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Common Fundamental Domains for Lattices of the Same Volume

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1 Introduction

Lattices are regular arrangement of points in Euclidean space. From a group theory viewpoint, lattices are some special subgroups of $\mathbb{R}^d$ which are of the form $M\mathbb{Z}^d$ where M is a nonsingular matrix of order d. They naturally occur in many areas such as Harmonic Analysis, Geometry and Number Theory. They have many applications in the representation of data, programming problems, cryptanalysis, multi antenna systems, and in the Theory of Communication. In the Theory of Communication, Dennis Gabor proposed in 1946 the use of a family of functions obtained from one Gaussian by time and frequency shifts (modulation). This collection of the shifted Gaussian is parametrized by a lattice in the time-frequency plane which is naturally identified with $\mathbb{R}^{2d}$. Together this collection of functions are meant to constitute a complete set of building blocks for a set of functions which satisfy some precise integrability condition. His proposal led to what is nowadays called Gabor analysis [3, 4, 6, 1, 2]

A starting point for the work presented in this thesis is due to a famous result in Gabor analysis proved by Han and Wang. Let $\Gamma_1 = M\mathbb{Z}^d$ and $\Gamma_2 = N\mathbb{Z}^d$ be two lattices such that M and N are some non-singular matrices of order d and $|\det M| = |\det N|$. Then there is at least one set E such that $\{E + \gamma : \gamma \in \Gamma_1\}$ and $\{E + \gamma : \gamma \in \Gamma_2\}$ both form a tiling of the Euclidean space $\mathbb{R}^d$. E is called a **common fundamental domain** for the lattices Γ_1 and Γ_2. A direct application of this remarkable result in Gabor theory (communication theory) is that once a common fundamental domain is explicitly constructed, then its characteristic function generates an orthonormal basis in the space of all square-integrable functions defined over $\mathbb{R}^d$. Moreover, this orthonormal basis is appealing in practice because it is parametrized by the lattice $\Gamma_1 \times \Gamma_2 \subset \mathbb{R}^{2d}$.

Since the proof given by Han and Wang is not constructive, in the present work we focus our attention on the particular case where $d = 2$. The main theme of this thesis is then to present explicit constructions of common fundamental domains for a class of rational lattices in a two-dimensional setting. Our work is organized as follows. In the second section, we review some important definitions and we state the results of Han and Wang which influenced our work. We also provide an example of a common fundamental domain for two lattices of the same volume. In the third section, we consider two distinct pairs of lattices of the same volume. Namely, for any natural number n, we consider the pairs

$$\Gamma_1 = \begin{bmatrix} 1/n & 0 \\ 0 & n \end{bmatrix} \mathbb{Z}^2 \text{ and } \Gamma_2 = \mathbb{Z}^2$$

and

$$\Gamma_1 = \begin{bmatrix} 1/n & 0 \\ n & n \end{bmatrix} \mathbb{Z}^2 \text{ and } \Gamma_2 = \mathbb{Z}^2$$

for which we construct explicit common fundamental domains.

2 Preliminaries

Let V be a d-dimensional $\mathbb{R}$-vector space. A lattice in V is a subgroup of the form

$$\Gamma = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_m$$

with linearly independent vectors $v_1, \cdots, v_m$ of V. The m-tuple $(v_1, \cdots, v_m)$ is called a basis for the lattice and the set

$$\Lambda = \{x_1 v_1 + \cdots x_m v_m : x_k \in \mathbb{R}, 0 \leq x_k < 1\}$$

is a fundamental domain of the lattice. The lattice is called complete or full-rank if $m = d$.

Proposition 1 *A lattice Γ is full-rank if and only if there exists a bounded subset $M \subset V$ such that the collection of all translates $M + \gamma, \gamma \in \Gamma$ covers the entire space V.*

Proof. If $\Gamma = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_d$ is complete then one may take M to be the fundamental domain $\Lambda = \{x_1 v_1 + \cdots x_d v_d : x_k \in \mathbb{R}, 0 \leq x_k < 1\}$. Conversely, let M be a bounded subset of V whose translates $\{M + \gamma : \gamma \in \Gamma\}$ cover V. Let W be the subspace spanned by Γ. We have to show that $V = W$. Let $v \in V$. Since

$$V = \bigcup_{\gamma \in \Gamma} M + \gamma$$

we may write for each $nv = a_n + \gamma_n$ where $a_n \in M$, $\gamma_n \in \Gamma \subset W$. Since M is bounded, $\frac{1}{n} a_n$ converges to zero and since W is closed, then

$$v = \lim_{n \to \infty} \frac{1}{n} a_n + \frac{1}{n} \gamma_n = \frac{1}{n} \gamma_n \in W$$

This concludes the proof. ■

Let us consider the commutative group $\mathbb{R}^2$ endowed with addition. A **full-rank lattice** Γ of $\mathbb{R}^2$ is a discrete subset of $\mathbb{R}^2$ which is defined as $\Gamma = M\mathbb{Z}^2$ where M is an invertible matrix. More precisely, we have

$$M\mathbb{Z}^2 = \left\{ M \begin{bmatrix} k \\ n \end{bmatrix} \in \mathbb{R}^2 : k, n \in \mathbb{Z} \right\}$$

where

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and $\det(M) = ad - bc \neq 0$. Moreover the following facts hold true.

1. The quantity $|\det(M)|$ is called the **volume** of $\Gamma = M\mathbb{Z}^d$.
2. The group Γ is cocompact in $\mathbb{R}^2$ in the sense that any set of representatives of the quotient group $\mathbb{R}^2/\Gamma$ is compact.

Definition 2 *A **fundamental domain** of a lattice of $\mathbb{Z}^2$ is a subset Λ of $\mathbb{R}^2$ such that the collection of sets*

$$\left\{ \Lambda + \begin{bmatrix} k \\ n \end{bmatrix} : (k, n) \in \mathbb{Z}^2 \right\}$$

forms a partition of $\mathbb{R}^2$. We say that Λ is a Γ-tiling set for $\mathbb{R}^2$

Proposition 3 *For every full-rank lattice Γ, there exists a countably infinite number of fundamental domains.*

Proof. Put $\Gamma = M\mathbb{Z}^d$ where M is a non-singular matrix. Now, let us define

$$E = M\,[0,1)^d = \left\{ Mx : x \in [0,1)^d \right\}.$$

Put

$$B_1 = \left\{ (x_1, \cdots, x_d) \in [0,1)^d : x_d \geq \frac{1}{2} \right\} \subset \mathbb{R}^d,$$
$$B_2 = \left\{ (x_1, \cdots, x_d) \in [0,1)^d : x_d < \frac{1}{2} \right\} \subset \mathbb{R}^d.$$

Clearly $\{B_1, B_2\}$ forms a partition of $[0,1)^d$. Writing

$$E = MB_1 \cup MB_2$$

such that

$$MB_k = \{Mx : x \in B_k\} \text{ for each } k \in \{1,2\}$$

then $\{MB_1, MB_2\}$ forms a partition of E. Next, for every natural number n, we define

$$A_n = (MB_1 + M(0, \cdots, 0, n)) \cup (MB_2 + M(0, \cdots, 0, -n)).$$

The collection of sets $\{A_n : n \in \mathbb{N}\}$ is a countably infinite collection of fundamental domains for the lattice Γ. ■

From the proposition above, it is obvious that there are several ways of constructing fundamental domains for a given full-rank lattice. Let us describe below a general procedure which is used to construct a fundamental domain for a given lattice subgroup of $\mathbb{R}^d$. Let $\Gamma = M\mathbb{Z}^d$ where M is some invertible matrix of order n.

1. Let $E = M[0,1)^d$. Then E is a fundamental domain for the lattice Γ. We call this fundamental domain the canonical fundamental domain for the lattice Γ.
2. Let $\{E_k : 1 \le k \le n\}$ be a finite partition of E consisting of sets of positive area
3. Define $I = \{z_k : 1 \le k \le n\} \subset \mathbb{Z}^d$
4. Put
$$E_I = \bigcup_{z_k \in I} (E_k + Mz_k) = \bigcup_{k=1}^{n} (E_k + Mz_k)$$

Example 4 *Let*

$$\Gamma = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \mathbb{Z}^2.$$

Since

$$\det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \neq 0$$

then a fundamental domain for Γ *is*

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} [0,1)^2 .$$

Moreover, let

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} [0,1)^2 = A_1 \cup A_2$$

such that each A_k,$k \in \{1,2\}$ *is a subset of positive area. Now ,let*

$$\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}$$

be two distinct elements of $\mathbb{Z}^2$. *Then*

$$\left(A_1 + \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}\right) \cup \left(A_2 + \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}\right)$$
$$= \left(A_1 + \begin{bmatrix} a+2b \\ 2a+3b \end{bmatrix}\right) \cup \left(A_2 + \begin{bmatrix} c+2d \\ 2c+3d \end{bmatrix}\right)$$

is a fundamental domain for Γ *as well.*

Example 5 *Let*

$$\Gamma_1 = M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \mathbb{Z}^2$$

then the fundamental domain of Γ_1 *is*

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} [0,1)^2$$

The picture of the fundamental domain is displayed below.

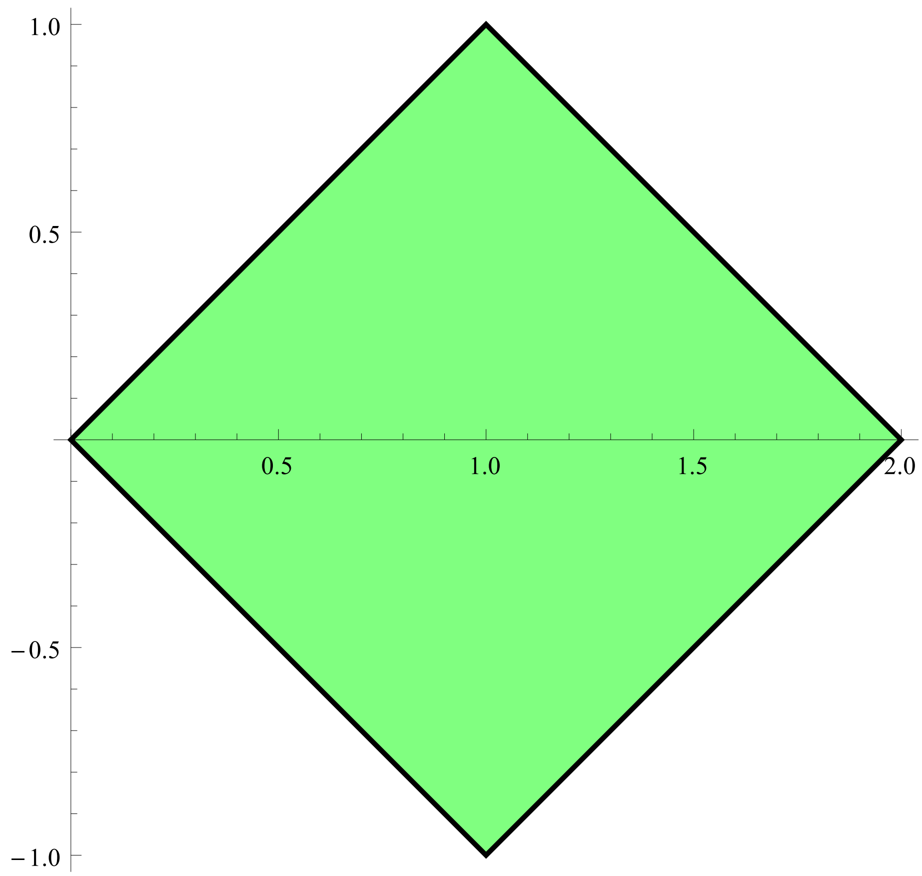

Figure 1:

Moreover, let

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} [0,1)^2 = A_1 \cup A_2$$

such that each A_k is a subset of positive area. Now ,let

$$\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}$$

be two distinct elements of $\mathbb{Z}^2$. Then

$$\left(A_1 + \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}\right) \cup \left(A_2 + \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}\right)$$
$$= \left(A_1 + \begin{bmatrix} a+b \\ -a+b \end{bmatrix}\right) \cup \left(A_2 + \begin{bmatrix} c+d \\ -c+d \end{bmatrix}\right)$$

is a fundamental domain for Γ *as well.*

2.1 Wang-Han Theorem

We will now present a remarkable result due to Wang and Han.

Theorem 6 *Let* Γ_1 *and* Γ_2 *be two full-rank lattices of* $\mathbb{R}^d$ *of the same volume. Then there is a set* Λ *which is a common fundamental domain for both* Γ_1 *and* Γ_2*. More precisely, there exists a set* $E \subset \mathbb{R}^d$ *such that*

$$\{E + \gamma : \gamma \in \Gamma_1\}, \textit{and } \{E + \gamma : \gamma \in \Gamma_2\}$$

both tile $\mathbb{R}^d$ *and area of* $E = |Vol\,(\Gamma_1)| = |Vol\,(\Gamma_2)|$ *.*

For a complete proof of the above, the interested reader is invited to refer to [2]

Being that the proof offered by Wang and Han is not constructive, we will explore the particular case where $d = 2$. We will study a class of lattices of the same volume. For the class of lattices considered in our work, we will construct some explicit common fundamental domains for the family of lattices. We present here an example when one lattice is quincunx and the other is $\mathbb{Z}^2$.

Example 7 *Let*

$$\Gamma_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbb{Z}^2 \textit{ and } \Gamma_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \mathbb{Z}^2$$

It is easy to see that Γ_1 *and* Γ_2 *have the same volume. Indeed,*

$$Vol(\Gamma_1) = \left| det \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right| = \left| det \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right| = 2$$

Let E_1 *and* E_2 *be fundamental domains for* Γ_1 *and* Γ_2 *respectively. Below, we display the fundamental domains for each.*

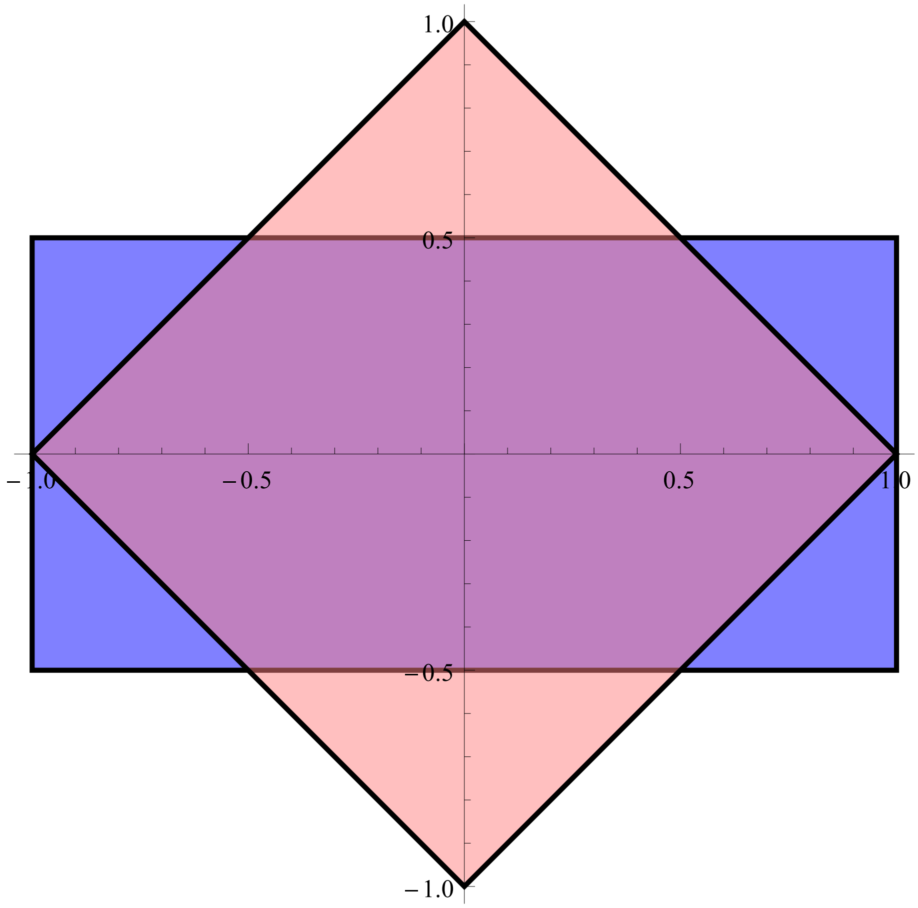

Figure 2:

The set in pink is E_1 *and the set in blue is* E_2*. Let*

$$E_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} [-\frac{1}{2}, \frac{1}{2})^2$$

and

$$E_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} [-\frac{1}{2}, \frac{1}{2})^2$$

Moreover, let

$$\begin{aligned}E_1 &= (E_1 \cap E_2) \cup (E_1 - E_2) \\ &= \bigcup_{k=1}^{5} (E_{1_k}) \\ &= E_{1_1} \cup (E_{1_2} \cup E_{1_3} \cup E_{1_4} \cup E_{1_5}) \\ &= (E_1 \cap E_2) \cup (E_{1_2} \cup E_{1_3} \cup E_{1_4} \cup E_{1_5})\end{aligned}$$

and let

$$\begin{aligned}E_2 &= (E_2 \cap E_1) \cup (E_2 - E_1) \\ &= \bigcup_{k=1}^{5} (E_{2_k}) \\ &= E_{2_1} \cup (E_{2_2} \cup E_{2_3} \cup E_{2_4} \cup E_{2_5}) \\ &= (E_1 \cap E_2) \cup (E_{2_2} \cup E_{2_3} \cup E_{2_4} \cup E_{2_5})\end{aligned}$$

such that the following pairs are congruent triangles:

$$E_{1_1}, E_{2_1}$$

$$E_{1_2}, E_{2_2}$$

$$E_{1_3}, E_{2_3}$$

$$E_{1_4}, E_{2_4}$$

$$E_{1_5}, E_{2_5}$$

Please observe that the boundary coordinates for each section are as follows:

$$E_{1_1}; E_{2_1} : \left(\frac{1}{2}, \frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}\right), (-1, 0), \left(-\frac{1}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right), (1, 0)$$

$$E_{1_2} : (-1, 0), \left(0, -\frac{1}{2}\right), \left(\frac{1}{2}, -\frac{1}{2}\right); E_{2_2} : (-1, 0), \left(-1, \frac{1}{2}\right), \left(-\frac{1}{2}, \frac{1}{2}\right)$$

$$E_{1_3}:\left(0,\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right),(0,1)\,;E_{2_3}:\left(-1,-\frac{1}{2}\right),\left(-\frac{1}{2},-\frac{1}{2}\right),(-1,0)$$
$$E_{1_4}:\left(0,\frac{1}{2}\right),(0,1),\left(-\frac{1}{2},\frac{1}{2}\right);E_{2_4}:\left(1,-\frac{1}{2}\right),(1,0),\left(\frac{1}{2},-\frac{1}{2}\right)$$
$$E_{1_5}:\left(0,-\frac{1}{2}\right),\left(-\frac{1}{2},-\frac{1}{2}\right),(0,-1)\,;E_{2_5}\left(1,\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{2}\right),(1,0)$$

To find the common fundamental domain, it is enough to find

$$\gamma_1,\gamma_2,\gamma_3,\gamma_4\in\Gamma_1,\alpha_1,\alpha_2,\alpha_3,\alpha_4\in\Gamma_2$$

such that

$$E_{2_1}+\alpha_1=E_{1_1}+\gamma_1$$

$$E_{2_2}+\alpha_2=E_{1_2}+\gamma_2$$

$$E_{2_3}+\alpha_3=E_{1_3}+\gamma_3$$

$$E_{2_4}+\alpha_4=E_{1_4}+\gamma_4$$

$$E_{2_5}+\alpha_5=E_{1_5}+\gamma_5$$

To find the values of each γ_k and α_k, it is necessary to find the coordinates of the vertices of each $E_{1_k}$and E_{2_k} so that we can set up different systems of equations to know how to shift each piece by the appropriate lattice point. Knowing that $k_1,k_2,j_1,j_2\in\mathbb{Z}$,as well as the fact that γ_k is in the form of $\begin{bmatrix} k_1-k_2 \\ k_1+k_2 \end{bmatrix}$ and α_k is in the form of $\begin{bmatrix} 2j_1 \\ j_2 \end{bmatrix}$ we can deduce the following information. We only need to examine one vertex from each fundamental domain because when looking at more than one, it becomes redundant.

$E_{2_1}$and $E_{1_1}$are already overlapping so we do not need to move them by $\gamma_1$and α_1. Therefore,$\gamma_1=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\alpha_1=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

For $E_{2_2}$and E_{1_2}we have:

$$\begin{cases} -1+2j_1=k_1-k_2 \\ \frac{1}{2}+j_2=-\frac{1}{2}+k_1+k_2 \end{cases}$$

From this, we have

$$j_1 = \frac{k_1 - k_2 + 1}{2}$$

$$j_2 = k_1 + k_2 - 1$$

It suffices to pick

$$k_1 = 2, k_2 = 1$$

so that

$$j_1 = 1, j_2 = 2$$

Therefore, $\gamma_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ *and* $\alpha_2 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

For E_{2_3} *and* E_{1_3} *we have:*

$$\begin{cases} -1 + 2j_1 = k_1 - k_2 \\ -\frac{1}{2} + j_2 = \frac{1}{2} + k_1 + k_2 \end{cases}$$

From this, we have

$$j_1 = \frac{k_1 - k_2 + 1}{2}$$

$$j_2 = k_1 + k_2 + 1$$

It suffices to pick

$$k_1 = 2, k_2 = 1$$

so that

$$j_1 = 1, j_2 = 4$$

Therefore, $\gamma_3 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ *and* $\alpha_3 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

For E_{2_4} *and* E_{1_4} *we have:*

$$\begin{cases} 1 + 2j_1 = k_1 - k_2 \\ -\frac{1}{2} + j_2 = \frac{1}{2} + k_1 + k_2 \end{cases}$$

From this, we have

$$j_1 = \frac{k_1 - k_2 - 1}{2}$$

$$j_2 = k_1 + k_2 + 1$$

It suffices to pick

$$k_1 = 6, k_2 = 3$$

so that

$$j_1 = 1, j_2 = 10$$

Therefore, $\gamma_4 = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$ *and* $\alpha_4 = \begin{bmatrix} 2 \\ 10 \end{bmatrix}$

For E_{2_5} *and* E_{1_5} *we have:*

$$\begin{cases} 1 + 2j_1 = k_1 - k_2 \\ \frac{1}{2} + j_2 = -\frac{1}{2} + k_1 + k_2 \end{cases}$$

From this, we have

$$j_1 = \frac{k_1 - k_2 - 1}{2}$$

$$j_2 = k_1 + k_2 - 1$$

It suffices to pick

$$k_1 = 6, k_2 = 3$$

so that

$$j_1 = 1, j_2 = 8$$

Therefore,$\gamma_5 = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$ *and* $\alpha_5 = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$

If we put

$$A_1 = E_{1_1}$$

$$A_2 = E_{1_2}$$

$$A_3 = E_{1_3}$$

$$A_4 = E_{1_4}$$

$$A_5 = E_{1_5}$$

then the common fundamental domain is given as follows

$$E = \bigcup_{k=1}^{5} (A_k + \gamma_1)$$

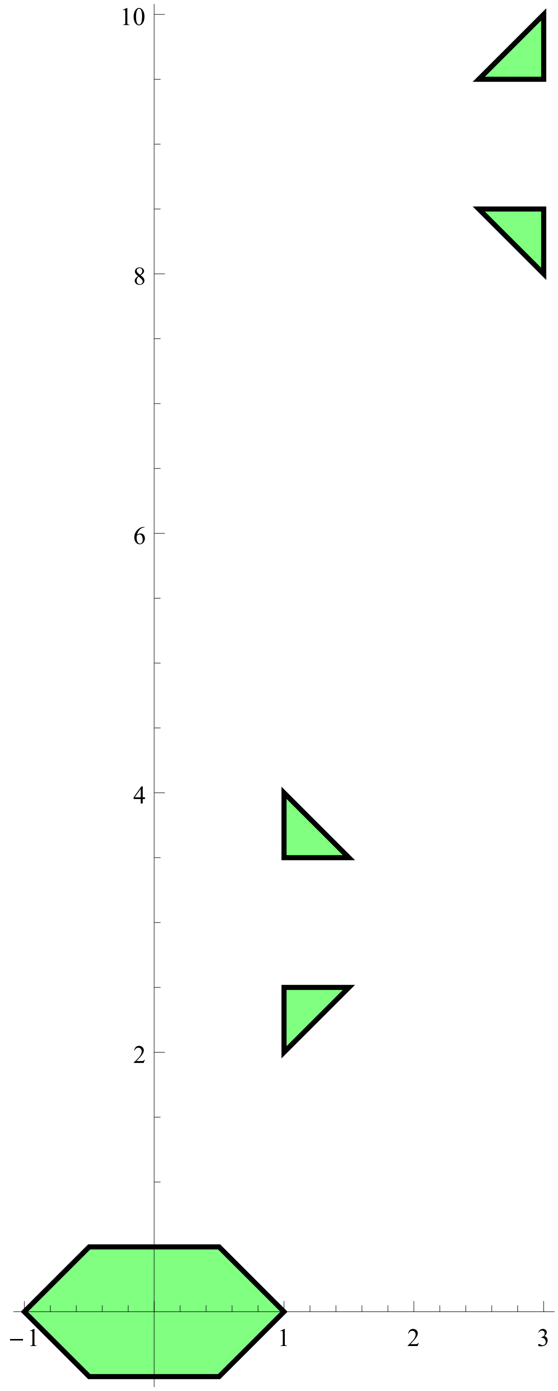

Figure 3:

3 Constructions of Common Fundamental Domains for Lattices

3.1 A Case of Two Rectangular Lattices

Let n be a natural number larger than one. Define two full-rank lattices

$$\Gamma_1 = \begin{bmatrix} 1/n & 0 \\ 0 & n \end{bmatrix} \mathbb{Z}^2 \text{ and } \Gamma_2 = \mathbb{Z}^2$$

of the same volume. We want to find a set $E \subset \mathbb{R}^2$ such that E is a common fundamental domain for both Γ_1 and Γ_2. We observe here that

$$\Gamma_1 = \begin{bmatrix} 1/n & 0 \\ 0 & n \end{bmatrix} \mathbb{Z}^2$$

is equal to the following set:

$$\left\{ \begin{bmatrix} \frac{k_1}{n} \\ nk_2 \end{bmatrix} : \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \in \mathbb{Z}^2 \right\}.$$

It is easy to see that Γ_1 and Γ_2 have the same volume. Indeed,

$$Vol(\Gamma_1) = \left| det \begin{bmatrix} 1/n & 0 \\ 0 & n \end{bmatrix} \right| = 1 = Vol(\Gamma_2)$$

Example 8 *Let*

$$\Gamma_1 = \begin{bmatrix} 1/3 & 0 \\ 0 & 3 \end{bmatrix} \mathbb{Z}^2 \text{ and } \Gamma_2 = \mathbb{Z}^2$$

be two lattices of the same volume. Below, we display two distinct fundamental domains for the lattices Γ_1, Γ_2. Γ_1 *is shown in pink and* Γ_2 *is shown in blue.*

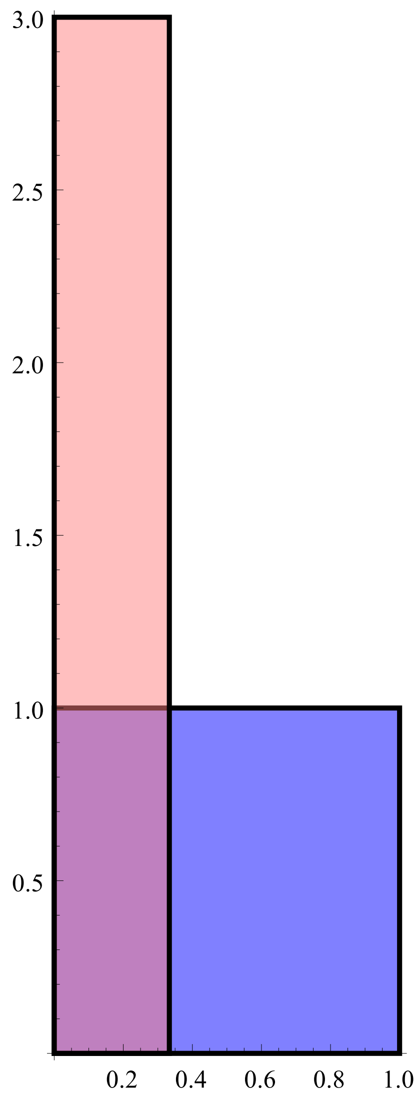

Figure 4:

We will show in this example that there exists a common fundamental domain for both Γ_1 and Γ_2. That is, there is a set $E \subset \mathbb{R}^2$ such that the collection of sets

$$\{E + \beta : \beta \in \Gamma_1\} \text{ and } \{E + \alpha : \alpha \in \Gamma_2\}$$

forms a tiling of the plane. First, it is fairly easy to see that

$$A = [0,1)^2$$

is a fundamental domain for the lattice $\mathbb{Z}^2$. Next, we also see that

$$A = [0,1)^2 = \left(\left[0, \frac{1}{3}\right) \times [0,1)\right) \cup \left(\left[\frac{1}{3}, \frac{2}{3}\right) \times [0,1)\right) \cup \left(\left[\frac{2}{3}, 1\right) \times [0,1)\right).$$

Defining

$$A_1 = \left[0, \frac{1}{3}\right) \times [0,1)$$
$$A_2 = \left[\frac{1}{3}, \frac{2}{3}\right) \times [0,1)$$
$$A_3 = \left[\frac{2}{3}, 1\right) \times [0,1)$$

then

$$A = A_1 \cup A_2 \cup A_3.$$

In general for any $k \in \{1,2,3\}$ we have that

$$A_k = \left[\frac{k-1}{3}, \frac{k}{3}\right) \times [0,1).$$

As a result of the above observation, we have that

$$A = \bigcup_{k=1}^{3} \left(\left[\frac{k-1}{3}, \frac{k}{3}\right) \times [0,1)\right).$$

Secondly, it is also not too hard to see that

$$\left[0, \frac{1}{3}\right) \times [0,3)$$

is a fundamental domain for the lattice

$$\Gamma_1 = \begin{bmatrix} 1/3 & 0 \\ 0 & 3 \end{bmatrix} \mathbb{Z}^2.$$

Moreover, we may partition $\left[0, \frac{1}{3}\right) \times [0,3)$ *as follows*

$$\begin{aligned} B &= \left[0, \frac{1}{3}\right) \times [0,3) \\ &= \left(\left[0, \frac{1}{3}\right) \times [0,1)\right) \cup \left(\left[0, \frac{1}{3}\right) \times [1,2)\right) \cup \left(\left[0, \frac{1}{3}\right) \times [2,3)\right). \end{aligned}$$

Put

$$B_1 = \left[0, \frac{1}{3}\right) \times [0,1)$$
$$B_2 = \left[0, \frac{1}{3}\right) \times [1,2)$$
$$B_3 = \left[0, \frac{1}{3}\right) \times [2,3)$$

so that

$$B = B_1 \cup B_2 \cup B_3.$$

In fact, in general for any $k \in \{1,2,3\}$ *we have that*

$$B_k = \left[0, \frac{1}{3}\right) \times [k-1, k)$$

and

$$B = \bigcup_{k=1}^{3} \left(\left[0, \frac{1}{3}\right) \times [k-1, k)\right)$$

is a fundamental domain for the lattice Γ_1*. We observe that for any* i, j *between one and three, the areas of the corresponding sets* A_i *and* B_j *are equal. Fix* $1 \leq k \leq 3$*. We want to show that there exits* $\beta_k \in \Gamma_1$ *and* $\alpha_k \in \Gamma_2$ *such that*

$$A_k + \alpha_k = B_k + \beta_k.$$

Since $\alpha_k \in \mathbb{Z}^2$ *we may write*

$$\alpha_k = \begin{bmatrix} j_k \\ i_k \end{bmatrix} \in \mathbb{Z}^2$$

such that

$$
\begin{aligned}
A_k + \alpha_k &= \left(\left[\frac{k-1}{3}, \frac{k}{3} \right) \times [0,1) \right) + \begin{bmatrix} j_k \\ i_k \end{bmatrix} \\
&= \left[\frac{k-1}{3} + j_k, \frac{k}{3} + j_k \right) \times [i_k, 1+i_k) .
\end{aligned}
$$

Next, since $\beta_k \in \Gamma_1$, *we may write*

$$
\beta_k = \begin{bmatrix} \frac{u_k}{3} \\ 3v_k \end{bmatrix} \text{ for } \begin{bmatrix} u_k \\ v_k \end{bmatrix} \in \mathbb{Z}^2.
$$

As a result of this

$$
\begin{aligned}
B_k + \beta_k &= \left(\left[0, \frac{1}{3} \right) \times [k-1, k) \right) + \begin{bmatrix} \frac{u_k}{3} \\ 3v_k \end{bmatrix} \\
&= \left[\frac{u_k}{3}, \frac{1}{3} + \frac{u_k}{3} \right) \times [k-1+3v_k, k+3v_k)
\end{aligned}
$$

Setting

$$
A_k + \alpha_k = B_k + \beta_k
$$

then

$$
\left[\frac{k-1}{3} + j_k, \frac{k}{3} + j_k \right) \times [i_k, 1+i_k) = \left[\frac{u_k}{3}, \frac{1}{3} + \frac{u_k}{3} \right) \times [k-1+3v_k, k+3v_k) .
$$

The above allows us to set the following system of equations

$$
\begin{cases}
\frac{k-1}{3} + j_k = \frac{u_k}{3} \\
\frac{k}{3} + j_k = \frac{1}{3} + \frac{u_k}{3} \\
i_k = k - 1 + 3v_k \\
1 + i_k = k + 3v_k
\end{cases}
$$

which we want to solve for

$$
i_k, j_k, u_k, v_k \in \mathbb{Z}.
$$

The solution of the above system of equations is

$$
\begin{aligned}
i_k &= k + 3v_k - 1, \\
j_k &= \frac{1}{3} u_k - \frac{1}{3} k + \frac{1}{3}.
\end{aligned}
$$

Clearly the solutions depend on $k \in \{1,2,3\}$. *We need to obtain some precise solution for each*

$$k \in \{1,2,3\}.$$

Indeed if $k = 1$

$$i_1 = 1 + 3v_1 - 1,$$
$$j_1 = \frac{1}{3}u_1$$

and we could pick for example

$$u_1 = v_1 = i_1 = j_1 = 0.$$

If $k = 2$ *then we have*

$$i_2 = 2 + 3v_2 - 1 = 1 + 3v_2$$
$$j_2 = \frac{1}{3}u_2 - \frac{2}{3} + \frac{1}{3} = \frac{u_2 - 1}{3}$$

and one could have for solutions the following:

$$v_2 = 0$$
$$i_2 = 1$$
$$u_2 = 4$$
$$j_2 = 1.$$

Finally, if $k = 3$ *then*

$$i_3 = 3 + 3v_3 - 1 = 2 + 3v_3$$
$$j_3 = \frac{u_3}{3} - 1 + \frac{1}{3} = \frac{u_3}{3} - \frac{2}{3} = \frac{u_3 - 2}{3}$$

So,

$$v_3 = 0$$
$$i_3 = 2$$
$$u_3 = 5$$
$$j_3 = 1.$$

In summary, a common fundamental domain for both Γ_1 *and* Γ_2 *is the set*

$$E = \bigcup_{k=1}^{3} \left(\left(\left[\frac{k-1}{3}, \frac{k}{3} \right) \times [0,1) \right) + \left[\begin{array}{c} j_k \\ i_k \end{array} \right] \right)$$

where (j_k, i_k) *are described above for*

$$k \in \{1,2,3\}.$$

More precisely,

$$E = \left(\left[0, \frac{1}{3} \right) \times [0,1) \right) \cup \left(\left[\frac{4}{3}, \frac{5}{3} \right) \times [1,2) \right) \cup \left(\left[\frac{5}{3}, 2 \right) \times [2,3) \right)$$

and this completes the proof.

Below, we display a common fundamental domain for Γ_1 *and* Γ_2*.*

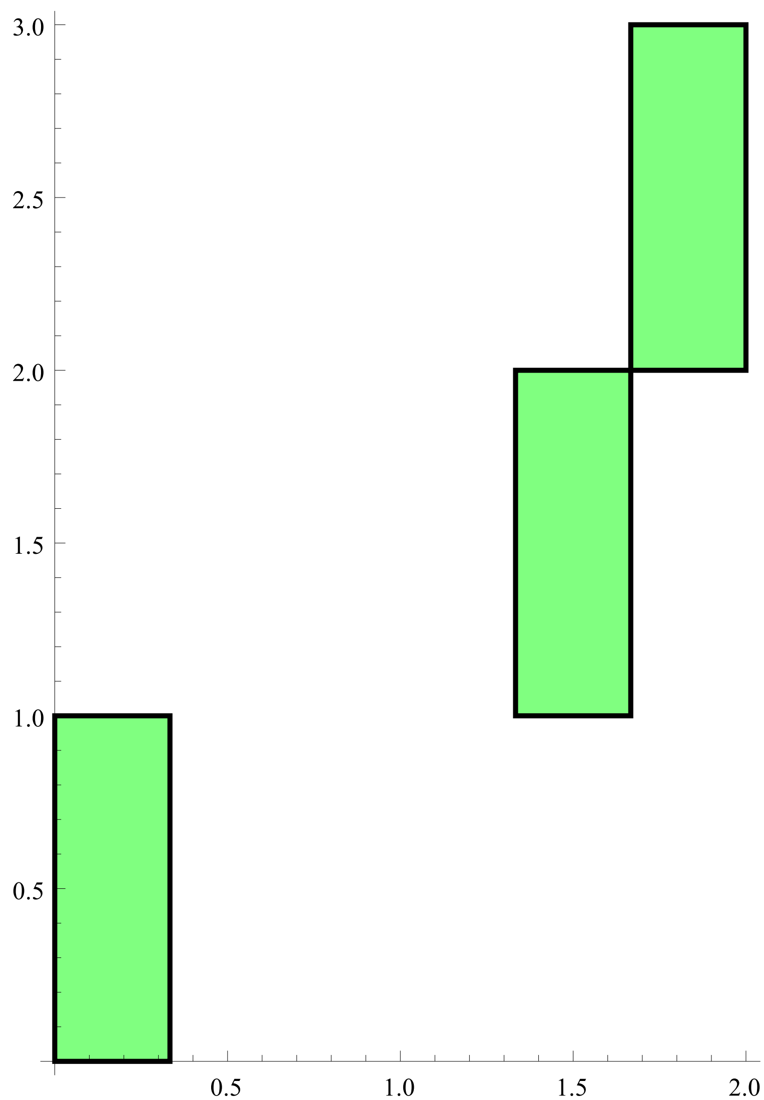

Figure 5:

Remark 9 *We remark here that Gotz E. Pfander, Peter Rashkov, Yang Wang proved in [6] Proposition* 5.1 *that the there exists a common* ***connected*** *fundamental domain for the lattices* Γ_1 *and* Γ_2 *described in Proposition 10. The common fundamental domain discovered in Proposition 10 is different from the one they discovered since our common fundamental domain is a disjoint union of disconnected sets.*

Proposition 10 *There exists a common fundamental domain for* Γ_1 *and* Γ_2*. That is, there exists a set* $E \subset \mathbb{R}^2$ *such that the collection of sets*

$$\{E + \beta : \beta \in \Gamma_1\}$$

and

$$\{E + \alpha : \alpha \in \Gamma_2\}$$

forms a tiling on the plane.

Proof. First of all, let $A = [0,1)^2$ be a fundamental domain for the lattice $\mathbb{Z}^2$. We partition A as follows.

$$A = \bigcup_{k=1}^{n} \left(\left[\frac{k-1}{n}, \frac{k}{n} \right) \times [0,1) \right)$$

such that

$$A_k = \left[\frac{k-1}{n}, \frac{k}{n} \right) \times [0,1)$$

for $k \in \{1, \cdots, n\}$. Secondly, it is also fairly easily to see that

$$B = \left[0, \frac{1}{n} \right) \times [0,n)$$

is a fundamental domain for the lattice Γ_1. Next, writing

$$B_k = \left[0, \frac{1}{n} \right) \times [k-1,k)$$

for $k \in \{1, 2, 3, \cdots, n\}$ then

$$B = \bigcup_{k=1}^{n} \left(\left[0, \frac{1}{n} \right) \times [k-1,k) \right)$$

For any $1 \leq i \leq n$ and $1 \leq j \leq n$, the areas of the corresponding sets A_i and B_j are equal. We want to show that there exists $\beta_k \in \Gamma_1$ and $\alpha_k \in \Gamma_2$ such that

$$A_k + \alpha_k = B_k + \beta_k$$

Since $\alpha_k \in \Gamma_2$, we can write $\alpha_k = \left\{ \begin{array}{c} j_k \\ i_k \end{array} \right\} \in \mathbb{Z}^2$ such that

$$A_k + \alpha_k = \left[\frac{k-1}{n} + j_k, \frac{k}{n} + j_k\right) \times [i_k, 1 + i_k)$$

Since $\beta_k \in \Gamma_1$, we can write $\beta_k = \left[\begin{array}{c} \frac{u_k}{n} \\ nv_k \end{array} \right]$ for $\left[\begin{array}{c} u_k \\ v_k \end{array} \right] \in \mathbb{Z}^2$ such that

$$B_k + \beta_k = \left[\frac{u_k}{n}, \frac{1}{n} + \frac{u_k}{n}\right) \times [k - 1 + nv_k, k + nv_k)$$

The above allows us to set up the following system of equations

$$\begin{aligned} \frac{k-1}{n} + j_k =& \frac{u_k}{n} \\ \frac{k}{n} + j_k =& \frac{1}{n} + \frac{u_k}{n} \\ i_k =& k - 1 + nv_k \\ 1 + i_k =& k + nv_k \end{aligned}$$

We want to solve this for $i_k, j_k, u_k, v_k \in \mathbb{Z}$. The solution for the above systems in terms of i_k, j_k are:

$$\begin{aligned} i_k =& k + nv_k - 1 \\ j_k =& \frac{u_k - k + 1}{n} \end{aligned}$$

for $k \in \{1, 2, 3, ..., n\}$. Clearly, the solutions depend on the value of k. Let us be more precise. Given any $v_k \in \mathbb{Z}$, then $i_k \in \mathbb{Z}$. Now, since j_k must also be an integer; for the second solution of the above system of equations, it suffices to pick $u_k = k - 1$, so that $j_k = 0$. As a result of the computations above, we can now obtain a common fundamental domain for both $\Gamma_1$and Γ_2 as follows. Put

$$E = \bigcup_{k=1}^{n} \left(A_k + \left[\begin{array}{c} j_k \\ i_k \end{array} \right] \right)$$

Then

$$E = \bigcup_{k=1}^{n} \left(\left[\frac{k-1}{n}, \frac{k}{n} \right) \times [k + nv_k - 1, k + nv_k) \right)$$

Clearly for any distinct k_1, k_2 the sets

$$\left[\frac{k-1}{n}, \frac{k}{n} \right) \times [k + nv_k - 1, k + nv_k)$$

do not overlap. Furthermore, by the construction procedure of fundamental domain, the set E is a common fundamental domain for both lattices Γ_1 and Γ_2 because

$$E = \bigcup_{k=1}^{n} \left(B_k + \begin{bmatrix} \frac{k-1}{n} \\ nv_k \end{bmatrix} \right) = \bigcup_{k=1}^{n} \left(\left[\frac{k-1}{n}, \frac{k}{n} \right) \times [k + nv_k - 1, k + nv_k) \right)$$

■

3.2 A Case of One Rectangular Lattice and One Non-Rectangular Lattice

Let n be a natural number larger than one. Define two full-rank lattices

$$\Gamma_1 = \begin{bmatrix} 1/n & 0 \\ n & n \end{bmatrix} \mathbb{Z}^2 \text{ and } \Gamma_2 = \mathbb{Z}^2$$

of the same volume. We want to find a set $E \subset \mathbb{R}^2$ such that E is a common fundamental domain for both Γ_1 and Γ_2. We observe here that

$$\Gamma_1 = \begin{bmatrix} 1/n & 0 \\ n & n \end{bmatrix} \mathbb{Z}^2$$

is equal to the following set:

$$\left\{ \begin{bmatrix} \frac{k_1}{n} \\ nk_1 + nk_2 \end{bmatrix} : \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \in \mathbb{Z}^2 \right\}.$$

It is easy to see that Γ_1 and Γ_2 have the same volume. Indeed,

$$Vol(\Gamma_1) = \left| det \begin{bmatrix} 1/n & 0 \\ n & n \end{bmatrix} \right| = 1 = Vol(\Gamma_2)$$

Example 11 *Let*

$$\Gamma_1 = \begin{bmatrix} 1/2 & 0 \\ 2 & 2 \end{bmatrix} \mathbb{Z}^2 \text{ and } \Gamma_2 = \mathbb{Z}^2$$

be two lattices of the same volume. Below, we show two distinct fundamental domains for the lattices Γ_1, Γ_2. Γ_1 *is shown in pink and* Γ_2 *is shown in blue.*

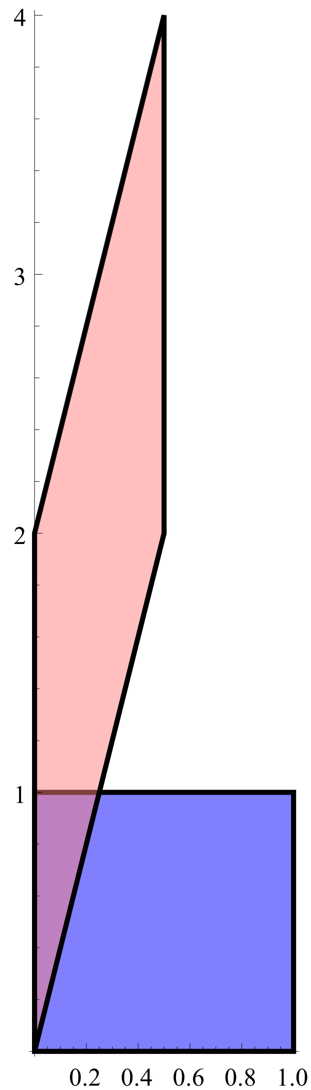

Figure 6:

We will show in this example that there exists a common fundamental domain for both Γ_1 and Γ_2. Let us suppose that E_1 and E_2 are fundamental domains for Γ_1 and Γ_2 respectively. Then,

$$E_1 = \bigcup_{k=1}^{4} (E_{1_k})$$

$$E_2 = \bigcup_{k=1}^{2} (E_{2_k})$$

$$= \left(E_{2_1}^1 \cup \left(E_{2_1} - E_{2_1}^1\right)\right) \cup \left(E_{2_2}^1 \cup \left(E_{2_2} - E_{2_2}^1\right)\right)$$

such that the following pairs are congruent.

$$E_{2_1}^1, E_{1_1}$$

$$E_{2_1} - E_{2_1}^1, E_{1_3}$$

$$E_{2_2}^1, E_{1_2}$$

$$E_{2_2} - E_{2_2}^1, E_{1_4}$$

To complete this, it is enough to find the following:

$$\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \Gamma_2$$

such that

$$E_{2_1}^1 + \alpha_1 = E_{1_1} + \gamma_1$$

$$\left(E_{2_1} - E_{2_1}^1\right) + \alpha_2 = E_{1_3} + \gamma_2$$

$$E_{2_2}^1 + \alpha_3 = E_{1_2} + \gamma_3$$

$$\left(E_{2_2} - E_{2_2}^1\right) + \alpha_4 = E_{1_4} + \gamma_4$$

To find the values of each γ_k and α_k, it is necessary to find the coordinates of the vertices of each E_{1_k} and E_{2_k} so that we can set up different *systems of equations to know how to shift each piece by the appropriate lattice point.Knowing that* $k_1, k_2, j_1, j_2 \in \mathbb{Z}$,*as well as the fact that* γ_k *is in the form of* $\begin{bmatrix} \frac{1}{2}k_1 \\ 2k_1 + 2k_2 \end{bmatrix}$ *and* α_k *is in the form of* $\begin{bmatrix} j_1 \\ j_2 \end{bmatrix}$ *we can deduce the following information. We only need to examine one vertex from each fundamental domain because when looking at more than one, it becomes redundant.*

For $E^1_{2_1}$ *and* E_{1_1}*, are already overlapping so we do not need to move them by* γ_1 *and* α_1. *Therefore,* $\gamma_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *and* $\alpha_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

For $E_{2_1} - E^1_{2_1}$ *and* E_{1_3},*we have:*

$$\begin{cases} \frac{1}{2} + \frac{1}{2}k_1 = \frac{1}{2} + j_1 \\ 2 + 2k_1 + 2k_2 = j_2 \end{cases}$$

From this, we have

$$j_1 = \frac{1}{2}k_1$$

$$j_2 = 2 + 2k_1 + 2k_2$$

It suffices to pick

$$k_1 = 2, k_2 = 0$$

so that

$$j_1 = 1, j_2 = 6.$$

Therefore, $\gamma_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ *and* $\alpha_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$.

For E_{2_2} *and* E_{1_2},*we have:*

$$\begin{cases} \frac{1}{4} + \frac{k_1}{2} = \frac{3}{4} + j_1 \\ 1 + 2k_1 + 2k_2 = j_2 \end{cases}$$

From this, we have

$$j_1 = \frac{k_1}{2} - \frac{1}{2}$$

$$j_2 = 1 + 2k_1 + 2k_2$$

It suffices to pick

$$k_1 = 1, k_2 = 0$$

so that

$$j_1 = 0, j_2 = 3.$$

Therefore, $\gamma_3 = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}$ *and* $\alpha_3 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

For $E_{2_2} - E^1_{2_2}$ *and* E_{1_4}*, we have:*

$$\begin{cases} \frac{1}{2} + \frac{1}{2}k_1 = 1 + j_1 \\ 4 + 2k_1 + 2k_2 = 1 + j_2 \end{cases}$$

From this, we have

$$j_1 = \frac{1}{2}k_1 - \frac{1}{2}$$

$$j_2 = 2k_1 + 2k_2 + 3$$

It suffices to pick

$$k_1 = 1, k_2 = 2$$

so that

$$j_1 = 0, j_2 = 9.$$

Therefore, $\gamma_4 = \begin{bmatrix} \frac{1}{2} \\ 6 \end{bmatrix}$ *and* $\alpha_4 = \begin{bmatrix} 0 \\ 9 \end{bmatrix}$.

If we put

$$A_1 = E^1_{2_1}$$

$$A_2 = E_{2_1} - E^1_{2_1}$$

$$A_3 = E^1_{2_2}$$

$$A_4 = E_{2_2} - E^1_{2_2}$$

then the common fundamental domain is given as follows

$$E = \bigcup_{k=1}^{4} (A_k + \alpha_k)$$

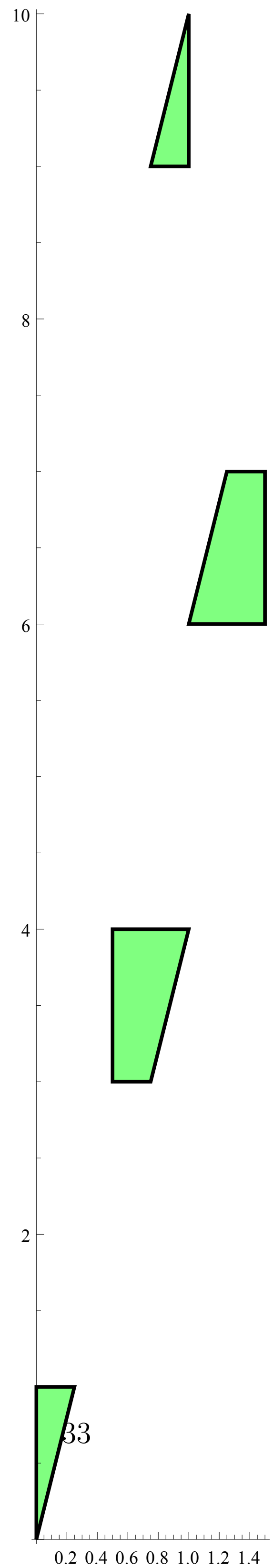

Figure 7:

Let n be a fixed natural number. Let

$$\Gamma_1 = \begin{bmatrix} \frac{1}{n} & 0 \\ n & n \end{bmatrix} \mathbb{Z}^2, \text{ and } \Gamma_2 = \mathbb{Z}^2$$

be two full-rank lattices.

Proposition 12 *There exists a common fundamental domain for Γ_1 and Γ_2 which consists of a disjoint union of $2n$ connected sets*

Proof. Let

$$E_1 = \begin{bmatrix} \frac{1}{n} & 0 \\ n & n \end{bmatrix} [0,1)^2$$

be a fundamental domain for Γ_1 and let

$$E_2 = [0,1)^2$$

be a fundamental domain for Γ_2. We write

$$E_1 = \bigcup_{k=1}^{2n} E_{1_k}$$

such that for $k \in \{1, 2, \cdots, 2n\}$

$$E_{1_k} = \left(\left[0, \frac{1}{n}\right) \times [k-1, k) \right) \cap E_1.$$

Furthermore, we observe that for each fixed k

$$\left(E_{1_{k+n}} + \begin{bmatrix} 0 \\ -n \end{bmatrix} \right) \cup E_{1_k}$$

is congruent to the rectangle

$$\left[0, \frac{1}{n}\right) \times [0,1)$$

and therefore has area $\frac{1}{n}$. Next, we write

$$E_2 = \bigcup_{k=1}^{n} E_{2_k}$$

where for each $k \in \{1, 2, \cdots, n\}$

$$E_{2_k} = \left[\frac{k-1}{n}, \frac{k}{n}\right) \times [0, 1) \,.$$

Furthermore, for each $k \in \{1, 2, \cdots, n\}$ we write that

$$E_{2_k} = E^1_{2_k} \cup \left(E_{2_k} - E^1_{2_k}\right)$$

such that

$$E^1_{2_k} \text{ and } E^1_{1_k}$$

are congruent and

$$E_{2_k} - E^1_{2_k} \text{ and } E^1_{1_{k+n}}$$

are congruent as well. To find a common fundamental domain for both lattices, it suffices then to find for each $k \in \{1, 2, \cdots, n\}$

$$\begin{aligned} \gamma_{1_k}, \gamma_{2_k} &\in \Gamma_1 \\ \alpha_{1_k}, \alpha_{2_k} &\in \Gamma_2 \end{aligned}$$

such that

$$\begin{aligned} E^1_{2_k} + \alpha_{1_k} &= E^1_{1_k} + \gamma_{1_k} \\ \left(E_{2_k} - E^1_{2_k}\right) + \alpha_{2_k} &= E^1_{1_{k+n}} + \gamma_{2_k}. \end{aligned}$$

A common fundamental domain would then be equal to

$$\begin{aligned} E &= \bigcup_{k=1}^{n} \left(\left(E^1_{1_k} + \gamma_{1_k}\right) \cup \left(E^1_{1_{k+n}} + \gamma_{2_k}\right)\right) \\ &= \bigcup_{k=1}^{n} \left(\left(E^1_{2_k} + \alpha_{1_k}\right) \cup \left(\left(E_{2_k} - E^1_{2_k}\right) + \alpha_{2_k}\right)\right). \end{aligned}$$

Now, to find the $\gamma_{1_k}, \alpha_{1_k}$, we define

$$\gamma_{1_k} = \begin{bmatrix} \frac{1}{n} & 0 \\ n & n \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix} \text{ and } \alpha_{1_k} = \begin{bmatrix} q_k \\ p_k \end{bmatrix}$$

where

$$\begin{bmatrix} u_k \\ v_k \end{bmatrix}, \begin{bmatrix} q_k \\ p_k \end{bmatrix} \in \mathbb{Z}^2.$$

Next, we consider the following equation:

$$\left(\begin{bmatrix} 0 \\ k \end{bmatrix} + \begin{bmatrix} \frac{1}{n} & 0 \\ n & n \end{bmatrix}\begin{bmatrix} u_k \\ v_k \end{bmatrix}\right) - \left(\begin{bmatrix} \frac{k-1}{n} \\ 1 \end{bmatrix} + \begin{bmatrix} q_k \\ p_k \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The above is equivalent to the system of equations

$$\begin{cases} -\frac{1}{n}(k - u_k + nq_k - 1) = 0 \\ k - p_k + nu_k + nv_k - 1 = 0 \end{cases}$$

which has for solution

$$\begin{aligned} p_k &= k + nu_k + nv_k - 1 \\ q_k &= \frac{u_k - k + 1}{n}. \end{aligned}$$

Put

$$u_k = k - 1$$

and $v_k = 0$. Then

$$\begin{aligned} p_k &= k + nu_k + nv_k - 1 \\ &= k + n(k-1) + nv_k - 1 \\ &= k - n + nv_k + kn - 1 \\ &= k - n + kn - 1 \end{aligned}$$

and

$$q_k = \frac{k - 1 - k + 1}{n} = 0.$$

Now, to find the γ_{2_k} and α_{2_k} we define

$$\gamma_{2_k} = \begin{bmatrix} \frac{1}{n} & 0 \\ n & n \end{bmatrix}\begin{bmatrix} w_k \\ z_k \end{bmatrix} \text{ and } \alpha_{2_k} = \begin{bmatrix} q_k \\ p_k \end{bmatrix}$$

where

$$\begin{bmatrix} w_k \\ z_k \end{bmatrix}, \begin{bmatrix} q_k \\ p_k \end{bmatrix} \in \mathbb{Z}^2$$

To find

$$\begin{bmatrix} q_k \\ p_k \end{bmatrix},$$

we consider the following equation.

$$\left(\begin{bmatrix} \frac{1}{n} \\ k+n \end{bmatrix} + \begin{bmatrix} \frac{1}{n} & 0 \\ n & n \end{bmatrix} \begin{bmatrix} w_k \\ z_k \end{bmatrix}\right) - \left(\begin{bmatrix} \frac{k}{n} \\ 1 \end{bmatrix} + \begin{bmatrix} q_k \\ p_k \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which is equivalent to the system of equations

$$\begin{cases} -\frac{1}{n}\left(k - w_k + nq_k - 1\right) \\ k + n - p_k + nw_k + nz_k - 1 \end{cases}.$$

With some straightforward calculations, we obtain that

$$\begin{aligned} p_k &= k + n + nw_k + nz_k - 1, \\ q_k &= \frac{1}{n}\left(w_k - k + 1\right). \end{aligned}$$

Picking

$$w_k = k - 1 + kn$$

then for $z_k = 0$

$$\begin{aligned} p_k &= k + n + n\left(k - 1 + kn\right) + nz_k - 1 \\ &= k + kn^2 + kn - 1 \\ q_k &= \frac{1}{n}\left(k - 1 + kn - k + 1\right) = k. \end{aligned}$$

Finally, we describe the common fundamental domain as follows.

$$E = \bigcup_{k=1}^{n}\left(\left(E_{2_k}^1 + \begin{bmatrix} 0 \\ k - n + kn - 1 \end{bmatrix}\right) \cup \left(\left(E_{2_k} - E_{2_k}^1\right) + \begin{bmatrix} k \\ k + kn^2 + kn - 1 \end{bmatrix}\right)\right)$$

■

4 Conclusion

The work presented in this thesis provides explicit construction of common fundamental domains for two classes of lattices of the same volume in a two-dimensional setting. For the classes considered in this thesis, we restrict ourselves to rational lattices (all points in the lattices have rational coordinates). We observe that our technique fails if we remove the assumption that the lattices are rational. It would be interesting to investigate if it is possible to provide explicit constructions of common fundamental domains for two lattices of the same volume where one is a rational lattice and the other is not. It would also be interesting to compare the topological properties of these fundamental domains with the type of fundamental domains constructed in this thesis. We anticipate that such common fundamental domain must be unbounded. We also remark here that our techniques suggest that it may be possible to construct similar rdomains in higher dimensional euclidean spaces.

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