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The Permutahedron π_n is Hamiltonian

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Abstract. The permutahedron $\pi_{n-1} \subseteq \Re^n$ is defined as the convex hull of all vertices obtained by permuting the coordinates of the vector <1, 2, …, n>.[11] Its vertices can be indentified with the permutations in S_n in such a way that two vertices are connected by an edge if and only if the corresponding permutations differ by an adjacent transposition (the permutation that maps $x_i \mapsto i$ corresponds to the vertex $\langle x_1, x_2, .., x_n \rangle$). In this paper, we prove by induction that π_n is Hamiltonian for $n \geq 2$. Our constructive proof finds a Hamiltonian cycle for π_n given any Hamiltonian cycle of π_{n-1}.

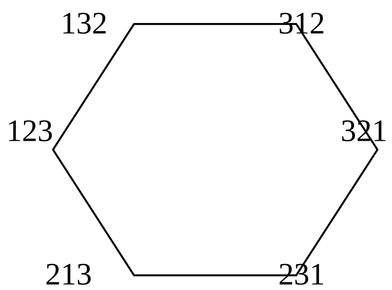
Mathematics subject classification: 05C45

Keywords: Hamiltonian cycle, Permutation, Permutahedron

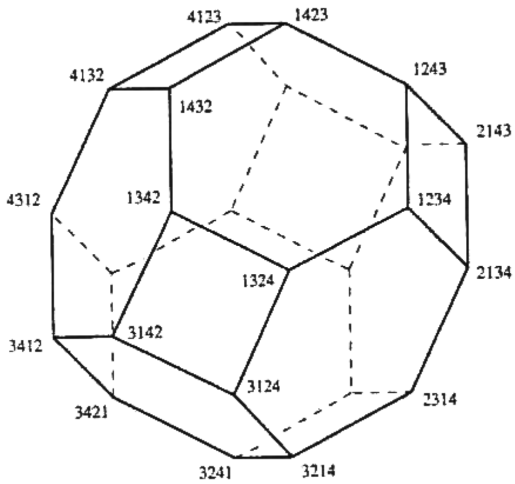
Introduction: The permutahedron is a beautiful geometric object, bringing together concepts from combinatorics, geometry and discrete mathematics. The problem of generating a list of permutations of 1, 2, …, n via a sequence of transpositions has been thoroughly explored (see the surveys by Savage [7] and Sedgewick [8], or any of [1], [2], [4], [6] or [10]). In the early 1960's, Johnson [5] and Trotter [9] independently discovered an elegant algorithm for generating this list of permutations using only *adjacent* permutations, thereby describing a method of finding a Hamiltonian cycle on the edge graph of the permutahedron. In this paper we present a method of finding such a Hamiltonian cycle which differs from that of Johnson and Trotter. In particular, our method exploits the polyhedral structure of the permutahedron in its inductive construction. In [3], we use our proof to find some lower bounds for the number of Hamiltonian cycles of the permutahedron.

Theorem: The edge graph of the permutahedron π_n is Hamiltonian (n>1).

Definition 1: The permutahedron $\pi_{n-1} \subseteq \Re^n$ is defined as the convex hull of all vertices obtained by permuting the coordinates of the vector <1, 2, …, n>.[11] Its vertices can be identified with the permutations in S_n in such a way that two vertices are connected by an edge if and only if the corresponding permutations differ by an adjacent transposition.



π_2



π_3

Definition 2: A sequence of transpositions

As we stated earlier, two vertices of a permutahedron are adjacent if they differ by an adjacent transposition. For example, 123 is adjacent to 132 by the transposition 23, and adjacent to 312 by the transposition 12. A cycle is determined by identifying a vertex and a sequence of those adjacent transpositions, for example if we start with the vertex 123 and the sequence 12-23-12-23-12-23, we get the cycle 123-213-231-321-312-132-123.

Proof: We prove that π_n is Hamiltonian by induction.

Base case: π_2 is a hexagon. The edges of the hexagon form a Hamiltonian cycle connecting its vertices.

Inductive assumption: Assume there is a Hamiltonian cycle connecting the vertices of π_{n-1}.

Inductive step: We shall use this cycle to show the existence of a Hamiltonian cycle connecting the vertices of π_n ($n>2$).

The permutahedron π_n has ($n+1$)! vertices, each of which is equated with a permutation of the numbers 1, 2, 3, ... n+1. List these in lexicographic order.
If n = 4 the vertices of π_3 are as follows:

1234 1243	2134 2143	3124 3142	4123 4132
1324 1342	2314 2341	3214 3241	4213 4231
1423 1432	2413 2431	3412 3421	4312 4321

For n = 5, the vertices of π_4 are:

12345 12354 12435 12453 12534 12543	21345 21354 21435 21453 21534 21543	31245 31254 31425 31452 31524 31542	41235 41253 41325 41352 41523 41532	51234 51243 51324 51342 51423 51432
13245 13254 13425 13452 13524 13542	23145 23154 23415 23451 23514 23541	32145 32154 32415 32451 32514 32541	42135 42153 42315 42351 42513 42531	52134 52143 52314 52341 52413 52431
14235 14253 14325 14352 14523 14532	24135 24153 24315 24351 24513 24531	34125 34152 34215 34251 34512 34521	43125 43152 43215 43251 43512 43521	53124 53142 53214 53241 53412 53421
15234 15243 15324 15342 15423 15432	25134 25143 25314 25341 25413 25431	35124 35142 35214 35241 35412 35421	45123 45132 45213 45231 45312 45321	54123 54132 54213 54231 54312 54321

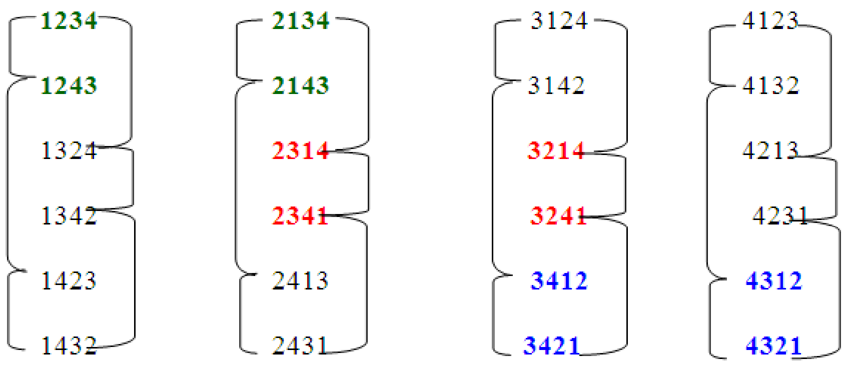
Note that the first *n!* vertices in the list (the first column in the array) have a **1** as the first digit in their permutation, the next *n!* vertices have a **2** as their first digit, etc.

Indeed, the lexicographic ordering sorts the vertices of π_n into *n+1* blocks (the columns of the array) of *n!* vertices, each block representing the vertices of an *n-1* dimensional facet of π_n ; each of these facets has the shape of π_{n-1}. We will call the *n-1* permutahedron formed by the vertices whose permutations start with 1 "facet 1" of our permutahedron, and the *n-1* permutahedron whose vertices start with *k* will be called "facet *k*" of our permutahedron

By the inductive hypothesis, we may connect the vertices of facet 1 in a Hamiltonian cycle.

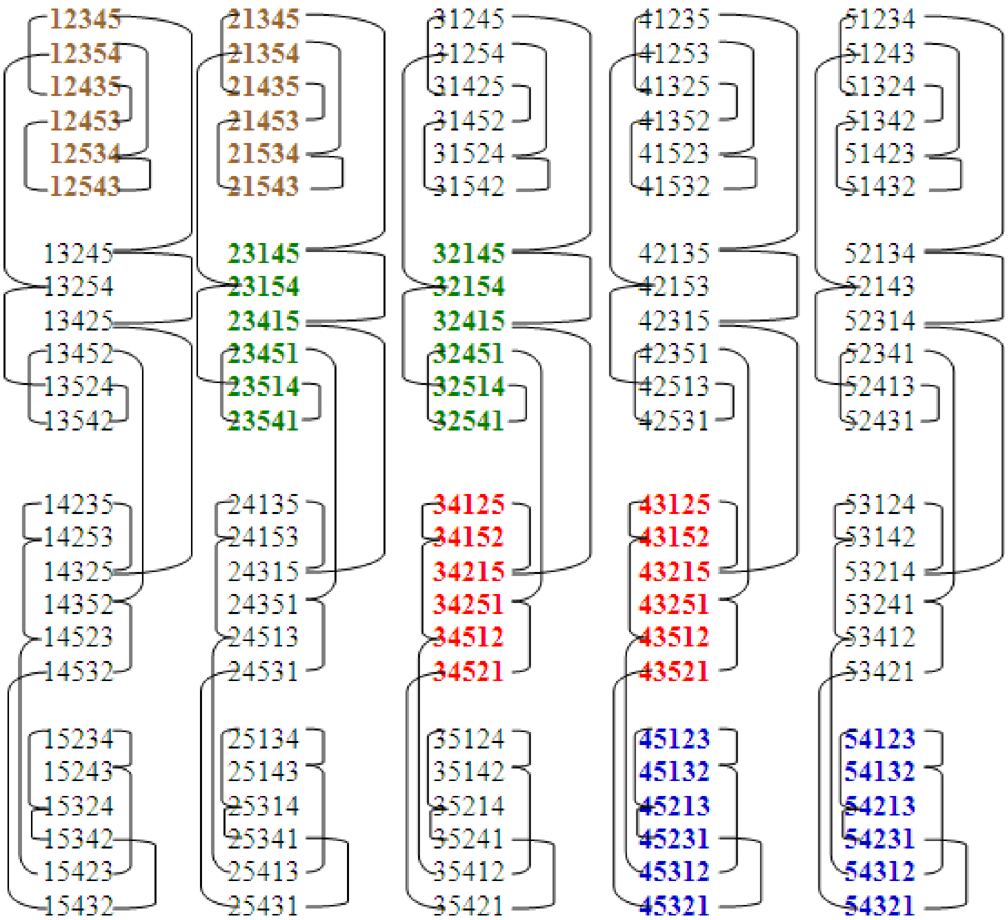
Connect the vertices of each of the remaining *(n-1)*-dimensional permutahedral facets (there are *n* of these facets, not including facet 1) in a similar fashion. In particular, we take care that *the Hamiltonian cycle visits the vertices of each facet in the same lexicographic order that the vertices of the first were visited in*.

An example: π_3



Notice that each facet (column in the array) is just π_2 and each cycle in each facet has the same sequence of transpositions 23-34-23-34-23-34-23.

An example: π_4



Each facet (column in the array) is just π_3 and each cycle in each facet has the same sequence of transpositions 23-34-23-34-45-34-23-34-45-34-45-34-23-34-23-34-45-34-23-34-45-34-45-34.

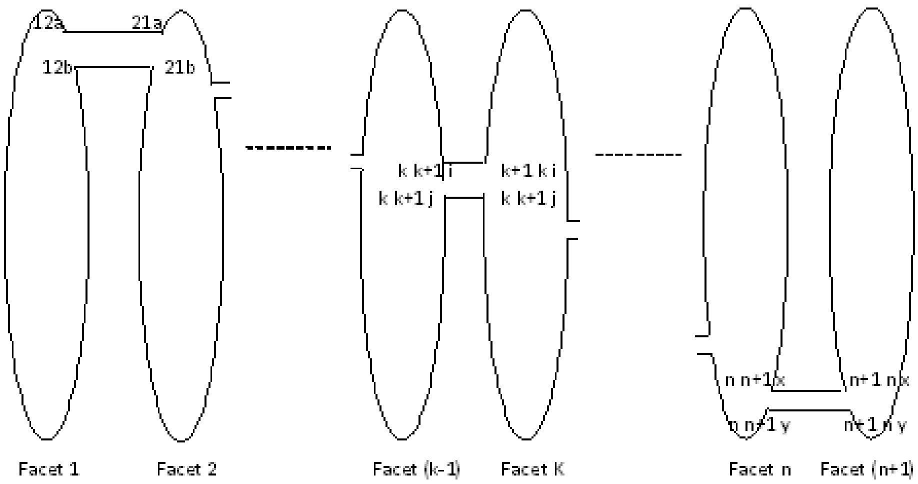
Now we join the Hamiltonian cycle connecting vertices of facet 1 to the one connecting the vertices of facet 2 to generate a Hamiltonian cycle connecting all vertices with first coordinate 1 or 2. To do this, we locate a "jumper" edge in the cycle that connects two vertices whose permutation starts 12; call it 12a.

<u>Claim:</u> Such a vertex must exist.

Consider any vertex whose permutation starts with 12. It has n neighbors. Of these, n-2 also start with 12, one starts with 21, and one starts with 1 and has a second digit that is not 2. The Hamiltonian cycle connecting the vertices of facet 1 does not visit the neighbor whose permutation starts with 21, so at most one of the edges in the cycle leads from 12a to a vertex whose permutation does not starting with 12. Hence, at least one edge of the Hamiltonian cycle must connect 12a to some vertex 12b whose permutation starts with 12. We will remove this jumper to create a "bridge" connecting the Hamiltonian cycles of facets 1 and 2.

Because the vertices of facet 2 are connected in the same lexicographic order as the vertices of facet 1, there is a pair of vertices 21a and 21b lexicographically correspondent with 12a and 12b that are joined by an edge of the Hamiltonian cycle connecting the vertices of facet 2. Deleting edges (12a, 12b) and (21a, 21b) and replacing them by (12a, 21a) and (12b, 21b) joins the two Hamiltonian cycles into one larger Hamiltonian cycle connecting all vertices of facets 1 and 2.

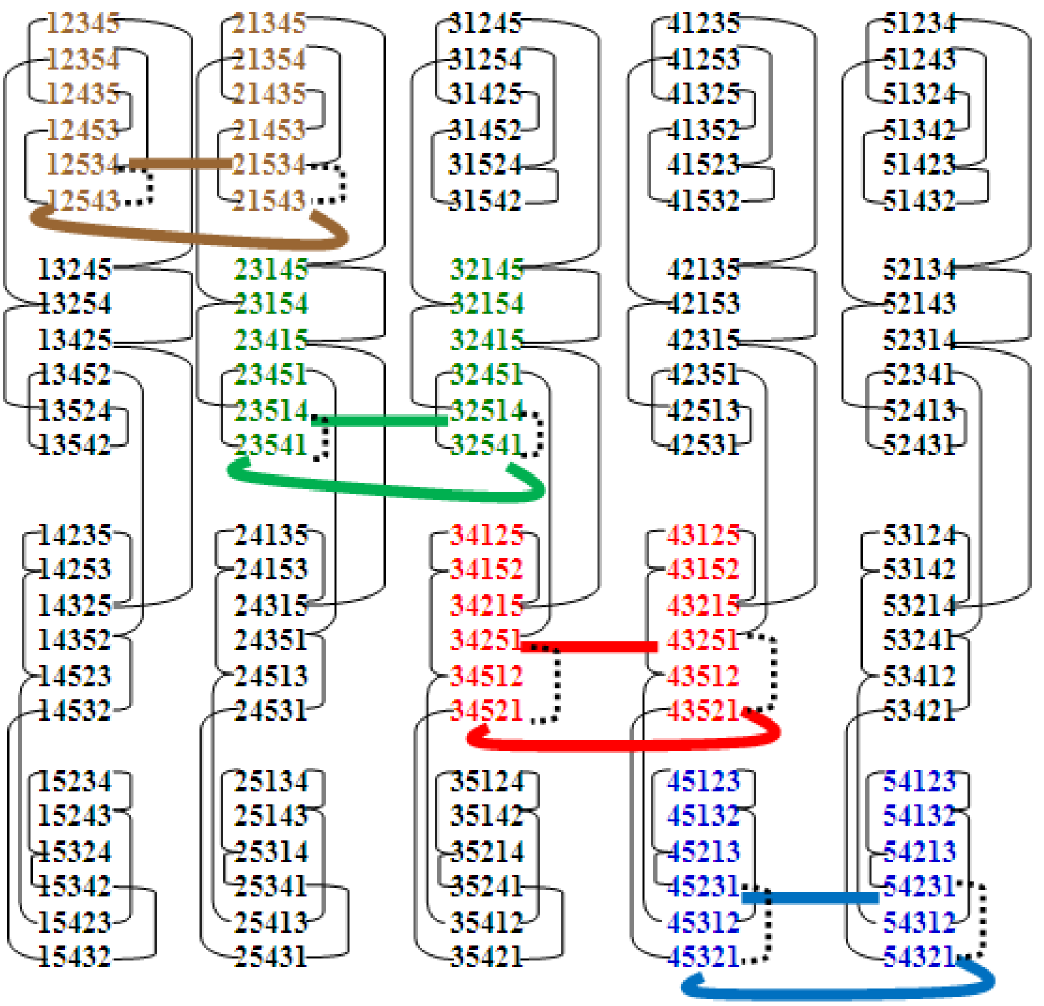
A similar argument serves to join facet k to facet $k+1$ by replacing "jumpers" (k $k+1$ a, k $k+1$ b) and ($k+1$ k a, $k+1$ k b) by the "bridge" (k $k+1$ a, $k+1$ k a) and (k $k+1$ b, $k+1$ k b).



Note that since facet k is connected to facet k-1 by edges joining vertices starting with k-1, k and k, k-1, these changes to the cycle on facet k do not affect the choice of jumpers available to form bridges from vertices starting with k, $k+1$ to those starting $k+1$, k.

Joining the Hamiltonian cycle traversing facet k to that traversing facet $k+1$ for k ranging from 1 to n creates a Hamiltonian cycle traversing the edge graph of π_n .

Here is a Hamiltonian cycle of π_4 generated by this construction:



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