



2013

An Investigation Relating Square and Triangular Numbers

Thomas Moore

Bridgewater State University, MOORE@bridgew.edu

Virtual Commons Citation

Moore, Thomas (2013). An Investigation Relating Square and Triangular Numbers. In *Mathematics Faculty Publications*. Paper 37.
Available at: http://vc.bridgew.edu/math_fac/37

This item is available as part of Virtual Commons, the open-access institutional repository of Bridgewater State University, Bridgewater, Massachusetts.

An investigation relating square and triangular numbers.

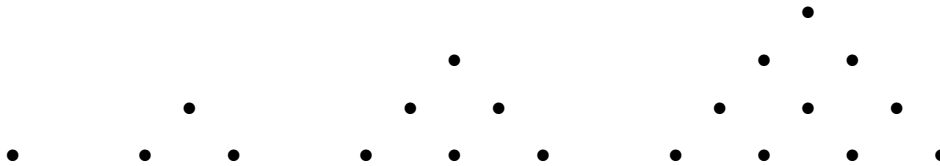
It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. (Gauss, letter to Bolyai, 1808)

Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime. (Polya, How to Solve It, 1945)

In yet another marginal note penned in his copy of Diophantus' *Arithmetica*, Fermat wrote, in 1636, speaking of the positive integers, that ... *every number is either a triangular number or the sum of two or three triangular numbers; every number is a square or the sum of two, three, or four squares; every number is a pentagonal number or the sum of two, three, four, or five pentagonal numbers; and so on ad infinitum, for hexagons, heptagons, and any polygons whatever...* He called it a beautiful result, a wonderful theorem. Now the case for four squares was proven correct by Lagrange in 1770. Later, on the day of 10 July, 1796, Gauss famously wrote "Eureka, num = $\Delta + \Delta + \Delta$ " in his diary. He shared his proof of this in 1801 in his famous and impactful book *Disquisitiones Arithmeticae*. More exactly, we have the

Theorem (Gauss). Every positive integer is the sum of at most three triangular numbers.

Now the notion of a polygonal or figurate or n -gonal number, such as the triangular and square numbers, goes back to ancient Greece and the Pythagoreans. Since unity, the number 1, had such special significance to these ancients, they did not count it as a polygonal number. However, it is now customary, as it was for Gauss, to do so and, in particular, 1 is an n -gonal number for all $n \geq 3$. Triangular numbers are so-called because they can be interpreted as the total number of dots in the pictures below, which images can be extended indefinitely.



Thus, the sequence of triangular numbers begins 1, 3, 6, 10, 15, 21, 28, Notice that each picture incorporates the previous picture with a new row of n dots below it.

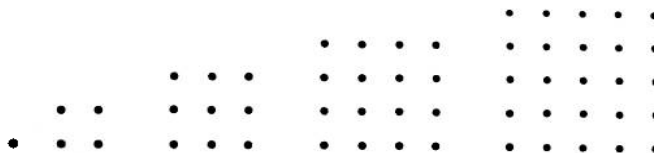
The phrase "at most" appears in our statement of Gauss' theorem because, after all, the only representations of 1, 2 and 4 in this way are as $1 = 1$, $2 = 1+1$ and $4 = 1 + 3$ each of which uses fewer than three triangular numbers. The number $5 = 1 + 1 + 3$ is the first to require three such.

In general, we let T_n be the n th triangular number, $n \geq 1$, so that $T_1 = 1, T_2 = 3, T_3 = 6$ and so on. We will refer to n as the index of T_n . If we add the number of dots from top to bottom in each triangular array of dots in the picture above, we can reveal that $T_n = 1 + 2 + 3 + \dots + n$.

This expression can be manipulated to show that $T_n = \frac{n(n+1)}{2}$. Indeed there is a well-known anecdote about Gauss as a young boy applying this formula. Perhaps he calculated like this:

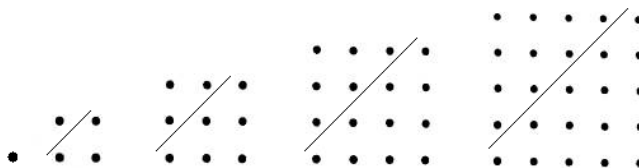
$$2T_n = T_n + T_n = (1 + 2 + 3 + \dots + n) + (n + (n-1) + (n-2) + \dots + 1) = (n+1) + (n+1) + \dots + (n+1) = n(n+1)$$

Of course the square numbers are the number of dots in each of the square arrays pictured below, which can be extended indefinitely.



So the squares are the familiar sequence 1, 4, 9, 16, ... with general term $S_n = n^2$.

There is a very nice result connecting these two sequences. It is suggested by these images:



Evidently each square number 4, 9, 16, ... is the sum of two consecutive triangular numbers. In fact it is easy to verify algebraically that $S_n = T_n + T_{n-1}$, but in this case the pictures convince us!

To the teacher: this activity we are engaging in fits under the Ontario curriculum guidelines of Mathematical Process Expectations [1]. There we find the statements that students will develop, select, apply, compare, and adapt a variety of problem-solving strategies as they pose and solve problems and conduct investigations, to help deepen their mathematical understanding. As well as, students will develop and apply problem-solving skills (e.g. the use of inductive reasoning, deductive reasoning, and counter-examples; construction of proofs) to make mathematical conjectures.

Even if you only take the time in class to introduce the polygonal numbers and to allow some experimentation with them on worksheets, perhaps based on this paper, your students will benefit from

the mathematical inductive process. For example coins are easy to use to construct the triangular numbers. Here is $T_4 = 10$.



I became curious as to which square numbers were sums of *exactly* three triangular numbers. This prompted me to write a very simple *Maple* (copyright, University of Waterloo) program to generate some data. (All programs referred to in the article are in an Appendix at the end of it.

Here is the output.

{9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961, 1024, 1089, 1156, 1225, 1296, 1369, 1444, 1521, 1600, 1681, 1764, 1849, 1936, 2025, 2116, 2209, 2304, 2401, 2500, 2601, 2704, 2809, 2916, 3025, 3136, 3249, 3364, 3481, 3600, 3721, 3844, 3969, 4096, 4225, 4356, 4624}

This set contains the squares of all the numbers from 3 to 66 as well as the square of 68. The omission of $67^2 = 4489$ is due to the range of values for a , b and c (see the appendix, program #1). Indeed, we have $4489 = T_{37} + T_{59} + T_{63} = 703 + 1770 + 2016$. Thus, surprisingly, the data suggests that *every square number beyond 4 is the sum of exactly three triangular numbers!* Naturally I wanted to prove this conjecture. In order to get a handle on it, I wrote another program (Appendix, program #2) to see the actual T_a , T_b and T_c involved in representing each square. Here, summarized in a table, is some of that output.

a	b	c	T_a	T_b	T_c	d
2	2	2	3	3	3	9
2	2	4	3	3	10	16
1	2	6	1	3	21	25
3	5	5	6	15	15	36
1	2	9	1	3	45	49
2	4	8	3	10	36	49
3	5	7	6	15	28	49
2	3	10	3	6	55	64
5	6	7	15	21	28	64
5	6	9	15	21	45	81
4	9	9	10	45	45	100
6	9	10	21	45	55	121
2	6	15	3	21	120	144
8	10	12	36	55	78	169
6	10	15	21	55	120	196
3	11	17	6	66	153	225
5	14	14	15	105	105	225
9	13	15	45	91	120	256
9	13	17	45	91	153	289
8	12	20	36	78	210	324
14	15	16	105	120	136	361

The output showed that many squares “d” have different representations as sums of three triangular numbers. I have included only a few examples of this in the above table, namely for the squares d equal to 49 and 64 and 225. In fact, I was struck by the proliferation of the different ways that a square number could be so written. These multiple representations presented a difficulty for me as I sought to examine the data for a pattern or patterns that would supply a direction of proof of the conjecture. Alas, I did not succeed in that grand endeavor. Consequently, I sought to prove a lesser result:

There are infinitely many squares each the sum of exactly three triangular numbers.

Combing through the data again and again, I eventually had my own EUREKA moment. I looked at the squares of just the triangular numbers and saw how they were expressible. The table above shows that

$$\begin{aligned}
 9 &= T_2^2 = 3 + 3 + 3 = T_2 + T_2 + T_2 \\
 36 &= T_3^2 = 15 + 15 + 6 = T_5 + T_5 + T_3 \\
 100 &= T_4^2 = 45 + 45 + 10 = T_9 + T_9 + T_4 \\
 225 &= T_5^2 = 105 + 105 + 15 = T_{14} + T_{14} + T_5
 \end{aligned}$$

Each expression is in the form $T_n^2 = T_a + T_a + T_x$, where the repeated subscript a is itself one less than a triangular number, that is, $a = T_n - 1$. In fact, looking closely at these four equations, a beautiful pattern emerges from the data. Our new claim is this:

$$(1) \quad T_n^2 = T_{T_n-1} + T_{T_n-1} + T_n, n \geq 2$$

The actual verification of this equation is easy, just straightforward algebra:

$$T_{T_n-1} + T_{T_n-1} + T_n = \frac{(T_n - 1)T_n}{2} + \frac{(T_n - 1)T_n}{2} + T_n = (T_n - 1)T_n + T_n = T_n^2 - T_n + T_n = T_n^2$$

To my eye it is certainly a beautiful relationship, and, perhaps a new discovery on the triangular numbers. The equation (1) proves the claim that there are infinitely many squares that are the sum of exactly three triangular numbers.

This discovery led me to experiment with variations involving the square numbers $S_n = n^2$ and the pentagonal numbers $P_n = \frac{n(3n-1)}{2}$. Again the pentagonal numbers may be introduced in the classroom using coins or counters, even candies: for example, here are $P_1 = 1, P_2 = 5$ and $P_3 = 12$:



To my surprise, I found that

$$(2) \quad S_n^2 = T_{S_n-1} + T_{S_n-1} + S_n, n \geq 2$$

and

$$(3) \quad P_n^2 = T_{P_n-1} + T_{P_n-1} + P_n, n \geq 2.$$

These equations (1), (2) and (3) are just like each other and suggest an obvious generalization. If we let

N_n be the nth N-gonal number, that is, $N_n = \left(\frac{N}{2} - 1\right)n^2 - \left(\frac{N}{2} - 2\right)n$, then

$$(4) \quad N_n^2 = T_{N_n-1} + T_{N_n-1} + N_n, n \geq 2$$

and, in fact, this generalization is true, algebraically verifiable, as was the verification of equation (1). Here is that verification:

$$T_{N_n-1} + T_{N_n-1} + N_n = \frac{(N_n-1)N_n}{2} + \frac{(N_n-1)N_n}{2} + N_n = (N_n-1)N_n + N_n = N_n^2 - N_n + N_n = N_n^2$$

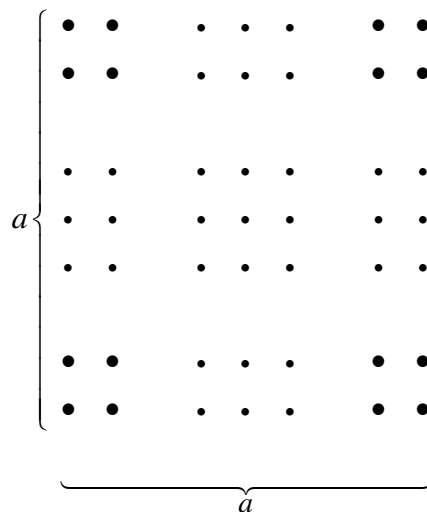
These relationships on polygonal numbers, while delightful, seemed to me a bit too flexible. I began to suspect that the truths expressed by these equations were in some way stating the obvious. And so I asked myself, is the relation (4) peculiar to the squares of polygonal numbers or is it more widely applicable?

Could it be that, for every positive integer $a > 1$, we have

$$(5) a^2 = T_{a-1} + T_{a-1} + a?$$

Well, yes! And this is easily proved algebraically, as above, but also geometrically, in the spirit of the Pythagoreans, as follows.

Consider this diagram:



If we section off the last column of a dots, then the figure decomposes into an oblong, that is, a rectangle a by $a - 1$ along with the column of a dots. The Pythagorean, Theon of Smyrna, circa 100CE, knew that an oblong number is the sum of two equal triangular numbers. So we have

$$(5) a^2 = (a-1)a + a = \frac{(a-1)a}{2} + \frac{(a-1)a}{2} + a = T_{a-1} + T_{a-1} + a$$

This investigation was a tale of numerical exploration, easily duplicated in the classroom and emphasizes the inductive process of gathering data, looking for a pattern and proving a result that may, indeed, be very pleasing, as the result (1) seems to the author, and may also lead to a generalization, as here we have in (4), although somewhat removed from the original question. The fact that (4) is a simple consequence of the last equation (5) above does not lessen the joy of this discovery process.

Reference:

1. The Ontario Curriculum Grades 11 and 12, Mathematics, 2007, p.112.

Appendix.

Maple program #1.

```
> S:=NULL: for a from 1 to 60 do
> for b from 1 to 60 do
> for c from 1 to 60 do
> d:=a*(a+1)/2 + b*(b+1)/2 + c*(c+1)/2 :
> if floor(sqrt(d)) = sqrt(d) then S := S union {d}: fi: od: od: od: print(S):
```

The last line of this code is checking to see if the square root of the sum d of three triangular numbers, indexed by a , b and c , is an integer. If so, then d is a square and we include it in the set S . At the end we print out the square numbers that have landed in S .

Maple program #2.

```
> for a from 1 to 20 do
> for b from 1 to 20 do
> for c from 1 to 20 do
> d:=a*(a+1)/2 + b*(b+1)/2 + c*(c+1)/2:
> if floor(sqrt(d)) = sqrt(d) then print(a, b, c, a*(a+1)/2 ,b*(b+1)/2 ,c*(c+1)/2,d): fi: od: od: od:
```


Tom Moore
Professor Emeritus
Mathematics Department
Conant Science and Mathematics Center
Bridgewater State University
Bridgewater, MA 02325

27 June, 2013