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Two Notes on Certain Relationships among Triangular Numbers

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We became interested as to what cubes of positive integers could be expressed as the difference of integer squares. A familiar algebraic maneuver came to mind. Suppose that $x^3 = a^2 - b^2$ and $y^3 = c^2 - d^2$. Then $(xy)^3 = x^3y^3 = (a^2 - b^2)(c^2 - d^2) = a^2c^2 - a^2d^2 - b^2c^2 + b^2d^2$, and so

$(xy)^3 = a^2c^2 + 2abcd + b^2d^2 - a^2d^2 - 2abcd - b^2c^2 = (ac + bd)^2 - (ad + bc)^2$. This mechanism provides a way to generate infinitely many cubes of positive integers as such a difference, perhaps necessitating the use of absolute values applied to the last expression. As a matter of fact, it is clear that nth powers may replace the cubes above. However, sticking to cubes will pay a dividend. Consider these particular examples without reading beyond them:

$$3^2 - 1^2 = 8$$
$$6^2 - 3^2 = 27$$
$$10^2 - 6^2 = 64$$
$$15^2 - 10^2 = 125$$

I hope the reader had the thrill of discovering that the squares used are the squares of consecutive *triangular numbers*. Recall that these numbers form the sequence 1, 3, 6, 10, 15, 21, … , with general term $t_n = \frac{n(n+1)}{2}, \forall n \geq 1$. This immediately suggests that $t_n{}^2 - t_{n-1}{}^2 = n^3, \forall n \geq 2$, a fact that is easily verified algebraically.

We also observed that

$$1^2 + 3^2 = 10 = T_4$$
$$3^2 + 6^2 = 45 = T_9$$
$$6^2 + 10^2 = 136 = T_{16}$$
$$10^2 + 15^2 = 325 = T_{25}$$

suggesting that $T_n{}^2 + T_{n+1}{}^2 = T_{T_n + T_{n+1}}, n \geq 1$, and this pleasant fact is easily verified algebraically.

Also, we can show that there are infinitely many triangular numbers satisfying $T_a + T_b = T_c^2$.

The computations

$$T_2 + T_3 = 3 + 6 = 9 = T_2^2$$
$$T_5 + T_6 = 15 + 21 = 36 = T_3^2$$
$$T_9 + T_{10} = 45 + 55 = 100 = T_4^2$$
$$T_{14} + T_{15} = 105 + 120 = 225 = T_5^2$$

suggest that $T_{T_n - 1} + T_{T_n} = T_n^2, n \geq 2$, again verifiable algebraically.