that the odd leg is a pentagonal number and the even leg is consecutive with the hypotenuse.

3. Submitted to *The Pentagon,* 2013 (used with permission from Pat Costello):

Prove that there are infinitely many Pythagorean triples (a, b, c) with "legs" a and b, one of which is an abundant number and the other a deficient number.

4. From *MathProblems Journal*, 2013 (used with permission from *MathProblems*):

The examples (3, 4, 5), (5, 12, 13), and (13, 84, 85) show that the same odd number may occur as the "hypotenuse" and as the "odd leg" of primitive Pythagorean triples. Provide explicit constructions of such triples to show that there are infinitely many such odd numbers.

5. For the *Girls' Angle Bulletin*:

Let (a, b, c) be a Pythagorean triple. (i) Prove that the highest power of 2 dividing a cannot equal the highest power of 2 dividing b. (ii) Prove the same for the highest power of 3 dividing a.

For More

See www.hps.cam.ac.uk/people/robson/neither-sherlock.pdf for more on Plimpton 322.

For a proof of the Pythagorean theorem, check out the Girls' Angle WIM Video featuring Ina Petkova. Also, check out the *Visual Proof of the Pythagorean Theorem* on the Girls' Angle YouTube channel.

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